A uniqueness result for 3-homogeneous latin trades

Nicholas J. Cavenagh

Abstract. A latin trade is a subset of a latin square which may be replaced with a disjoint mate to obtain a new latin square. A k-homogeneous latin trade is one which intersects each row, each column and each entry of the latin square either 0 or k times. In this paper, we show that a construction given by Cavenagh, Donovan and Drápal for 3-homogeneous latin trades in fact classifies every minimal 3-homogeneous latin trade. We in turn classify all 3-homogeneous latin trades. A corollary is that any 3-homogeneous latin trade may be partitioned into three, disjoint, partial transversals.

Keywords: latin square, latin trade, critical set

Classification: 05B15

1. Introduction

The earliest study of latin trades may be found in [12], where they are referred to as *exchangeable partial groupoids*. Later, latin trades were re-examined via research into *critical sets* (minimal defining sets of latin squares) and the study of intersections between the operation tables of *quasigroups* (which are exactly latin squares) ([14]). (See [17] for an up-to-date survey paper of critical sets.) Until recently, critical set researchers were unaware of this earlier research, causing some overlap of results.

For example, some basic properties of latin trades are given in both [12] and [7]. In both these papers latin trades of small size (up to at most 11) are classified. Recently a number of geometric and algebraic interpretations of latin trades have emerged ([9], [10], [11]). Results on other kinds of combinatorial trades may be found in [9] and [18].

In [8], a construction is given that decomposes any latin trade into a "sum" of intercalates (latin trades of size 4). In some sense this result is analagous to the decomposition of any permutation into involutions. Indeed, it may be profitable to think of a latin trade as a permutation generalized into two dimensions.

Some reasons for interest in k-homogeneous latin trades are: (1) they often partition into disjoint partial transversals, and thus have implications for partial orthogonality; (2) the k-homogeneous property is invariant under conjugacies (see the next section) and (3) they often have, compared with other minimal latin trades, large size with respect to the order of the latin square they are contained in (see, for example, [2]). The last property indicates that k-homogneous latin trades may be useful in locating small critical sets, as a critical set must intersect every latin trade in the latin square.

A survey of results on critical sets may be found in [16]. Currently the best known lower bound for the size of a critical set in a latin square of order n is $\lfloor (4n-8)/3 \rfloor$ ([15]). The smallest critical sets of order n so far constructed have size $\lfloor n^2/4 \rfloor$ and exist in the back circulant latin square B_n, the latin square based on the addition table for the integers modulo n ([6]). It is conjectured, for example in [1], that no smaller critical sets exist.

Our bank of knowledge on latin trades is probably strongest when considering latin trades that occur in B_n. Here we can exploit the cyclic properties of the underlying group $(\mathbb{Z}_n, +)$. It has been shown ([13], or [3] for an alternative proof), that the size of a latin trade in B_n is at least $O(\log p)$, where p is the least prime that divides n. In [9], it is shown that certain decompositions of equilateral triangles into smaller equilateral triangles can be used to construct latin trades in B_n. These constructions give latin trades of size $O((\log n)^2)$ in B_n for any integer n.

Recently Drápal [10] has shown that latin trades may be considered as coherently orientable digraphs, and thus may be associated with a non-negative, integer genus. Furthermore there seems to be some connection between the genus of a latin trade and the type of latin square which contains that latin trade. For example, every minimal latin trade with genus equal to 0 known to the author can be embedded in B_n.

A minimal, 3-homogeneous latin trade will always have genus equal to 1. There is an abundance of 3-homogeneous latin trades in the operation table for $((\mathbb{Z}_2)^n, +)$; in fact [Ca4] gives an infinite family of such latin trades. It is conjectured that for $n \geq 2$, there exists a minimal 3-homogeneous latin trade that occurs in $((\mathbb{Z}_2)^n, +)$ but not in $((\mathbb{Z}_2)^{n-1}, +)$. A construction for 4-homogeneous latin trades is given in [5].

The uniqueness result in this paper means we have a construction that gives all possible 3-homogeneous latin trades. This contributes to the classification of latin trades with genus 1 and latin trades that occur in $((\mathbb{Z}_2)^n, +)$. A corollary of this result is that every 3-homogeneous latin trade may be partitioned into disjoint partial transversals.

2. Definitions

We start with basic definitions which allow us to state and prove our main results.

Let $N = N(n)$ be some finite set of size n. (Unless otherwise stated, assume that $N(n) = \{0, 1, \ldots, n-1\}$.) Let $R(N) = \{r_i \mid i \in N\}$, $C(N) = \{c_i \mid i \in N\}$ and $E(N) = \{e_i \mid i \in N\}$.

A *partial latin square* P of order n is a set of ordered triples of the form (r_i, c_j, e_k), where $r_i \in R(N)$, $c_j \in C(N)$ and $e_k \in E(N)$ with the following properties:

- if $(r_i, c_j, e_k) \in P$ and $(r_i, c_j, e_{k'}) \in P$ then $k = k'$,
- if $(r_i, c_j, e_k) \in P$ and $(r_i, c_{j'}, e_k) \in P$ then $j = j'$ and
- if $(r_i, c_j, e_k) \in P$ and $(r_{i'}, c_j, e_k) \in P$ then $i = i'$.

We may also represent a partial latin square P as an $n \times n$ array with entries chosen from the set $E(N)$ such that if $(r_i, c_j, e_k) \in P$, the *entry* e_k occurs in cell (r_i, c_j).

A partial latin square has the property that each entry occurs at most once in each row and at most once in each column. If all the cells of the array are filled then the partial latin square is termed a latin square. For a given partial latin square P the set of cells

$$\mathcal{S}_P = \{(r_i, c_j) \mid (r_i, c_j, e_k) \in P, \text{ for some } e_k \in E(N)\}$$

is said to determine the *shape* of P and $|\mathcal{S}_P| = |P|$ is said to be the *size* of the partial latin square. That is, the size of P is the number of non-empty cells in the array. For each $i \in N$, let $\mathcal{R}_P^i$ denote the set of entries occurring in row r_i of P. Formally, $\mathcal{R}_P^i = \{e_k \mid (r_i, c_j, e_k) \in P\}$. For each $j \in N$, we define $\mathcal{C}_P^j = \{e_k \mid (r_i, c_j, e_k) \in P\}$. Finally, for each $k \in N$, we define $\mathcal{E}_P^k = \{(r_i, c_j) \mid (r_i, c_j, e_k) \in P\}$.

A non-empty partial latin square P is a *partial transversal* if for each $i \in N$, $|\mathcal{R}_P^i|, |\mathcal{C}_P^i|, |\mathcal{E}_P^i| \leq 1$.

A partial latin square T of order n is said to be a *latin trade* (or *latin interchange*) if $T \neq \emptyset$ and there exists a partial latin square T' (called a *disjoint mate* of T) of order n, such that

- $\mathcal{S}_T = \mathcal{S}_{T'}$,
- if $(r_i, c_j, e_k) \in T$ and $(r_i, c_j, e_{k'}) \in T'$ then $k \neq k'$,
- for each $i \in N(n)$, $\mathcal{R}_T^i = \mathcal{R}_{T'}^i$,
- for each $j \in N(n)$, $\mathcal{C}_T^j = \mathcal{C}_{T'}^j$.

We often refer to T and T' together as a *latin bitrade* (T, T'). Note that (T, T') is a latin bitrade if and only if (T', T) is a latin bitrade.

A latin trade is said to be *minimal* if it contains no smaller latin trade. A latin trade T of order n is said to be k-homogeneous if

- for each $i \in N(n)$, $|\mathcal{R}_T^i| = 0$ or k, and
- for each $i \in N(n)$, $|\mathcal{C}_T^i| = 0$ or k, and
- for each $i \in N(n)$, $|\mathcal{E}_T^i| = 0$ or k.

Clearly if T is a k-homogeneous latin trade of order n, its size is equal to km for some integer m, where $n \geq m \geq k$. A minimal 2-homogeneous latin trade is uniquely a 2×2 latin subsquare.

A *critical set* in a latin square L (of order n) is a partial latin square $P \subseteq L$, such that

(1) L is the only latin square of order n which has element e_k in cell (r_i, c_j) for each $(r_i, c_j, e_k) \in P$; and
(2) no proper subset of P satisfies (1).

It follows that a critical set P in a latin square L must intersect every latin trade in L; and is minimal with respect to this property.

Two latin squares are *isotopic* if they are equivalent under some relabelling of rows, columns and entries. If P is a partial latin square, we define a *conjugate* (sometimes *parastrophy*) of P to be one of the partial latin squares given by:

$$\begin{gathered}\{(r_j, c_k, e_i) \mid (r_i, c_j, e_k) \in P\}, \{(r_k, c_i, e_j) \mid (r_i, c_j, e_k) \in P\},\\ \{(r_i, c_k, e_j) \mid (r_i, c_j, e_k) \in P\}, \{(r_k, c_j, e_i) \mid (r_i, c_j, e_k) \in P\},\\ \{(r_j, c_i, e_k) \mid (r_i, c_j, e_k) \in P\}, \{(r_i, c_j, e_k) \mid (r_i, c_j, e_k) \in P\}.\end{gathered}$$

Taking a conjugate preserves many of the combinatorial properties of a partial latin square; a fact we exploit throughout this paper. For example, the k-homogeneous property is invariant under conjugacies.

A construction for 3-homogeneous latin trades based on a hexagonal packing of circles in the plane is given in [4]. The next theorem, from [2], gives an equivalent definition of such latin trades. In this paper we show that any minimal 3-homogeneous latin trade meets this definition. This verifies the uniqueness of the construction in [4].

Theorem 1 ([2])**.** *Let m_1, m_2 be integers such that $m_1 \geq 2$, $m_2 \geq 1$. Let $k \in N(m_1)$ and if $m_2 = 1$ then $k \geq 2$. Let $r_{(i,j)}$, $c_{(i,j)}$ and $e_{(i,j)}$ be distinct rows, columns and entries of a partial latin square, for each $i \in N(m_2)$ and $j \in N(m_1)$. Define the following sets of ordered triples:*

1. $T_1 = \{(r_{(i,j)}, c_{(i,j)}, e_{(i,j)}) \mid (i,j) \in N(m_2) \times N(m_1)\}$;
2. $T_2 = \{(r_{(i,j)}, c_{(i,j+1 (\mathrm{mod}\, m_1))}, e_{(i+1,j)}) \mid 0 \leq i < m_2 - 1, j \in N(m_1)\}$;
3. $T_3 = \{(r_{(i,j)}, c_{(i,j+1 (\mathrm{mod}\, m_1))}, e_{(0,j+k (\mathrm{mod}\, m_1))}) \mid i = m_2 - 1, j \in N(m_1)\}$;
4. $T_4 = \{r_{(i,j)}, c_{(m_2-1,j-k+1 (\mathrm{mod}\, m_1))}, e_{(i,j+1 (\mathrm{mod}\, m_1))}) \mid i = 0, j \in N(m_1)\}$;
5. $T_5 = \{(r_{(i,j)}, c_{(i-1,j+1 (\mathrm{mod}\, m_1))}, e_{(i,j+1 (\mathrm{mod}\, m_1))}) \mid 0 < i \leq m_2 - 1, j \in N(m_1)\}$.

Then $T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$ is a 3-homogeneous latin trade.

3. Properties

In this section we establish some properties common to every 3-homogeneous latin trade. These properties will ultimately show that Theorem 1 identifies every possible minimal 3-homogeneous latin trade.

Lemma 2. *Let T be a latin trade with disjoint mate T'. Let r_i be a row of T, c_j and $c_{j'}$ distinct columns of T and e_k, $e_{k'}$ distinct entries of T such that*

$$(r_i, c_j, e_k), (r_i, c_{j'}, e_{k'}) \in T \text{ and } (r_i, c_j, e_{k'}), (r_i, c_{j'}, e_k) \in T'.$$

Then T is not a 3-homogeneous latin trade.

Proof: Suppose that T is 3-homogeneous. Then there are three entries in row r_i, and thus there exist $j'' \notin \{j, j'\}$ and $k'' \notin \{k, k'\}$ such that $(r_i, c_{j''}, e_{k''}) \in T$. But since $\mathcal{R}^i_T = \mathcal{R}^i_{T'}$, we must have $(r_i, c_{j''}, e_{k''}) \in T'$. This contradicts the fact that T and T' are disjoint. □

Lemma 3. *Let T be a latin trade with disjoint mate T'. Let r_i, $r_{i'}$ be distinct rows of T, c_j a column of T and e_k, $e_{k'}$ distinct entries of T such that $(r_i, c_j, e_k), (r_{i'}, c_j, e_{k'}) \in T$ and $(r_i, c_j, e_{k'}), (r_{i'}, c_j, e_k) \in T'$. Then T is not a 3-homogeneous latin trade.*

Proof: The proof is similar to the previous lemma. □

Lemma 4. *Let T be a 3-homogeneous latin trade with disjoint mate T'. Let r_i, $r_{i'}$ be distinct rows of T, c_j and $c_{j'}$ distinct columns of T and e_k an entry of T such that $(r_i, c_j, e_k), (r_{i'}, c_{j'}, e_k) \in T$. Then either $(r_i, c_{j'}, e_k) \in T'$ or $(r_{i'}, c_j, e_k) \in T'$ but not both.*

Proof: Suppose first that neither $(r_i, c_{j'}, e_k) \in T'$ nor $(r_{i'}, c_j, e_k) \in T'$. Since T and T' are disjoint, neither $(r_i, c_j, e_k) \in T'$ nor $(r_{i'}, c_{j'}, e_k) \in T'$. Moreover as $\mathcal{R}^i_T = \mathcal{R}^i_{T'}$ and $\mathcal{R}^{i'}_T = \mathcal{R}^{i'}_{T'}$, there must exist two distinct columns $c_{j''}$ and $c_{j'''}$, with $j'', j''' \notin \{j, j'\}$, such that $(r_i, c_{j''}, e_k) \in T'$ and $(r_{i'}, c_{j'''}, e_k) \in T'$. However this implies that e_k occurs in at least four columns of T, contradicting the definition of 3-homogeneous.

Next suppose that both $(r_i, c_{j'}, e_k) \in T'$ and $(r_{i'}, c_j, e_k) \in T'$. Then there exist $i'' \notin \{i, i'\}$ and $j'' \notin \{j, j'\}$ such that $(r_{i''}, c_{j''}, e_k) \in T'$. Moreover we must have $(r_{i''}, c_{j''}, e_k) \in T$, contradicting the fact that T and T' are disjoint. □

Definition 5. Let $m \geq 2$ be an integer. Let $\{r_1, r_2, \dots r_m\}$ be a set of distinct rows, $\{c_1, c_2, \dots c_m\}$ be a set of distinct columns and $\{e_1, e_2, \dots e_m\}$ be a set of distinct entries. Let (T, T') be a latin bitrade of order at least m such that:

1. $(r_i, c_i, e_i) \in T$, for each $1 \leq i \leq m-1$;
2. $(r_i, c_i, e_{i+1}) \in T'$, for each $1 \leq i \leq m-1$;
3. $(r_1, c_m, e_m) \in T$ and $(r_1, c_m, e_1) \in T'$.

We say that such a latin bitrade (T, T') has the *inappropriate* property (with respect to m).

	c_1	c_2	$\dots$	c_{m-1}	c_m
r_1	e_1		$\dots$		e_m
r_2		e_2	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r_{m-1}			$\dots$	e_{m-1}	

Elements of latin trade T

	c_1	c_2	$\dots$	c_{m-1}	c_m
r_1	e_2		$\dots$		e_1
r_2		e_3	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r_{m-1}			$\dots$	e_m	

Elements of disjoint mate T'

We will soon show that a latin bitrade with the inappropriate property must not be 3-homogeneous. We first need a lemma.

Lemma 6. *Let (T, T') be a 3-homogeneous latin bitrade with the inappropriate property. Let $1 \leq i \leq m-1$ and $2 \leq j \leq m$ be integers such that $i \neq j$ and $(i, j) \neq (1, m)$. Then each cell of the form (r_i, c_j) is empty in T and T'.*

PROOF: Our proof is by induction on i.

So first suppose that $i = 1$. Examining row r_1, there must exist $j' \notin \{1, m\}$ such that $(r_1, c_{j'}, e_2) \in T$ and $(r_1, c_{j'}, e_m) \in T'$. We cannot have $j' \in \{2, m-1\}$ as entry e_2 occurs in column c_2 of T and entry e_m occurs in column c_{m-1} of T'. Moreover we cannot have $3 \leq j' \leq m-2$, as this would imply the existence of four entries within a column. Thus the lemma is true for $i = 1$.

Next assume that the lemma is true for $i-1$, for some i such that $2 \leq i < m-1$. Suppose that (r_i, c_j) is non-empty for some j, where $2 \leq j \leq m$ and $j \neq i$. The entries e_i and e_{i+1} already occur in row i of T or T'. Since (T, T') is 3-homogeneous, we must have either $(r_i, c_j, e_{i+1}) \in T$, or $(r_i, c_j, e_i) \in T'$. We thus spilt our proof into two cases.

Case 1: $(r_i, c_j, e_{i+1}) \in T$. Now, $(r_{i+1}, c_{i+1}, e_{i+1}) \in T$, so $j \neq i+1$. Moreover, from Lemma 4, we either have $(r_{i+1}, c_j, e_{i+1}) \in T'$ or $(r_i, c_{i+1}, e_{i+1}) \in T'$. The latter is impossible as $(r_i, c_i, e_{i+1}) \in T'$. Examining column c_j and the fact that $(r_j, c_j, e_j) \in T$ gives us $(r_i, c_j, e_j) \in T'$. Applying Lemma 4 and the fact that $(r_{j-1}, c_{j-1}, e_j) \in T'$ gives $(r_i, c_{j-1}, e_j) \in T$ or $(r_{j-1}, c_j, e_j) \in T$. The latter is impossible, as $(r_j, c_j, e_j) \in T$. Examining row r_i implies that $(r_i, c_{j-1}, e_i) \in T'$.

As $(r_{i-1}, c_{i-1}, e_i) \in T'$, once again we can apply Lemma 4, deducing that either $(r_i, c_{i-1}, e_i) \in T$ or $(r_{i-1}, c_{j-1}, e_i) \in T$. The former contradicts $(r_i, c_i, e_i) \in T$ and the latter contradicts our inductive hypothesis.

Case 2: $(r_i, c_j, e_i) \in T'$. Now, $(r_{i-1}, c_{i-1}, e_i) \in T'$, so $j \neq i-1$. Moreover, from Lemma 4, we either have $(r_{i-1}, c_j, e_i) \in T$ or $(r_i, c_{i-1}, e_i) \in T$. The latter contradicts $(r_i, c_i, e_i) \in T$ and the former contradicts our inductive hypothesis.

□

The following lemma is critical in obtaining the main result of this paper.

Lemma 7. *Let (T, T') be a latin bitrade with the inappropriate property with respect to some integer m. Then, T is not a 3-homogeneous latin trade.*

Proof: Assume that T is a 3-homogeneous latin trade with the inappropriate property respect to m. We will create a contradiction, using induction on m.

If $m = 2$, we have a contradiction from Lemma 2. If $m = 3$, there must exist c_4 such that $(r_1, c_4, e_2) \in T$ and $(r_1, c_4, e_3) \in T'$. From Lemma 6, $c_4 \notin \{c_1, c_2, c_3\}$. From Lemma 4, either $(r_1, c_2, e_2) \in T'$ or $(r_2, c_4, e_2) \in T'$. But from Lemma 6, (r_1, c_2) is an empty cell, so we must have $(r_2, c_4, e_2) \in T'$. Similarly applying Lemma 4 to T' rather than T, we have that $(r_2, c_4, e_3) \in T$. This creates a contradiction to T being 3-homogeneous by applying Lemma 3 to column c_4. The case $m = 3$ is illustrated below.

	c_1	c_2	c_3	c_4
r_1	e_1		e_3	e_2
r_2		e_2		e_3

Elements of T

	c_1	c_2	c_3	c_4
r_1	e_2		e_1	e_3
r_2		e_3		e_2

Elements of T'

Otherwise $m \geq 4$, and we assume the lemma is true for $m - 1$.

Claim: There exist distinct columns $c_{m+1}, c_{m+2}, \ldots, c_{2m-2}$ (each not elements of $\{c_2, c_3, \ldots c_m\}$) such that $(r_x, c_{m+x}, e_{x+1}) \in T$ and $(r_{x+1}, c_{m+x}, e_{x+1}) \in T'$, for each x, where $1 \leq x \leq m-2$.

Since $(r_x, c_x, e_{x+1}) \in T'$ for each x, $1 \leq x \leq m-2$, there must exist a column, denoted by c_{m+x}, such that $(r_x, c_{m+x}, e_{x+1}) \in T$. Lemma 6 tells us that c_{m+x} is distinct from columns c_2 through to c_m. From Lemma 4 and the fact that $(r_{x+1}, c_{x+1}, e_{x+1}) \in T$, either $(r_{x+1}, c_{m+x}, e_{x+1}) \in T'$ or $(r_x, c_{x+1}, e_{x+1}) \in T'$. However the latter is not possible because of Lemma 6.

It remains to show that the columns $c_{m+1}, c_{m+2}, \ldots, c_{2m-2}$ are pairwise distinct. So suppose that $c_{m+j} = c_{m+k}$, where $1 \leq j < k \leq m-2$. If $k - j > 1$, we have at least four non-empty cells $((r_j, c_{m+j}), (r_{j+1}, c_{m+j}), (r_k, c_{m+j}), (r_{k+1}, c_{m+j}))$ in column c_{m+j}, a contradiction to T being 3-homogeneous. Otherwise $k = j+1$. This implies that $(r_{j+1}, c_{m+j}, e_{j+1}) = (r_k, c_{m+k}, e_k) \in T'$ and

$(r_k, c_{m+k}, e_{k+1}) \in T$. Together with $(r_k, c_k, e_k) \in T$ and $(r_k, c_k, e_{k+1}) \in T'$, we have a contradiction in row r_k from Lemma 2.

So our claim is true. Examining row r_1 tells us that $(r_1, c_{m+1}, e_m) \in T'$. Applying Lemma 4 and $(r_{m-1}, c_{m-1}, e_m) \in T'$ implies either $(r_1, c_{m-1}, e_m) \in T$ (contradicting Lemma 6) or $(r_{m-1}, c_{m+1}, e_m) \in T$. The situation in columns c_{m+1} through to c_{2m-2} is shown in the diagram below.

	c_{m+1}	c_{m+2}	$\cdots$	c_{2m-3}	c_{2m-2}
r_1	e_2		$\cdots$		
r_2		e_3	$\cdots$		
r_3			$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r_{m-2}			$\cdots$		e_{m-1}
r_{m-1}	e_m		$\cdots$		

Latin trade T

	c_{m+1}	c_{m+2}	$\cdots$	c_{2m-3}	c_{2m-2}
r_1	e_m		$\cdots$		
r_2	e_2		$\cdots$		
r_3		e_3	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
r_{m-2}			$\cdots$	e_{m-2}	
r_{m-1}			$\cdots$		e_{m-1}

Disjoint mate T'

Next, consider a conjugate of the above partial latin squares where the rows become entries, the entries become columns and the columns become rows (see below).

	e_2	e_3	$\cdots$	e_{m-1}	e_m
c_{m+1}	r_1		$\cdots$		r_{m-1}
c_{m+2}		r_2	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
c_{2m-2}			$\cdots$	r_{m-2}	

Latin trade T

	e_2	e_3	$\dots$	e_{m-1}	e_m
c_{m+1}	r_2		$\dots$		r_1
c_{m+2}		r_3	$\dots$		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
c_{2m-2}			$\dots$	r_{m-1}	

Disjoint mate T'

Now, any conjugate of T remains 3-homogeneous, and this conjugate has the inappropriate property with respect to $m-1$, contradicting our inductive assumption. $\square$

Lemma 8. *Let (T,T') be a 3-homogeneous latin bitrade. Let $(r_i, c_j, e_k) \in T$. Then there exist unique $j' \neq j$ and unique $k' \neq k$ such that $(r_i, c_{j'}, e_{k'}) \in T'$.*

PROOF: Let $(r_i, c_j, e_k) \in T$. By the definition of a latin trade, there exist $j'' \neq j$ and $k'' \neq k$ such that $(r_i, c_{j''}, e_k) \in T'$ and $(r_i, c_j, e_{k''}) \in T'$. Since (T,T') is 3-homogeneous, there must be one further element in row r_i of T'. So we have $(r_i, c_{j'}, e_{k'}) \in T'$, for some column $c_{j'} \notin \{c_j, c_{j''}\}$ and entry $e_{k'}$. Since T' is a partial latin square, $k' \notin \{k, k''\}$. $\square$

4. Uniqueness

Using the properties of 3-homogeneous latin trades demonstrated in the previous section, we will ultimately show that any minimal 3-homogeneous latin trade must have the structure given in Theorem 1. Throughout this section, (T,T') refers to a 3-homogeneous latin bitrade.

Definition 9. Let (r_0, c_0, e_0) be a fixed element of T. We define r_h, c_h and e_h for each integer $h > 0$ recursively as follows. Assume that $(r_h, c_h, e_h) \in T$ is defined. Then define $c_{h+1} (\neq c_h)$, $e_{h+1} (\neq e_h)$ so that $(r_h, c_{h+1}, e_{h+1}) \in T'$. Finally, define $r_{h+1} (\neq r_h)$ so that $(r_{h+1}, c_{h+1}, e_{h+1}) \in T$.

Lemma 10. *For each $h \geq 0$, r_h, c_h, e_h, as given in the previous definition, are uniquely defined.*

PROOF: This follows mainly from Lemma 8. $\square$

Lemma 11. *Let r_h, c_h and e_h be defined for each integer $h > 0$ as in the previous definition. Let h and l be two integers such that at least one of $r_h = r_l$, $c_h = c_l$ and $e_h = e_l$ is true. Then all three must be true.*

PROOF: Choose $l > h$ such that at least one of $r_h = r_l$, $c_h = c_l$ and $e_h = e_l$ is true. We may further assume that $l - h$ is minimized.

If at least two of the above statements are true, then by the definition of a partial latin square, all three must be true. So assume that exactly one of these three are true. Note that we cannot have $l-h=1$ because of Lemma 8 and Definition 9.

Suppose that $r_h = r_l$ but $c_h \neq c_l$ and $e_h \neq e_l$. Then we have (r_h, c_h, e_h), $(r_h, c_l, e_l) \in T$ and $(r_h, c_h, e_{h+1}) \in T'$. We cannot have $e_{h+1} = e_l$ from the minimality of $l-h$. It follows, by analysis of row r_h, that $(r_h, c_l, e_h) \in T'$. But then (T, T'), under an isotopism which sets $h=0$, has the inappropriate property, as defined in the previous section. However this contradicts Lemma 7.

The remaining cases follow by considering conjugates. □

Definition 12. Let r_h, c_h and e_h be defined for each integer $h > 0$ as in Definition 9. Let h be the least integer such that there exists $l' > h$ such that $(r_h, c_h, e_h) = (r_{l'}, c_{l'}, e_{l'})$. Next, let l be the least l' with this property. Then we define $m_1 = l - h$ and $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)}) = (r_{h+\alpha}, c_{h+\alpha}, e_{h+\alpha})$, for each $\alpha \in N(m_1)$.

Lemma 13. *Let m_1 and $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)})$ be as in the previous definition, for each $\alpha \in N(m_1)$. Then, $m_1 \geq 2$ and each of the following are true:*

1. $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)}) \in T$, *for each* $\alpha \in N(m_1)$;
2. $(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T'$, *for each* $\alpha \in N(m_1)$ *(with subscripts calculated modulo* m_1*)*;
3. *if* $r_{(0,\alpha)} = r_{(0,\alpha')}$, *where* $\alpha, \alpha' \in N(m_1)$, *then* $\alpha = \alpha'$;
4. *if* $c_{(0,\alpha)} = c_{(0,\alpha')}$, *where* $\alpha, \alpha' \in N(m_1)$, *then* $\alpha = \alpha'$;
5. *if* $e_{(0,\alpha)} = e_{(0,\alpha')}$, *where* $\alpha, \alpha' \in N(m_1)$, *then* $\alpha = \alpha'$.

Proof: This follows from Lemma 11. □

Henceforth the second co-ordinates of the subscripts of row, column and entry labels are always calculated modulo m_1. Note that since $(r_{(0,0)}, c_{(0,1)}, e_{(0,1)}) \in T'$, there must be an entry in cell $(r_{(0,0)}, c_{(0,1)})$ of T. The next lemma deals with the case when this entry is equal to $e_{(0,k)}$, for some $k \in N(m_1)$.

Lemma 14. *Suppose that $(r_{(0,0)}, c_{(0,1)}, e_{(0,k)}) \in T$ for some $k \in N(m_1)$. Then, $k \notin \{0,1\}$. Moreover, for each integer $\alpha \in N(m_1)$,*

$$(r_{(0,\alpha-k)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}), (r_{(0,\alpha-1)}, c_{(0,\alpha-k)}, e_{(0,\alpha)}) \in T$$
$$(r_{(0,\alpha-k)}, c_{(0,\alpha-k)}, e_{(0,\alpha)}), (r_{(0,\alpha)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}) \in T'.$$

Proof: We know that $(r_{(0,0)}, c_{(0,1)}, e_{(0,1)}) \in T'$ and $(r_{(0,0)}, c_{(0,0)}, e_{(0,0)}) \in T$. Thus $k \notin \{0,1\}$. We now show the lemma to be true for $\alpha = k$. We have

$$(r_{(0,0)}, c_{(0,1)}, e_{(0,k)}) \in T, \tag{1}$$

so the three entries in row $r_{(0,0)}$ must be $e_{(0,0)}$, $e_{(0,1)}$ and $e_{(0,k)}$. Neither $e_{(0,0)}$ nor $e_{(0,1)}$ may be in cell $(r_{(0,0)}, c_{(0,0)})$ of T' so we must have

$$(2) \qquad (r_{(0,0)}, c_{(0,0)}, e_{(0,k)}) \in T'.$$

Next, applying Lemma 4 to $(r_{(0,k)}, c_{(0,k)}, e_{(0,k)}) \in T$ and (1), we have either $(r_{(0,k)}, c_{(0,1)}, e_{(0,k)}) \in T'$ or $(r_{(0,0)}, c_{(0,k)}, e_{(0,k)}) \in T'$. As $(r_{(0,k-1)}, c_{(0,k)}, e_{(0,k)}) \in T'$, we cannot have the latter, so

$$(3) \qquad (r_{(0,k)}, c_{(0,1)}, e_{(0,k)}) \in T'.$$

Next, $(r_{(0,k-1)}, c_{(0,k)}, e_{(0,k)}) \in T'$ and (2) imply, by Lemma 4, that either $(r_{(0,k-1)}, c_{(0,0)}, e_{(0,k)}) \in T$ or $(r_{(0,0)}, c_{(0,k)}, e_{(0,k)}) \in T$. The latter cannot be true because $(r_{(0,k)}, c_{(0,k)}, e_{(0,k)}) \in T$. Thus,

$$(4) \qquad (r_{(0,k-1)}, c_{(0,0)}, e_{(0,k)}) \in T.$$

Now, equations (1), (2), (3) and (4) together show our lemma to be true for $\alpha = k$.

Next, assume the lemma is true for a particular value of α. We will show it is then true for $\alpha + 1$ (modulo m_1), thus proving the lemma in its entirity.

Consider row $r_{(0,\alpha)}$. We have: $(r_{(0,\alpha)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}) \in T'$ from the inductive assumption and $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)}) \in T$, $(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T'$ from Lemma 13. We may deduce that:

$$(5) \qquad (r_{(0,\alpha)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T.$$

So, entry $e_{(0,\alpha+1)}$ must occur somewhere in column $c_{(0,\alpha-k+1)}$ of T'. We have:

$(r_{(0,\alpha-k)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}) \in T$ (from our inductive assumption) and
$(r_{(0,\alpha-k+1)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha-k+1)}) \in T$, (from Lemma 13).

But if $(r_{(0,\alpha-k)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T'$, we have a contradiction, from Lemma 3, in column $c_{(0,\alpha-k+1)}$. So we must have

$$(6) \qquad (r_{(0,\alpha-k+1)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T'.$$

Next, entry $e_{(0,\alpha+1)}$ must occur somewhere in row $r_{(0,\alpha-k+1)}$ of T. From Lemma 13, $(r_{(0,\alpha-k+1)}, c_{(0,\alpha-k+2)}, e_{(0,\alpha-k+2)}) \in T'$. We can thus infer that there is some entry in cell $(r_{(0,\alpha-k+1)}, c_{(0,\alpha-k+2)})$ of T. This entry can be neither $e_{(0,\alpha-k+2)}$ nor $e_{(0,\alpha-k+1)}$, so we must have

$$(7) \qquad (r_{(0,\alpha-k+1)}, c_{(0,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T.$$

Finally, the above element, together with $(r_{(0,\alpha+1)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T$ (from Lemma 13), imply, from Lemma 4, that either $(r_{(0,\alpha-k+1)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)})$ $\in T'$ or $(r_{(0,\alpha+1)}, c_{(0,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T'$. However, equation (6) above contradicts the former, so we must have

$$(r_{(0,\alpha+1)}, c_{(0,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T'. \tag{8}$$

Equations (5), (6), (7) and (8) together prove our inductive step. □

Observe that in Lemma 13 if we add any β (modulo m_1) to the second coordinate of the subscripts, the lemma is unchanged. We thus may generalize Lemma 14 to the following:

Corollary 15. *Suppose that $(r_{(0,\beta)}, c_{(0,\beta+1)}, e_{(0,\beta+k)}) \in T$ for some $\beta, k \in N(m_1)$. Then, $k \notin \{0,1\}$. Moreover, for each integer $\alpha \in N(m_1)$,*

$$\begin{gathered}(r_{(0,\alpha-k)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}), (r_{(0,\alpha-1)}, c_{(0,\alpha-k)}, e_{(0,\alpha)}) \in T \\ (r_{(0,\alpha-k)}, c_{(0,\alpha-k)}, e_{(0,\alpha)}), (r_{(0,\alpha)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}) \in T'.\end{gathered}$$

In particular, $(r_{(0,0)}, c_{(0,1)}, e_{(0,k)}) \in T$.

Theorem 16. *Suppose that $(r_{(0,0)}, c_{(0,1)}, e_{(0,k)}) \in T$ for some $k \in N(m_1)$ and that T is a minimal 3-homogeneous latin trade. Then T is precisely equivalent to the latin trade given in Theorem 1 with $m_2 = 1$ and $k \geq 2$.*

Proof: Since $m_2 = 1$, the partial latin squares T_2 and T_5 from Theorem 1 are empty. We know that $T_1 \subset T$ from Lemma 13. From the previous lemma, for each integer $\alpha \in N(m_1)$,

$$(r_{(0,\alpha-k)}, c_{(0,\alpha-k+1)}, e_{(0,\alpha)}), (r_{(0,\alpha-1)}, c_{(0,\alpha-k)}, e_{(0,\alpha)}) \in T.$$

Equivalently, for each $j \in N(m_1)$,

$$(r_{(0,j)}, c_{(0,j+1)}, e_{(0,j+k)}), (r_{(0,j)}, c_{(0,j-k+1)}, e_{(0,j+1)}) \in T,$$

where second subscripts are calculated modulo m_1. Thus T_3, T_4 from Theorem 1 are both subsets of T.

From Theorem 1, $T_1 \cup T_3 \cup T_4$ is a latin trade when $m_2 = 1$. Since $T_1 \cup T_3 \cup T_4 \subseteq T$ and T is a minimal latin trade, we must have that $T_1 \cup T_3 \cup T_4 = T$. □

For the rest of this section, we assume that the entry in cell $(r_{(0,0)}, c_{(0,1)})$ of T is not equal to $e_{(0,k)}$, for all $k \in N(m_1)$.

Lemma 17. *For each $\alpha \in N(m_1)$, there exist a unique row (denoted by $r_{(1,\alpha)}$), a unique column (denoted by $c_{(1,\alpha)}$), and a unique entry (denoted by $e_{(1,\alpha)}$) such that:*

1. $(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}) \in T$, $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(1,\alpha)}) \in T'$;
2. $e_{(1,\alpha)} \neq e_{(0,\beta)}$, *for all* $\alpha, \beta \in N(m_1)$;
3. $e_{(1,\alpha)} = e_{(1,\beta)}$ *implies that* $\alpha = \beta$;
4. $(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}) \in T$, $(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}) \in T'$;
5. $r_{(1,\alpha)} \neq r_{(0,\beta)}$, *for all* $\alpha, \beta \in N(m_1)$;
6. $r_{(1,\alpha)} = r_{(1,\beta)}$ *implies that* $\alpha = \beta$;
7. $(r_{(1,\alpha)}, c_{(1,\alpha)}, e_{(1,\alpha)}) \in T$, $(r_{(1,\alpha)}, c_{(1,\alpha+1)}, e_{(1,\alpha+1)}) \in T'$;
8. $c_{(1,\alpha)} \neq c_{(0,\beta)}$, *for all* $\alpha, \beta \in N(m_1)$; *and*
9. $c_{(1,\alpha)} = c_{(1,\beta)}$ *implies that* $\alpha = \beta$.

Proof: For each $\alpha \in N(m_1)$, we know from Lemma 13 that $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)}) \in T$ and $(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T'$. Let $e_{(1,\alpha)}$ be the third entry that occurs in row $r_{(0,\alpha)}$. Analysis of row $r_{(0,\alpha)}$ then yields that Condition 1 is true. From Corollary 15, if $e_{(1,\alpha)} = e_{(0,\alpha+k)}$ for some $\alpha, k \in N(m_1)$, then $(r_{(0,0)}, c_{(0,1)}, e_{(0,k)}) \in T$, a contradiction. Condition 2 of this lemma then follows.

Next suppose that $e_{(1,\alpha)} = e_{(1,\beta)}$ and $\alpha > \beta$. If $\alpha = \beta + 1$, then we have, from Condition 1 and Lemma 13:

$$(r_{(0,\beta)}, c_{(0,\beta+1)}, e_{(1,\beta)}) \in T, (r_{(0,\beta)}, c_{(0,\beta+1)}, e_{(0,\beta+1)}) \in T',$$
$$(r_{(0,\beta+1)}, c_{(0,\beta+1)}, e_{(0,\beta+1)}) \in T, (r_{(0,\beta+1)}, c_{(0,\beta+1)}, e_{(1,\beta)}(= e_{(1,\beta+1)})) \in T'.$$

So we get a contradiction in column $c_{(0,\beta+1)}$ from Lemma 3. Otherwise $\alpha \notin \{\beta, \beta+1\}$. Then we have:

$$(r_{(0,\beta)}, c_{(0,\beta+1)}, e_{(1,\beta)}) \in T, (r_{(0,\beta)}, c_{(0,\beta)}, e_{(1,\beta)}) \in T',$$
$$(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\beta)}(= e_{(1,\alpha)})) \in T, (r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(1,\beta)}(= e_{(1,\alpha)})) \in T'.$$

Thus entry $e_{(1,\beta)}$ occurs in at least 4 columns, a contradiction. So Condition 3 of this lemma is correct.

So far in column $c_{(0,\alpha+1)}$ we have:

$$(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}), (r_{(0,\alpha+1)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T,$$
$$(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}), (r_{(0,\alpha+1)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}) \in T'.$$

So there must exist a row $r_{(1,\alpha)}$ such that Condition 4 holds. Suppose that $r_{(1,\alpha)} = r_{(0,\beta)}$, for some $\alpha, \beta \in N(m_1)$. Then, from Condition 4, we have

$(r_{(0,\beta)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}) \in T'$, which implies that $e_{(1,\alpha)}$ must occur somewhere in row $r_{(0,\beta)}$ of T. But from Lemma 13 $(r_{(0,\beta)}, c_{(0,\beta)}, e_{(0,\beta)}) \in T$ and from Condition 1 $(r_{(0,\beta)}, c_{(0,\beta+1)}, e_{(1,\beta)}) \in T$, so we have a contradiction. Thus Condition 5 holds.

Next suppose that $r_{(1,\alpha)} = r_{(1,\beta)}$, where $\beta < \alpha$. Then, considering row $r_{(1,\alpha)}$, we have:

$$(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}), (r_{(1,\alpha)}(= r_{(1,\beta)}), c_{(0,\beta+1)}, e_{(1,\beta+1)}) \in T,$$
$$(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}), (r_{(1,\alpha)}, c_{(0,\beta+1)}, e_{(1,\beta)}) \in T'.$$

As we can have at most three distinct entries in a row, this implies that $\beta+1 = \alpha$. But then

$$(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(1,\alpha)}), (r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}) \in T',$$
$$(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}), (r_{(1,\alpha)}, c_{(0,\alpha)}, e_{(1,\alpha)}) = (r_{(1,\beta)}, c_{(0,\beta+1)}, e_{(1,\beta+1)}) \in T,$$

together contradict Lemma 4. This proves Condition 6.

Next, from Condition 4, we can conclude that $e_{(1,\alpha)}$ occurs somewhere in row $r_{(1,\alpha)}$ of T. So for each $\alpha \in N(m_1)$, we let $c_{(1,\alpha)}$ be the unique column such that $(r_{(1,\alpha)}, c_{(1,\alpha)}, e_{(1,\alpha)}) \in T$. Then apply Lemma 4 to $(r_{(1,\alpha+1)}, c_{(1,\alpha+1)}, e_{(1,\alpha+1)}) \in T$ and $(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}) \in T$ (from Condition 4) we get

$$(r_{(1,\alpha+1)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}) \in T' \quad \text{or} \quad (r_{(1,\alpha)}, c_{(1,\alpha+1)}, e_{(1,\alpha+1)}) \in T'.$$

The former contradicts $(r_{(0,\alpha+1)}, c_{(0,\alpha+1)}, e_{(1,\alpha+1)}) \in T'$ (from Condition 1). Thus we have demonstrated Condition 7 is true.

Condition 8 follows from the fact that there are already three distinct entries in columns of the form $c_{(0,\beta)}$ (namely, $e_{(0,\beta)}$, $e_{(1,\beta-1)}$ and $e_{(1,\beta)}$) for each $\beta \in N(m_1)$. Finally, assume that $c_{(1,\alpha)} = c_{(1,\beta)}$, for some $\alpha < \beta$. Then,

$$(r_{(1,\alpha)}, c_{(1,\alpha)}, e_{(1,\alpha)}), (r_{(1,\beta)}, c_{(1,\alpha)}(= c_{(1,\beta)}), e_{(1,\beta)}) \in T \quad \text{and}$$
$$(r_{(1,\alpha-1)}, c_{(1,\alpha)}, e_{(1,\alpha)}), (r_{(1,\beta-1)}, c_{(1,\alpha)}(= c_{(1,\beta)}), e_{(1,\beta)}) \in T'.$$

As a column may intersect at most three distinct rows, we must have $\alpha = \beta - 1$. Then, in row $r_{(1,\alpha)}$:

$$(r_{(1,\alpha)}, c_{(1,\alpha)}, e_{(1,\alpha)}), (r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\beta)} = (e_{(1,\alpha+1)})) \in T \quad \text{and}$$
$$(r_{(1,\alpha)}, c_{(0,\alpha+1)}, e_{(1,\alpha)}), (r_{(1,\alpha)}, c_{(1,\alpha)}(= c_{(1,\alpha+1)}), e_{(1,\beta)}(= e_{(1,\alpha+1)})) \in T',$$

which contradicts Lemma 2. $\square$

Definition 18. Let x be an integer greater than or equal to 1. We say that the 3-homogeneous latin bitrade (T, T') has the x-property if for each β, $1 \leq \beta \leq x$ and for each $\alpha \in N(m_1)$, there exist a unique row (denoted by $r_{(\beta,\alpha)}$), a unique column (denoted by $c_{(\beta,\alpha)}$), and a unique entry (denoted by $e_{(\beta,\alpha)}$) such that:

1. $(r_{(\beta-1,\alpha)}, c_{(\beta-1,\alpha+1)}, e_{(\beta,\alpha)}) \in T, (r_{(\beta-1,\alpha)}, c_{(\beta-1,\alpha)}, e_{(\beta,\alpha)}) \in T'$,
2. $e_{(\beta,\alpha)} = e_{(\beta',\alpha')}$ implies that $\beta = \beta'$ and $\alpha = \alpha'$,
3. $(r_{(\beta,\alpha)}, c_{(\beta-1,\alpha+1)}, e_{(\beta,\alpha+1)}) \in T$, $(r_{(\beta,\alpha)}, c_{(\beta-1,\alpha+1)}, e_{(\beta,\alpha)}) \in T'$,
4. $r_{(\beta,\alpha)} = r_{(\beta',\alpha')}$ implies that $\beta = \beta'$ and $\alpha = \alpha'$,
5. $(r_{(\beta,\alpha)}, c_{(\beta,\alpha)}, e_{(\beta,\alpha)}) \in T$, $(r_{(\beta,\alpha)}, c_{(\beta,\alpha+1)}, e_{(\beta,\alpha+1)}) \in T'$,
6. $c_{(\beta,\alpha)} = c_{(\beta',\alpha')}$ implies that $\beta = \beta'$ and $\alpha = \alpha'$.

From Lemma 17, we know already that the 3-homogeneous latin bitrade (T, T') has the 1-property. The next lemma has a proof similar to that of Lemma 14.

Lemma 19. *Suppose that the 3-homogeneous latin bitrade (T, T') has the x-property. Let e_i be the entry such that $(r_{(x,0)}, c_{(x,1)}, e_i) \in T$ and suppose that $i = (0, k)$, for some $k \in N(m_1)$. Then, for each $\alpha \in N(m_1)$:*

1. $(r_{(0,\alpha)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha)}) \in T'$,
2. $(r_{(0,\alpha-1)}, c_{(x,\alpha-k)}, e_{(0,\alpha)}) \in T$,
3. $(r_{(x,\alpha-k)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha)}) \in T$ *and*
4. $(r_{(x,\alpha-k)}, c_{(x,\alpha-k)}, e_{(0,\alpha)}) \in T'$.

Proof: We first show the lemma to be true for $\alpha = k$. We have

$$(r_{(x,0)}, c_{(x,1)}, e_{(0,k)}) \in T, \tag{9}$$

$(r_{(x,0)}, c_{(x,0)}, e_{(x,0)}) \in T$, $(r_{(x,0)}, c_{(x,1)}, e_{(x,1)}) \in T'$. So the three entries in row $r_{(x,0)}$ must be $e_{(x,0)}$, $e_{(x,1)}$ and $e_{(0,k)}$. Neither $e_{(x,0)}$ nor $e_{(x,1)}$ may be in cell $(r_{(x,0)}, c_{(x,0)})$ of T' so we must have

$$(r_{(x,0)}, c_{(x,0)}, e_{(0,k)}) \in T'. \tag{10}$$

Next, as $(r_{(0,k)}, c_{(0,k)}, e_{(0,k)}) \in T$ and from (9) above, applying Lemma 4, either $(r_{(0,k)}, c_{(x,1)}, e_{(0,k)}) \in T'$ or $(r_{(x,0)}, c_{(0,k)}, e_{(0,k)}) \in T'$. As $(r_{(0,k-1)}, c_{(0,k)}, e_{(0,k)}) \in T'$ we cannot have the latter, so

$$(r_{(0,k)}, c_{(x,1)}, e_{(0,k)}) \in T'. \tag{11}$$

Next, $(r_{(0,k-1)}, c_{(0,k)}, e_{(0,k)}) \in T'$ and equation (10) imply, by Lemma 4, that either $(r_{(0,k-1)}, c_{(x,0)}, e_{(0,k)}) \in T$ or $(r_{(x,0)}, c_{(0,k)}, e_{(0,k)}) \in T$. The latter cannot be true because $(r_{(0,k)}, c_{(0,k)}, e_{(0,k)}) \in T$. Thus,

$$(r_{(0,k-1)}, c_{(x,0)}, e_{(0,k)}) \in T. \tag{12}$$

Now, equations (9), (10), (11) and (12) together show our lemma to be true for $\alpha = k$.

Next, assume the lemma is true for a particular value of α. We will show it is then true for $\alpha + 1$ (modulo m_1), thus proving the lemma.

Consider row $r_{(0,\alpha)}$. We have: $(r_{(0,\alpha)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha)}) \in T'$ from the inductive assumption and $(r_{(0,\alpha)}, c_{(0,\alpha)}, e_{(0,\alpha)}) \in T$, $(r_{(0,\alpha)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T'$ from Lemma 13. We may deduce that:

$$(r_{(0,\alpha)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T. \tag{13}$$

So, entry $e_{(0,\alpha+1)}$ must occur somewhere in column $c_{(x,\alpha-k+1)}$ of T'. We have:

$(r_{(x,\alpha-k)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha)}) \in T$ (from our inductive assumption) and
$(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+1)}, e_{(x,\alpha-k+1)}) \in T$ (from Condition 5 of Definition 18).

But if $(r_{(x,\alpha-k)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T'$, we have a contradiction, in column $c_{(x,\alpha-k+1)}$, from Lemma 3. So we must have

$$(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T'. \tag{14}$$

Next, entry $e_{(0,\alpha+1)}$ must occur somewhere in row $r_{(x,\alpha-k+1)}$ of T. From Condition 5 of Definition 18, $(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+2)}, e_{(x,\alpha-k+2)}) \in T'$. We can thus infer that there is some entry in cell $(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+2)})$ of T. This entry is neither $e_{(0,\alpha-k+2)}$ nor $e_{(x,\alpha-k+1)}$, so we must have

$$(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T. \tag{15}$$

Finally, the above element, together with $(r_{(0,\alpha+1)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T$ (from Lemma 13), imply, from Lemma 4, that either $(r_{(x,\alpha-k+1)}, c_{(0,\alpha+1)}, e_{(0,\alpha+1)}) \in T'$ or $(r_{(0,\alpha+1)}, c_{(x,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T'$. But $(r_{(x,\alpha-k+1)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha+1)}) \in T'$ (from (14)) contradicts the former, so we must have

$$(r_{(0,\alpha+1)}, c_{(x,\alpha-k+2)}, e_{(0,\alpha+1)}) \in T'. \tag{16}$$

Equations (13), (14), (15) and (16) together prove our inductive step. □

Observe that in Definition 18, if we add a constant integer $(\bmod\, m_1)$ to the second co-ordinate of the subscripts of all the rows, columns and entries, the x-property still holds. It follows that we can generalize the previous lemma to the following:

Corollary 20. *Suppose that the* 3*-homogeneous latin bitrade* (T, T') *has the* x*-property and let* $(r_{(x,\beta)}, c_{(x,\beta+1)}, e_{(0,k+\beta)}) \in T$ *for some* $k, \beta \in N(m_1)$*. Then, for each* $\alpha \in N(m_1)$*, Conditions* 1 *through to* 4 *of Lemma* 19 *hold. In particular,* $(r_{(x,0)}, c_{(x,1)}, e_{(0,k)}) \in T$.

Theorem 21. *Suppose that* (T, T') *is a minimal* 3*-homogeneous latin bitrade with the* x*-property. Let* e_i *be the entry such that* $(r_{(x,0)}, c_{(x,1)}, e_i) \in T$ *and suppose that* $i = (0, k)$*, for some* $k \in N(m_1)$*. Then,* T *is precisely equivalent to the latin trade given in Theorem* 1 *with* $m_2 - 1 = x$.

Proof: We will show that each of the partial latin squares T_1 through to T_6 from Theorem 1 are a subset of T. For T_1 we have Condition 5 of Definition 18 plus Condition 1 from Lemma 13. For T_2 we have Condition 1 of Definition 18. Next, Condition 3 of Lemma 19 states that $(r_{(x,\alpha-k)}, c_{(x,\alpha-k+1)}, e_{(0,\alpha)}) \in T$. So setting $j = \alpha - k$ gives T_3. Condition 2 of Lemma 19 states that $(r_{(0,\alpha-1)}, c_{(x,\alpha-k)}, e_{(0,\alpha)}) \in T$. Setting $j = \alpha - 1$ gives T_4. Finally, Condition 3 of Definition 18 gives T_5.

Thus $T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \subset T$. But since T is a minimal latin trade and Theorem 1 states that $T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5$ is a latin trade, $T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 = T$. $\square$

The proof of the following lemma is similar to that of Lemma 17.

Lemma 18. *Suppose that the* 3*-homogeneous latin bitrade* (T, T') *has the* x*-property. Let* e_i *be the entry such that* $(r_{(x,0)}, c_{(x,1)}, e_i) \in T$ *and suppose that* $i \neq (0, k)$*, for all* $k \in N(m_1)$*. Then* (T, T') *has the* $(x+1)$*-property.*

Proof: We know that $i \neq (0, k)$, for all $k \in N(m_1)$. Suppose that $i = (\beta, \alpha)$, for some $0 < \beta \leq x$ and $\alpha \in N(m_1)$. Conditions 1, 3 and 5 from Definition 18 give the three occurrences of $e_{(\beta,\alpha)}$ in T: $(r_{(\beta-1,\alpha)}, c_{(\beta-1,\alpha+1)}, e_{(\beta,\alpha)}) \in T$, $(r_{(\beta,\alpha-1)}, c_{(\beta-1,\alpha)}, e_{(\beta,\alpha)}) \in T$ and $(r_{(\beta,\alpha)}, c_{(\beta,\alpha)}, e_{(\beta,\alpha)}) \in T$. By observation, none of these can be equal to $(r_{(x,0)}, c_{(x,1)}, e_i)$. So we can assume that $i \neq (\beta, \alpha)$, for all $\alpha \in N(m_1)$ and $0 \leq \beta \leq x$.

To show that (T, T') has the $x+1$-property, we must show that for each $\alpha \in N(m_1)$, there exist a unique row (denoted by $r_{(x+1,\alpha)}$), a unique column (denoted by $c_{(x+1,\alpha)}$), and a unique entry (denoted by $e_{(x+1,\alpha)}$) such that:

A. $(r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T$, $(r_{(x,\alpha)}, c_{(x,\alpha)}, e_{(x+1,\alpha)}) \in T'$,
B. $e_{(x+1,\alpha)} = e_{(\beta,\alpha')}$ implies that $\beta = x+1$ and $\alpha = \alpha'$,
C. $(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T$, $(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T'$,
D. $r_{(x+1,\alpha)} = r_{(\beta,\alpha')}$ implies that $\beta = x+1$ and $\alpha = \alpha'$,
E. $(r_{(x+1,\alpha)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha)}) \in T$, $(r_{(x+1,\alpha)}, c_{(x+1,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T'$,
F. $c_{(x+1,\alpha)} = c_{(\beta,\alpha')}$ implies that $\beta = x+1$ and $\alpha = \alpha'$.

Letting $\beta = x$ in Condition 5 of Definition 18, there must be an entry in cell $(r_{(x,\alpha)}, c_{(x,\alpha)})$ of T' and an entry in cell $(r_{(x,\alpha)}, c_{(\beta,\alpha+1)})$ of T. Neither entry can be equal to $e_{(x,\alpha)}$ or $e_{(x,\alpha+1)}$, so we must have the same entry in both cells (as there are exactly three entries per row); we define this entry to be $e_{(x+1,\alpha)}$. Thus Condition A is verified.

Suppose that $e_{(x+1,\alpha)} = e_{(\beta,\alpha')}$ for some $\beta < x+1$ and α'. Let $\beta = 0$. Then, from Corollary 20, $(r_{(x,0)}, c_{(x,1)}, e_{(0,k)}) \in T$, for some integer k, a contradiction. Thus we cannot have $\beta = 0$.

Next let $0 < \beta < x+1$. But, as observed in the first paragraph of this proof, Conditions 1, 3 and 5 from Definition 18 give the three occurrences of $e_{(\beta,\alpha)}$ in T. Finally let $\beta = x+1$ and $\alpha < \alpha'$. If $\alpha' = \alpha + 1$, then we have, from (A) and Definition 18:

$$(r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T, (r_{(x,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}(= e_{(x+1,\alpha+1)})) \in T',$$
$$(r_{(x,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x,\alpha+1)}) \in T, (r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x,\alpha+1)}) \in T'.$$

So we get a contradiction in column $c_{(x,\alpha+1)}$ from Lemma 3. Otherwise assume that $\alpha' \notin \{\alpha, \alpha+1\}$. Then we have:

$$(r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T, (r_{(x,\alpha)}, c_{(x,\alpha)}, e_{(x+1,\alpha)}) \in T',$$
$$(r_{(x,\alpha')}, c_{(x,\alpha'+1)}, e_{(x+1,\alpha)}) \in T, (r_{(x,\alpha')}, c_{(x,\alpha')}, e_{(x+1,\alpha)}) \in T'.$$

Thus entry $e_{(x+1,\alpha)}$ occurs in at least 4 columns, a contradiction to T being 3-homogeneous. So we have verified Condition B.

By observation of column $c_{(x,\alpha+1)}$ and from (A) and Definition 18,

$$(r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T, (r_{(x,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T',$$
$$(r_{(x,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x,\alpha+1)}) \in T, (r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x,\alpha+1)}) \in T'.$$

So there must exist a row $r_{(x+1,\alpha)}$ such that Condition C holds. Suppose that $r_{(x+1,\alpha)} = r_{(\beta,\alpha')}$, for some $0 \leq \beta \leq x+1$ and $\alpha, \alpha' \in m_1$.

Let $\beta = 0$. Then the distinct entries $e_{(0,\alpha')}$, $e_{(0,\alpha'+1)}$ (from Lemma 13) and $e_{(x+1,\alpha+1)}$, $e_{(x+1,\alpha)}$ (from (C)) all occur in row $r_{(0,\alpha')}$, a contradiction. We cannot have $0 < \beta < x+1$, as all occurrences of such rows $r_{(\beta,\alpha')}$ are given in Definition 18.

Otherwise let $\beta = x+1$. Then, considering row $r_{(x+1,\alpha)}$, from (C), we have:

$$(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}), (r_{(x+1,\alpha)}(= r_{(x+1,\alpha')}), c_{(x,\alpha'+1)}, e_{(x+1,\alpha'+1)}) \in T,$$
$$(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}), (r_{(x+1,\alpha)}(= r_{(x+1,\alpha')}), c_{(x,\alpha'+1)}, e_{(x+1,\alpha')}) \in T'.$$

As we can have at most three distinct entries in a row, this implies, without loss of generality, that $\alpha = \alpha' + 1$. Thus

$$(r_{(x+1,\alpha)}, c_{(x,\alpha)}, e_{(x+1,\alpha)}) = (r_{(x+1,\alpha')}, c_{(x,\alpha'+1)}, e_{(x+1,\alpha'+1)})$$

and

$$(r_{(x,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T, (r_{(x,\alpha)}, c_{(x,\alpha)}, e_{(x+1,\alpha)}) \in T' \quad \text{(from (A))},$$
$$(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T', (r_{(x+1,\alpha)}, c_{(x,\alpha)}, e_{(x+1,\alpha)}) \in T \quad \text{(from (C))}$$

together contradict Lemma 4. This proves Condition D.

Next, from Condition C, we can conclude that $e_{(x+1,\alpha)}$ occurs somewhere in row $r_{(x+1,\alpha)}$ of T. So for each $\alpha \in N(m_1)$, we let $c_{(x+1,\alpha)}$ be the unique column such that $(r_{(x+1,\alpha)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha)}) \in T$. Then, applying Lemma 4 to

$$(r_{(x+1,\alpha+1)}, c_{(x+1,\alpha+1)}, e_{(x+1,\alpha+1)}), (r_{(x+1,\alpha)}, c_{(x+1,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T \text{(by (C))}$$

we get $(r_{(x+1,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T'$ or $(r_{(x+1,\alpha)}, c_{(x+1,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T'$. The former contradicts $(r_{(x,\alpha+1)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T'$ (from Condition A). Thus we have demonstrated Condition E.

Finally suppose that $c_{(x+1,\alpha)} = c_{(\beta,\alpha')}$ for some β, $0 \le \beta \le x+1$ and $\alpha, \alpha' \in N(m_1)$. We cannot have $\beta < x+1$, as from Lemma 13 and Definition 18, there are already three distinct entries in columns of the form $c_{(\beta,\alpha')}$ (namely, $e_{(\beta,\alpha')}$, $e_{(\beta+1,\alpha'-1)}$ and $e_{(\beta+1,\alpha')}$) for each $\alpha' \in N(m_1)$. So assume that $c_{(x+1,\alpha)} = c_{(x+1,\alpha')}$, for some $\alpha < \alpha'$. Then, from (E),

$$(r_{(x+1,\alpha)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha)}), (r_{(x+1,\alpha')}, c_{(x+1,\alpha)}(= c_{(x+1,\alpha')}), e_{(x+1,\alpha')}) \in T \text{ and}$$
$$(r_{(x+1,\alpha-1)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha)}), (r_{(x+1,\alpha'-1)}, c_{(x+1,\alpha)}(= c_{(x+1,\alpha')}), e_{(x+1,\alpha')}) \in T'.$$

As a column may intersect at most three distinct rows, we have $\alpha = \alpha' - 1$. Then, in row $r_{(x+1,\alpha)}$:

$$(r_{(x+1,\alpha)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha)}) \in T, (r_{(x+1,\alpha)}, c_{(x+1,\alpha)}, e_{(x+1,\alpha+1)}) \in T' \text{ (from (E))},$$
$$(r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha+1)}) \in T, (r_{(x+1,\alpha)}, c_{(x,\alpha+1)}, e_{(x+1,\alpha)}) \in T' \text{ (from (C))},$$

which contradicts Lemma 2. □

Theorem 23. *Suppose that (T, T') is a finite, minimal 3-homogeneous latin bitrade. Then, T is precisely equivalent to the latin trade given in Theorem 1, where m_1 is given as in Definition 12 and m_2 is the greatest integer such that (T, T') has the m_2-property (see Definition 18).*

Proof: We know that (T, T') always has the 1-property. As (T, T') is finite, there must exist an integer x such that (T, T') does not have the $x+1$-property. Lemma 22 then implies that the entry in cell $(r_{(x,0)}, c_{(x,1)})$ is equal to $e_{(0,k)}$ for some k. Thus we may apply Theorem 21 (if $x > 1$) or Theorem 16 (if $x = 1$). □

5. Classification

In [4] it is shown that if T is a latin trade given by Theorem 1, then T cannot be expressed as the union of two, disjoint latin trades. The following lemma leads to the classification of all 3-homogeneous latin trades.

Lemma 24. *Let T be a non-minimal 3-homogeneous latin trade. Suppose furthermore that T cannot be written as the union of two, non-empty, disjoint latin trades. Then $|T| = 9$, and T is isotopic to a latin square of order 3.*

PROOF: Suppose that $T_1 \subset T$, where T_1 is a minimal latin trade. If T_1 is 3-homogeneous, then $T \backslash T_1$ must also be a 3-homogeneous latin trade, contradicting the conditions of this lemma. So we may assume that T_1 is not 3-homogeneous.

Let T_1' and T' be disjoint mates of T_1 and T, respectively. (Note that $T_1 \subset T$ does not necessarily imply $T_1' \subset T'$.) Without loss of generality (and considering conjugacies), let r_1 be a row with exactly two entries in T_1. So let $(r_1, c_1, e_1), (r_1, c_2, e_2) \in T_1 \subset T$ and $(r_1, c_1, e_2), (r_1, c_2, e_1) \in T_1'$. Let $(r_1, c_3, e_3) \in T \setminus T_1$ be the third element in row r_1 of T. Without loss of generality, let $(r_1, c_1, e_2), (r_1, c_2, e_3), (r_1, c_3, e_1) \in T'$. Let r_2 be the row such that $(r_2, c_2, e_1) \in T_1 \subset T$ and r_3 the row such that $(r_3, c_3, e_1) \in T$. Analysing T and T', we can infer that $(r_2, c_2, e_2), (r_3, c_2, e_1), (r_2, c_1, e_1) \in T'$ and $(r_3, c_2, e_3) \in T$. If $(r_2, c_1, e_2) \in T$, then we have a contradiction in row r_2 from Lemma 2. Thus $(r_2, c_1, e_2) \notin T$.

Let x be the entry such that $(r_2, c_2, x) \in T_1'$. What we know so far about T, T' and T_1' is illustrated below:

	c_1	c_2	c_3
r_1	e_1	e_2	e_3
r_2		e_1	
r_3		e_3	e_1

T

	c_1	c_2	c_3
r_1	e_2	e_3	e_1
r_2	e_1	e_2	
r_3		e_1	

T'

	c_1	c_2	c_3
r_1	e_1	e_2	
r_2		e_1	
r_3			

T_1

	c_1	c_2	c_3
r_1	e_2	e_1	
r_2		x	
r_3			

T_1'

Case 1: $x = e_3$. Then entry e_3 is in row r_2 of T_1 (and thus also T). As $(r_2, c_1, e_2) \notin T$, we must have $(r_2, c_1, e_3) \in T_1 \subset T$. By applying Lemma 4, we can show that $(r_2, c_3, e_3) \in T'$ and $(r_3, c_1, e_3) \in T'$. Thus, analysing row r_2, $(r_2, c_3, e_2) \in T$, implying $(r_3, c_3, e_2) \in T'$, implying furthermore that $(r_3, c_1, e_2) \in T$. Thus T contains a 3-homogeneous latin trade of size 9, which is isotopic to a latin square of order 3. Since we assumed that T contains no smaller 3-homogeneous latin trades, T must be exactly equal to this latin trade, and $m = 3$, a contradiction.

Case 2: $x = e_2$. We know from T' that cell (r_2, c_1) is non-empty in T.

Case 2a: (r_2, c_1) is empty in T_1. Now e_1 must occur somewhere in row r_2 of T_1' and by observation of T, e_1 can only occur in columns c_1, c_2 and c_3 of T_1 (and hence T_1'). So we have $(r_2, c_3, e_1) \in T_1'$. It follows that (r_2, c_3) is non-empty in T_1, and thus non-empty in T.

Now entry e_2 must occur somewhere in row r_2 of T. But $(r_2, c_1, e_2) \notin T$, so $(r_2, c_3, e_2) \in T$. Next, analysing column c_3, $(r_2, c_3, e_3), (r_3, c_3, e_2) \in T'$. We may further infer that $(r_2, c_1, e_3) \in T$. Applying Lemma 4 to $(r_1, c_1, e_2), (r_3, c_3, e_2) \in T'$ gives $(r_3, c_1, e_2) \in T$. So we have the same conclusion as in Case 1.

Case 2b:. (r_2, c_1) is non-empty in T_1. Since $(r_2, c_1, e_2) \notin T_1$, we must have $(r_2, c_1, e_k) \in T_1$ for some entry $e_k \notin \{e_1, e_2\}$. Thus the three distinct entries in row r_2 of T_1 are e_1, e_2 and e_k. We may infer that $(r_2, c_1, e_1) \in T_1'$. Now, let

$$U = T_1 \setminus \{(r_1, c_1, e_1), (r_1, c_2, e_2), (r_2, c_2, e_1)\},$$
$$V = T_1' \setminus \{(\{(r_1, c_1, e_2), (r_1, c_2, e_1), (r_2, c_2, e_2), (r_2, c_1, e_1)\}) \cup \{(r_2, c_1, e_2)\}.$$

Clearly U and V are disjoint and have the same shape. Also,

$$\mathcal{R}_U^{r_1} = \mathcal{R}_{T_1}^{r_1} \setminus \{e_1, e_2\} = \mathcal{R}_V^{r_1}, \qquad \mathcal{R}_U^{r_2} = \mathcal{R}_{T_1}^{r_2} \setminus \{e_1\} = \mathcal{R}_V^{r_1},$$
$$\mathcal{C}_U^{c_1} = \mathcal{C}_{T_1}^{c_1} \setminus \{e_1\} = \mathcal{C}_V^{c_1}, \qquad \mathcal{C}_U^{c_2} = \mathcal{C}_{T_1}^{c_2} \setminus \{e_1, e_2\} = \mathcal{C}_V^{c_2}.$$

Thus (U, V) is a latin bitrade (see the Definitions section). But $U \subset T_1$, contradicting the fact that T_1 is minimal. □

Corollary 25. *If T is a 3-homogeneous latin trade, T may be written as a disjoint sum of latin trades isotopic to those given in Theorem 1.*

Corollary 26. *If T is a 3-homogeneous latin trade, T may be partitioned into three, disjoint partial transversals.*

PROOF: The latin trade from Theorem 1 may be partitioned into three, disjoint partial transversals: T_1, $T_2 \cup T_3$ and $T_4 \cup T_5$. □

We finally note that Theorem 1 will, in certain cases, give isotopic latin trades for different values of m_1, m_2 and k. However, given the results in this paper, the enumeration of all non-isotopic 3-homogeneous latin trades is certainly a possible task.

References

[1] Bate J.A., van Rees G.H.J., *Minimal and near-minimal critical sets in back circulant latin squares*, Australas. J. Combin. **27** (2003), 47–61.

[2] Cavenagh N.J., *Embedding 3-homogeneous latin trades into abelian 2-groups*, Comment. Math. Univ. Carolin. **45** (2004), 191–212.

[3] Cavenagh N.J., *The size of the smallest latin trade in a back circulant latin square*, Bull. Inst. Combin. Appl. **38** (2003), 11–18.
[4] Cavenagh N.J., Donovan D., Drápal A., 3-*homogeneous latin trades*, Discrete Math. **300** (2005), 57–70.
[5] Cavenagh N.J., Donovan D., Drápal A., 4-*homogeneous latin trades*, Australas. J. Combin. **32** (2005), 285–303.
[6] Cooper J., Donovan D., Seberry J., *Latin squares and critical sets of minimal size*, Australas. J. Combin. **4** (1991), 113–120.
[7] Donovan D., Howse A., Adams P., *A discussion of latin interchanges*, J. Combin. Math. Combin. Comput. **23** (1997), 161–182.
[8] Donovan D., Mahmoodian E.S., *An algorithm for writing any latin interchange as the sum of intercalates*, Bull. Inst. Combin. Appl. **34** (2002), 90–98.
[9] Drápal A., *On a planar construction of quasigroups*, Czechoslovak Math. J. **41** (1991), 538–548.
[10] Drápal A., *Hamming distances of groups and quasi-groups*, Discrete Math. **235** (2001), 189–197.
[11] Drápal A., *Geometry of latin trades*, manuscript circulated at the conference Loops'03, Prague, 2003.
[12] Drápal A., Kepka T., *Exchangeable Groupoids I*, Acta Univ. Carolin. Math. Phys. **24** (1983), 57–72.
[13] Drápal A., Kepka T., *On a distance of groups and latin squares*, Comment. Math. Univ. Carolin. **30** (1989), 621–626.
[14] Fu C.-M., Fu H.-L., *The intersection problem of latin squares*, J. Combin. Inform. System Sci. **15** (1990), 89–95.
[15] Horak P., Aldred R.E.L., Fleischner H., *Completing Latin squares: critical sets*, J. Combin. Designs **10** (2002), 419–432.
[16] Keedwell A.D., *Critical sets for latin squares, graphs and block designs: A survey*, Congr. Numer. **113** (1996), 231–245.
[17] Keedwell A.D., *Critical sets in latin squares and related matters: an update*, Util. Math. **65** (2004), 97–131.
[18] Khosrovshahi G.B., Maimani H.R., Torabi R., *On trades: an update*, Discrete Appl. Math. **95** (1999), 361–376.
[19] Street A.P., *Trades and defining sets*, in: C.J. Colbourn and J.H. Dinitz, ed., CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL., USA, 1996, 474–478.

School of Mathematics, The University of New South Wales, Sydney NSW 2052, Australia

(*Received* March 29, 2005, *revised* February 28, 2006)