A note on D-spaces

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Abstract. Every semi-stratifiable space or strong Σ-space has a σ-cushioned (mod$k$)-network. In this paper it is showed that every space with a σ-cushioned (mod$k$)-network is a D-space, which is a common generalization of some results about D-spaces.

Keywords: D-spaces, (mod$k$)-networks, cushioned collections, semi-stratifiable spaces

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A *neighborhood assignment* for a topological space (X, τ) is a function $\phi : X \to \tau$ such that $x \in \phi(x)$ for each $x \in X$. A space X is said to be a D-space if, for each neighborhood assignment ϕ for X, there exists a closed discrete subset D of X such that $\{\phi(d) : d \in D\}$ covers X. The first published results on D-spaces appear in [5], where it is proved that finite products of Sorgenfrey lines are D-spaces. Several interesting questions on D-spaces were raised by E.K. van Douwen and W.F. Pfeffer in [6]. It is still an open problem whether every regular Lindelöf space is a D-space. It is also asked whether there exists a subparacompact or metacompact space which is not a D-space. These questions are still open.

In [1] the authors study D-property in classes of generalized metric spaces. It is known that semi-stratifiable spaces are D-spaces [2], and strong Σ-spaces are D-spaces [3]. Semi-stratifiable spaces need not be strong Σ-spaces, and strong Σ-spaces need not be semi-stratifiable spaces [8]. It is therefore natural to look for a "common denominator" to the results mentioned above.

Throughout this paper, all spaces are assumed to be T_1. We refer the reader to [7] for notations and terminology not explicitly given here.

Definition 1. A space X is said to be a *semi-stratifiable space* [4] if, for each open set U of X, one can assign a sequence $\{F(n, U)\}_{n\in\mathbb{N}}$ of closed subsets of X such that

(1) $U = \bigcup_{n\in\mathbb{N}} F(n, U)$;
(2) $F(n, U) \subset F(n, V)$ whenever $U \subset V$.

A correspondence $U \to \{F(n, U)\}_{n\in\mathbb{N}}$ is a *semi-stratification* for X whenever it satisfies the conditions (1) and (2).

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A collection $\mathcal{P}$ of pairs of subsets of a space X is called a *pair-network* for X if whenever $x \in U$ with U open in X, $x \in P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$. A collection $\mathcal{P}$ of pairs of subsets of a space X is called *cushioned* if $\overline{\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}} \subset \bigcup\{P_2 : (P_1, P_2) \in \mathcal{P}'\}$ for each $\mathcal{P}' \subset \mathcal{P}$.

Lemma 2. *A space is a semi-stratifiable space if and only if it is a space with a σ-cushioned network.*

PROOF: Let (X, τ) be a semi-stratifiable space, and $U \to \{F(n, U)\}_{n\in\mathbb{N}}$ be a semi-stratification for X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{(F(n, U), U) : U \in \tau\}$. If $\tau' \subset \tau$, then $\overline{\bigcup_{U\in\tau'} F(n, U)} \subset F(n, \bigcup\tau') \subset \bigcup\tau'$. Thus $\mathcal{P}_n$ is a cushioned collection of X. Hence, $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ is a σ-cushioned network for X.

Conversely, suppose that a space X has a σ-cushioned network $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, where each $\mathcal{P}_n$ is cushioned in X. For each $n \in \mathbb{N}$, $U \in \tau$, put $F(n, U) = \overline{\bigcup\{P_1 : (P_1, P_2) \in \bigcup_{i\le n} \mathcal{P}_i, P_2 \subset U\}}$. It is obvious that $F(n, U) \subset U$ and $F(n, U) \subset F(n, V)$ whenever $U \subset V$. Let $x \in U \in \tau$. There exist $n \in \mathbb{N}$ and $(P_1, P_2) \in \mathcal{P}_n$ such that $x \in P_1 \subset P_2 \subset U$, thus $x \in F(n, U)$. Hence $U = \bigcup_{n\in\mathbb{N}} F(n, U)$. Therefore, $U \to \{F(n, U)\}_{n\in\mathbb{N}}$ is a semi-stratification for X. □

Definition 3. A collection $\mathcal{P}$ of pairs of subsets of a space X is called a $(modk)$-*network* for X if, there is a cover $\mathcal{K}$ of compact subsets of X such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X, then $K \subset P_1 \subset P_2 \subset U$ for some $(P_1, P_2) \in \mathcal{P}$.

Theorem 4. *Spaces with a σ-cushioned $(modk)$-network are D-spaces.*

PROOF: Suppose a space X has a σ-cushioned (mod$k$)-network $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ with respect to a cover $\mathcal{K}$ of compact subsets, where each $\mathcal{P}_n$ is cushioned and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$. Assume each $\mathcal{P}_n$ is well-ordered.

Let ϕ be an arbitrary neighborhood assignment for X. First, a subset D_n of X is defined inductively as follows for each $n \in \mathbb{N}$.

Set $D_0 = \emptyset$. Assume $D_m \subset X$ is defined for all $0 < m < n$.

A finite subset D^n_α of X is defined inductively as follows for order numbers α.

Set $D^n_0 = \emptyset$. Assume D^n_β is defined for all $0 < \beta < \alpha$. Put

$$U = \bigcup\left\{\phi(d) : d \in \left(\bigcup\{D^n_\beta : \beta < \alpha\}\right) \cup D_{n-1}\right\}.$$

Denote R^n_α by the following requirement: There exist $K \in \mathcal{K}$, $(P_1, P_2) \in \mathcal{P}_n$ and $\{x_1, x_2, \dots, x_k\} \subset K \setminus U$ such that

$$K \subset P_1 \subset P_2 \subset U \cup \phi(x_1) \cup \phi(x_2) \cup \cdots \cup \phi(x_k).$$

If R^n_α does not hold, the induction on α stops. Otherwise, let (P_1, P_2) be the first pair in $\mathcal{P}_n$ that satisfies R^n_α, and put $D^n_\alpha = \{x_1, x_2, \dots, x_k\}$.

Let $D_n = (\bigcup_{\alpha<\gamma_n} D^n_\alpha) \cup D_{n-1}$ for some order number γ_n, where γ_n is the first ordinal number for which R^n_α does not hold.

Secondly, put $D = \bigcup\{D_n : n \in \mathbb{N}\}$, and we shall prove that X is a D-space by D.

Claim 1. $X = \bigcup_{d\in D} \phi(d)$.

If not, then $K \setminus \bigcup_{d\in D} \phi(d) \neq \emptyset$ for some $K \in \mathcal{K}$. Put $L = K \setminus \bigcup_{d\in D} \phi(d)$. L is a non-empty, compact subset of X. There is a finite subset $\{x_1, x_2, \ldots, x_k\}$ of L such that $L \subset \phi(x_1) \cup \phi(x_2) \cup \cdots \cup \phi(x_k)$ because ϕ is a neighborhood assignment for X. Denote $M = K \setminus (\phi(x_1) \cup \phi(x_2) \cup \cdots \cup \phi(x_k))$. Then $M \subset K \setminus L \subset \bigcup_{d\in D} \phi(d)$. By the compactness of M, $M \subset \bigcup_{d\in D_j} \phi(d)$ for some $j \in \mathbb{N}$, thus $K \subset \phi(x_1) \cup \phi(x_2) \cup \cdots \cup \phi(x_k) \cup (\bigcup_{d\in D_j} \phi(d))$. Since $\mathcal{P}$ is a (mod$k$)-network for X with respect to $\mathcal{K}$, there are $m \in \mathbb{N}$ and $(P_1, P_2) \in \mathcal{P}_m$ such that

$$K \subset P_1 \subset P_2 \subset \phi(x_1) \cup \phi(x_2) \cup \cdots \cup \phi(x_k) \cup \Big(\bigcup_{d\in D_j} \phi(d)\Big).$$

Put $n = \max\{j+1, m\}$, $U = \bigcup\{\phi(d) : d \in D_{n-1}\}$. Then $\bigcup_{d\in D_j} \phi(d) \subset U$, $(P_1, P_2) \in \mathcal{P}_n$ and $\{x_1, x_2, \ldots, x_k\} \subset K \setminus U$. Hence R^n_1 holds, thus $D^n_1 = \{x_1, x_2, \ldots, x_k\}$, and $K \subset P_1 \subset P_2 \subset \bigcup_{d\in D} \phi(d)$, a contradiction.

Claim 2. Each D_n is a closed discrete subset of X.

D_0 is closed discrete in X. Suppose that D_{n-1} is closed discrete in X. It suffices to prove that $\bigcup_{\alpha<\gamma_n} D^n_\alpha$ is closed discrete in X in order to show that D_n is closed discrete in X. For each $\alpha < \gamma_n$, let $(P^n_{1\tilde{\alpha}}, P^n_{2\tilde{\alpha}})$ be the first element in $\mathcal{P}_n$ satisfying R^n_α. Assume that $x \in \overline{\bigcup_{\alpha<\gamma_n} D^n_\alpha}$. Since $\bigcup_{\alpha<\gamma_n} D^n_\alpha \subset \bigcup_{\alpha<\gamma_n} P^n_{1\tilde{\alpha}}$, then $x \in \bigcup_{\alpha<\gamma_n} P^n_{2\tilde{\alpha}} \subset \bigcup\{\phi(d) : d \in (\bigcup\{D^n_\alpha : \alpha < \gamma_n\}) \cup D_{n-1}\}$. And since each $D^n_\alpha \cap \bigcup_{d\in D_{n-1}} \phi(d) = \emptyset$, then $x \in \bigcup\{\phi(d) : d \in \bigcup_{\alpha<\gamma_n} D^n_\alpha\}$. Let α_x be the minimal element α in γ_n satisfying $x \in \bigcup\{\phi(d) : d \in D^n_\alpha\}$. Put $V = \bigcup\{\phi(d) : d \in D^n_{\alpha_x}\} \setminus \overline{\bigcup_{\alpha<\alpha_x} P^n_{1\tilde{\alpha}}}$. Then V is an open neighborhood of x in X, and $V \cap (\bigcup_{\alpha<\gamma_n} D^n_\alpha) \subset D^n_{\alpha_x}$ is finite. Hence, $\bigcup_{\alpha<\gamma_n} D^n_\alpha$ is closed discrete in X.

Claim 3. D is closed discrete in X.

For each $x \in X$, there is $n \in \mathbb{N}$ such that $x \in \bigcup_{d\in D_n} \phi(d)$ by Claim 1. Since D_n is closed discrete in X, there is an open neighborhood W of x in X such that $W \subset \bigcup_{d\in D_n} \phi(d)$ and W contains at most a point in D_n. Thus $W \cap D \subset ((\bigcup_{d\in D_n} \phi(d)) \cap (D \setminus D_n)) \cup (W \cap D_n)$. For each $y \in D \setminus D_n$, $y \in D^m_\alpha$ for some $m > n$ and some $\alpha < \gamma_m$. Put $U = \bigcup\{\phi(d) : d \in (\bigcup\{D^m_\beta : \beta < \alpha\}) \cup D_{m-1}\}$. Then $\bigcup_{d\in D_n} \phi(d) \subset U$, and $D^m_\alpha \cap U = \emptyset$, thus $y \notin \bigcup_{d\in D_n} \phi(d)$. Hence, $(D \setminus D_n) \cap (\bigcup_{d\in D_n} \phi(d)) = \emptyset$. This shows that W contains at most a point in D. Therefore, D is closed discrete in X. $\square$

A collection $\mathcal{F}$ of closed subsets of a space X is called *closure-preserved* if $\bigcup \mathcal{F}'$ is closed for each $\mathcal{F}' \subset \mathcal{F}$. A space is called a *strong* Σ*-space* (*strong* $\Sigma^{\#}$*-space*) [8] if there exists a σ-locally finite (σ-closure-preserved) collection $\mathcal{F}$ of closed subsets, and a cover $\mathcal{K}$ of X by compact subsets, such that, whenever $K \in \mathcal{K}$ and $K \subset U$ with U open in X, then $K \subset F \subset U$ for some $F \in \mathcal{F}$. Every strong Σ-space is a strong $\Sigma^{\#}$-space. If a collection $\mathcal{F}$ of closed subsets of a space X is closure-preserved, then a collection $\{(F, F) : F \in \mathcal{F}\}$ of pairs of subsets of X is cushioned. Hence every strong $\Sigma^{\#}$-space or semi-stratifiable space is a space with a σ-cushioned (mod$k$)-network. Thus Theorem 4 is a common generalization of Buzyakova's result [3] and Borges-Wehrly's result [2]. The following corollary is an unpublished result obtained by Liang-xue Peng.

Corollary 5. *Every strong* $\Sigma^{\#}$*-space is a D-space.*

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