Topologies generated by ideals

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Abstract. A topological space X is said to be *generated by an ideal* $\mathcal{I}$ if for all $A \subseteq X$ and all $x \in \overline{A}$ there is $E \subseteq A$ in $\mathcal{I}$ such that $x \in \overline{E}$, and is said to be *weakly generated by* $\mathcal{I}$ if whenever a subset A of X contains $\overline{E}$ for every $E \subseteq A$ with $E \in \mathcal{I}$, then A itself is closed. An important class of examples are the so called weakly discretely generated spaces (which include sequential, scattered and compact Hausdorff spaces). Another paradigmatic example is the class of Alexandroff spaces which corresponds to spaces generated by finite sets. By considering an appropriate topology on the power set of X we show that τ is weakly generated by $\mathcal{I}$ iff τ is a $\mathcal{I}$-closed subset of $\mathcal{P}(X)$. The class of spaces weakly generated by an ideal behaves as the class of sequential spaces, in the sense that their closure operator can be characterized as the sequential closure and moreover there is a natural notion of a convergence associated to them. We also show that the collection of topologies weakly generated by $\mathcal{I}$ is lattice isomorphic to a lattice of pre-orders over $\mathcal{I}$.

Keywords: lattices of topologies, hyperspaces, tightness, Alexandroff spaces, Fréchet and sequential spaces, discretely generated spaces, sequential convergence

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1. Introduction

Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal over X containing all finite subsets of X. We say that τ is *generated by* $\mathcal{I}$ if for every $A \subseteq X$ and every $x \in \overline{A}$ there is $E \in \mathcal{I}$ such that $E \subseteq A$ and $x \in \overline{E}$. We will say τ is *weakly generated by* $\mathcal{I}$ if whenever a subset A of X contains $\overline{E}$ for every $E \subseteq A$ with $E \in \mathcal{I}$, then A itself is closed. There are several notions that have been studied in the literature which motivate the above definitions. For instance: (1) A space is generated by finite sets iff it is an Alexandroff space (that is to say, the intersection of arbitrary open sets is open). (2) If $\mathcal{I}$ is the ideal generated by the τ-discrete sets, then we obtain the notions of a (weakly) discretely generated space, which has recently received considerable attention [1], [2], [7]. (3) If $\mathcal{I}$ is the collection of countable subsets of X, then τ is generated by $\mathcal{I}$ iff τ is countable tight. (4) Let τ be a T_1 topology and $\mathcal{I}$ be the ideal generated by the range of all τ-convergent sequences. Then τ is Fréchet (sequential) iff τ is (weakly) generated by $\mathcal{I}$.

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In order to state our results we need to recall some facts and introduce some terminology. It is well known ([15]) that Alexandroff topologies are given by pre-orders over X, namely for every pre-order $\leq$ over X let $\Gamma(\leq)$ be the topology generated by the sets $\{y \in X : \ x \leq y\}$ for $x \in X$. Equivalently, A is $\Gamma(\leq)$-closed iff A contains every y such that $y \leq x$ for some $x \in A$. Then τ is an Alexandroff topology iff there is a (unique) pre-order $\leq$ such that $\tau = \Gamma(\leq)$. For a given topology τ such pre-order is called the specialization pre-order of τ and is usually denoted by $\leq_\tau$. Moreover, the collection of Alexandroff topologies over X is a lattice and the map $\tau \mapsto \leq_\tau$ is a lattice isomorphism. We generalize these results. We will work with pre-orders $\sqsubseteq$ (called specialization relations) over $\mathcal{I}$ and associate to them a topology $\tau(\sqsubseteq)$ in a natural way: A is $\tau(\sqsubseteq)$-closed iff $A \supseteq F$ for all $F \sqsubseteq E$ with $E \subseteq A$ and $E, F \in \mathcal{I}$. Alexandroff topologies then correspond to topologies of the form $\tau(\sqsubseteq)$ where $\sqsubseteq$ is a pre-order over the ideal of finite subsets of X.

We introduce a notion of an $\mathcal{I}$-convergence analogous to that of a sequential convergence where sequences are substituted by sets in $\mathcal{I}$. So an $\mathcal{I}$-convergence will be a collection $\mathbb{A} \subseteq \mathcal{I} \times X$ satisfying some natural axioms. To each $\mathcal{I}$-convergence $\mathbb{A}$, a topology $\tau(\mathbb{A})$ is associated as follows: a set $A \subseteq X$ is $\tau(\mathbb{A})$-closed iff $x \in A$ for all $(E, x) \in \mathbb{A}$ with $E \subseteq A$. We present a version of the Urysohn axiom for sequential convergence in this more general context.

It was shown in [17] that a topology τ over X is Alexandroff iff it is a closed subset of the power set $\mathcal{P}(X)$ endowed with the product topology (i.e. identifying a subset of X with its characteristic function). We define a topology on $\mathcal{P}(X)$ called the $\mathcal{I}$-topology. When $\mathcal{I}$ is the ideal of finite sets, the $\mathcal{I}$-topology is the product topology. We will show that the $\mathcal{I}$-closure of a topology is again a topology.

Our main results are the following:

Theorem 1.1. *Let $\mathcal{I}$ and τ be, respectively, an ideal* (*containing all finite sets*) *and a topology over X. The following are equivalent:*

1. *τ is weakly generated by $\mathcal{I}$;*
2. *τ is closed in $\mathcal{P}(X)$ with respect to the $\mathcal{I}$-topology;*
3. *there is a specialization pre-order $\sqsubseteq$ over $\mathcal{I}$ such that $\tau = \tau(\sqsubseteq)$;*
4. *there is an $\mathcal{I}$-convergence $\mathbb{A}$ such that $\tau = \tau(\mathbb{A})$.*

Theorem 1.2. *Let $\mathcal{I}$ be an ideal over X* (*containing all finite sets*)*. The collection of all topologies weakly generated by $\mathcal{I}$ is lattice isomorphic to a lattice of pre-orders over $\mathcal{I}$.*

The paper is organized as follows. In Section 3 we study the $\mathcal{I}$-topology and show that the $\mathcal{I}$-closure of a topology is a topology. In Section 4 we show that a topology is weakly generated by $\mathcal{I}$ iff it is closed with respect to the $\mathcal{I}$-topology. In Section 5 we present a characterization of weakly generated topologies in terms of specialization pre-orders over $\mathcal{I}$ and show that the collection of weakly generated topologies is a lattice. Finally, in Section 6 we study the notion of an $\mathcal{I}$-convergence.

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2. Notation and terminology

An ideal over a set X is a collection of subsets of X closed under taking subsets and finite unions. We will always assume that an ideal contains every finite subset of X. The ideal of finite sets is denoted by Fin. For a given family $\mathcal{A}$ of subsets of X, we denote by $\mathcal{A}^*$ the collection of complements of sets in $\mathcal{A}$. Recall that $\mathcal{I}$ is an ideal iff $\mathcal{I}^*$ is a filter. A *Čech closure operator* ([13]) on a set X is a map $C : \mathcal{P}(X) \to \mathcal{P}(X)$ (where $\mathcal{P}(X)$ denotes the power set of X) such that $C(\emptyset) = \emptyset$, $A \subseteq C(A)$ and $C(A \cup B) = C(A) \cup C(B)$ for any $A, B \subseteq X$. Notice that under this conditions C is also monotone, i.e. $C(A) \subseteq C(B)$ whenever $A \subseteq B \subseteq X$. If C is also idempotent, it is called a *Kuratowski closure operator*. A Čech operator C can be iterated transfinitely and its limit C^∞ is a Kuratowski closure operator. Thus each Čech operator has a topology associated with it. For the undefined notions used in this paper we refer to [9].

3. Ochan closures of families of subsets

Given $A \subseteq B \subseteq X$ we denote by $[A, B]$ the interval determined by A and B with respect to the order $\subseteq$, that is to say,

$$[A, B] = \{C \in \mathcal{P}(X) :\ A \subseteq C \subseteq B\}.$$

We will write $[x, B]$ instead of $[\{x\}, B]$. Let $\mathcal{A}$ and $\mathcal{B}$ be two collections of subsets of X such that $\mathcal{A}$ contains $\emptyset$ and is closed under finite unions and $\mathcal{B}$ contains X and is closed under finite intersections. Since $[A, B] \cap [A', B'] = [A \cup A', B \cap B']$ it is clear that the collection of intervals $[A, B]$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ forms a base for a topology on $\mathcal{P}(X)$ which will be called the $(\mathcal{A}, \mathcal{B})$-topology. Thus we talk about a subset $T \subseteq \mathcal{P}(X)$ being $(\mathcal{A}, \mathcal{B})$-open, $(\mathcal{A}, \mathcal{B})$-closed, etc. We will focus mainly on the case where $\mathcal{A}$ is an ideal and $\mathcal{B}$ is a filter. In particular, every basic interval $[A, B]$ is clopen and thus the $(\mathcal{A}, \mathcal{B})$-topology is zero-dimensional.

This topologies of family are generically called Ochan topologies ([12], [14]) or Pixley-Roy topologies ([4], [5]). The product topology on $\mathcal{P}(X)$ clearly corresponds to the $(\mathsf{Fin}, \mathsf{Fin}^*)$-topology. The $(\mathsf{Fin}, \mathcal{P}(\mathbb{N}))$-topology has played a prominent role in Ramsey Theory, since the subspace $[\mathbb{N}]^\omega$ of $\mathcal{P}(\mathbb{N})$ of all infinite subsets of $\mathbb{N}$ is Ellentuck's space ([8]). The approach of regarding a topology as a subspace of $\mathcal{P}(X)$ with the product topology has been used in [17], [18] [19].

Let $\mathcal{A}$ and $\mathcal{B}$ be respectively an ideal and a filter over X. In this section we present some general results about the $(\mathcal{A}, \mathcal{B})$-closure (called *Ochan closure*) of families of sets. For instance, we show that the $(\mathcal{A}, \mathcal{B})$-closure $\overline{\tau}$ of a topology τ is a topology. We will be most interested in the case where $\mathcal{A}$ is Fin and $\mathcal{B}$ is the dual filter $\mathcal{I}^*$ of some ideal $\mathcal{I}$. The $(\mathsf{Fin}, \mathcal{I}^*)$-topology will be simply called the $\mathcal{I}$-topology and consequently we will talk about $\mathcal{I}$-closed sets, $\mathcal{I}$-closure, etc.

The propositions that follow contain some basic facts about Ochan closures. The first one says that union, intersection and complementation are continuous and open when they are viewed as maps on Ochan spaces.

Proposition 3.1. *Let $\mathcal{I}, \mathcal{F}$ be respectively an ideal and a filter of subsets of X. Consider the functions $f, g : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ and $h = \mathcal{P}(X) \to \mathcal{P}(X)$ given by $f(A, B) = A \cup B$, $g(A, B) = A \cap B$ and $h(A) = X \setminus A$.*

(i) *Then f and g are continuous and open maps when $\mathcal{P}(X)$ is given the $(\mathcal{I}, \mathcal{F})$-topology.*
(ii) *h is a homeomorphism regarded as a map from $\mathcal{P}(X)$ with the $(\mathcal{I}, \mathcal{F})$-topology onto $\mathcal{P}(X)$ with the $(\mathcal{F}^*, \mathcal{I}^*)$-topology.*

Proof: (ii) To see that h is a homeomorphism just notice that $X - A \in [K, L]$ iff $A \in [X \setminus L, X \setminus K]$.

(i) Let $K \in \mathcal{I}$ and $E \in \mathcal{F}$ and $A, B \subseteq X$ such that $A \cup B \in [K, E]$. Then it is easy to check that $f([K \cap A, E] \times [K \cap B, E]) \subseteq [K, E]$. Analogously, if $A \cap B \in [K, E]$, then $g([K, E \cup (A \setminus E)] \times [K, E \cup (B \setminus E)]) \subseteq [K, E]$. Thus f and g are continuous. To see that they are open just notice that f sends $[K, E] \times [K', E']$ onto $[K \cup K', E \cup E']$ and for g use (ii) and DeMorgan laws. $\square$

Proposition 3.2. *Let $\mathcal{I}, \mathcal{F}$ be respectively an ideal and a filter of subsets of X. Let $\mathcal{B} \subseteq \mathcal{P}(X)$ and $\overline{\mathcal{B}}$ denote the $(\mathcal{I}, \mathcal{F})$-closure of $\mathcal{B}$.*

(i) *If $\mathcal{B}$ is closed under finite (arbitrary) unions, then $\overline{\mathcal{B}}$ is closed under finite (arbitrary) unions. Analogously, if $\mathcal{B}$ is closed under finite (arbitrary) intersection, then $\overline{\mathcal{B}}$ is closed under finite (arbitrary) intersections.*
(ii) *If $\mathcal{I} = \mathsf{Fin}$ and $\mathcal{B}$ is closed under finite unions, then $\overline{\mathcal{B}}$ is closed under arbitrary unions.*
(iii) *If $\mathcal{F}$ is Fin^* and $\mathcal{B}$ is closed under finite intersections, then $\overline{\mathcal{B}}$ is closed under arbitrary intersections.*
(iv) *If $\mathcal{B}$ is closed under subsets (resp. supersets), then so is $\overline{\mathcal{B}}$.*

Proof: First we claim that it suffices to show (i) for unions. In fact, suppose that (i) holds for unions and let $\mathcal{B}$ be a family closed under intersections. Consider the collection $\mathcal{A}$ of complements of sets in $\mathcal{B}$. Then $\mathcal{A}$ is closed under unions. Then the $(\mathcal{F}^*, \mathcal{I}^*)$-closure of $\mathcal{A}$ is closed under unions. The result now follows from Proposition 3.1(ii), since the $(\mathcal{I}, \mathcal{F})$-closure of $\mathcal{B}$ is mapped by h to the $(\mathcal{F}^*, \mathcal{I}^*)$-closure of $\mathcal{A}$.

Now we show (i) for finite unions. Let f be the function given in Proposition 3.1. For any family $\mathcal{A}$ of subsets of X we have $\mathcal{A} \subseteq f(\mathcal{A} \times \mathcal{A})$. And $\mathcal{A}$ is closed under finite unions iff $f(\mathcal{A} \times \mathcal{A}) = \mathcal{A}$. Let $\mathcal{B}$ be a family of sets closed under finite unions. Since f is continuous, we have that $\overline{\mathcal{B}} \subseteq f(\overline{\mathcal{B}} \times \overline{\mathcal{B}}) \subseteq \overline{f(\mathcal{B} \times \mathcal{B})} = \overline{\mathcal{B}}$. The case of arbitrary unions follows from the argument given below for (ii).

To see (ii), suppose $\mathcal{B}$ is closed under finite unions and $V_i \in \overline{\mathcal{B}}$ for all $i \in I$. Let $K \in \mathcal{I}$ and $E \in \mathcal{F}$ such that $K \subseteq \bigcup_i V_i \subseteq E$. Then $K \cap V_i \subseteq V_i \subseteq E$ for all i. Thus there is $V_i' \in \mathcal{B}$ such that $K \cap V_i \subseteq V_i' \subseteq E$ for all $i \in I$. Then $K \subseteq \bigcup_i V_i' \subseteq E$. Now suppose that K is finite, then for some finite $J \subseteq I$ we have that $K \subseteq \bigcup_{i \in J} V_i' \subseteq E$ and $\bigcup_{i \in J} V_i' \in \mathcal{B}$. Therefore $[K, E] \cap \mathcal{B} \neq \emptyset$ and thus $\bigcup_i V_i \in \overline{\mathcal{B}}$.

(iii) follows from (ii) and Proposition 3.1(ii). Finally, to see (iv), suppose $\mathcal{B}$ is closed under subsets and let $B \subseteq A \in \overline{\mathcal{B}}$. Fix $K \in \mathcal{I}$ and $F \in \mathcal{F}$ such that $K \subseteq B \subseteq F$. Then $K \subseteq A \subseteq F \cup A$, therefore there is $V \in \mathcal{B}$ such that $K \subseteq V \subseteq F \cup A$. Hence $V \cap F \in \mathcal{B}$ and $K \subseteq V \cap F \subseteq F$. An analogous argument shows that when $\mathcal{B}$ is closed under supersets, so is $\overline{\mathcal{B}}$. □

It is clear from the previous result that the $\mathcal{I}$-closure of a filter (ideal) is again a filter (ideal). The $\mathcal{I}$-closure of an ideal $\mathcal{J}$ (containing every finite set) is $\mathcal{P}(X)$, since $\mathcal{J}$ is clearly $\mathcal{I}$-dense. However, for the closure of a filter we have the following characterization.

Proposition 3.3. *Let $\mathcal{I}$ and $\mathcal{F}$ be respectively an ideal and a filter over X. Let $\overline{\mathcal{F}}$ be the $\mathcal{I}$-closure of $\mathcal{F}$. Then*

$$A \in \overline{\mathcal{F}} \quad \text{iff} \quad A \cup B \in \mathcal{F} \quad \text{for all} \quad B \in \mathcal{I}^*.$$

Proof: Let $\mathcal{F}$ be a filter over X. Suppose $A \in \overline{\mathcal{F}}$ and let $B \in \mathcal{I}^*$. Then $A \in [\emptyset, A \cup B]$ and $A \cup B \in \mathcal{I}^*$. Therefore there is $C \in \mathcal{F}$ contained in $A \cup B$ and thus $A \cup B \in \mathcal{F}$. Conversely, suppose $A \cup B \in \mathcal{F}$ for all $B \in \mathcal{I}^*$. Let K be a finite set and $B \in \mathcal{I}^*$ such that $A \in [K, B]$. Then $B = A \cup B \in \mathcal{F}$ and hence $[K, B] \cap \mathcal{F} \neq \emptyset$. Therefore $A \in \overline{\mathcal{F}}$. □

3.1 The $\mathcal{I}$-closure of a topology

In this section we will analyze the Ochan closure of a topology.

Theorem 3.4. *Let $\mathcal{I}, \mathcal{F}$ be respectively an ideal and a filter on X.*

(i) *Every topology τ over X is $(\mathsf{Fin}, \mathcal{P}(X))$-closed.*
(ii) *If τ is a topology over X, then the $(\mathcal{I}, \mathcal{F})$-closure of τ is a topology.*

(iii) *If $\mathcal{B}$ is a collection of subsets of X closed under finite unions and intersections, then $\overline{\mathcal{B}}$ the $(\mathsf{Fin}, \mathcal{F})$-closure of $\mathcal{B}$ is a topology. Moreover, when $\mathcal{F} = \mathcal{P}(X)$, then $\overline{\mathcal{B}}$ is the topology generated by $\mathcal{B}$.*

PROOF: (i) Let A be a set which is not τ-open. Let $x \in A$ such that $x \notin \mathrm{int}_\tau(A)$. Then $A \in [x, A]$ and $[x, A] \cap \tau = \emptyset$. (ii) and (iii) follow from Proposition 3.2. □

The $(\mathcal{A}, \mathcal{B})$-closure of a topology with respect to two arbitrary families $\mathcal{A}$ and $\mathcal{B}$ is not in general a topology as we will show in the following example. Thus the assumption that $\mathcal{I}$ is an ideal and $\mathcal{F}$ is a filter is important when dealing with Ochan closures.

Example 3.5. Let τ be a topology over X, $\mathcal{B}$ be the collection of τ-closed sets and $\overline{\tau}$ be the $(\mathsf{Fin}, \mathcal{B})$-closure of τ. Then

$$A \in \overline{\tau} \;\text{ iff }\; A \subseteq \mathrm{int}_\tau(cl_\tau(A)).$$

In fact, suppose $A \subseteq \mathrm{int}_\tau(cl_\tau(A))$ and let $K \subseteq A \subseteq F$ be such that K is finite and F is τ-closed. Then $K \subseteq \mathrm{int}_\tau(cl_\tau(A)) \subseteq F$. Thus $A \in \overline{\tau}$. Conversely, suppose $A \in \overline{\tau}$ and let $x \in A$. Since $A \in [x, cl_\tau(A)]$, there is a τ-open set O such that $x \in O \subseteq cl_\tau(A)$ and we are done. To see that $\overline{\tau}$ is not in general closed under intersections consider in $\mathbb{R}$ two dense sets with finite and non empty intersection.

Proposition 3.6. *Let $\mathcal{I}$ and τ be respectively an ideal and a topology over X. Let $\overline{\tau}$ denote the $\mathcal{I}$-closure of τ. Then*

(i) $\mathrm{int}_\tau(F) = \mathrm{int}_{\overline{\tau}}(F)$ *for all $F \in \mathcal{I}^*$ or equivalently $cl_\tau(E) = cl_{\overline{\tau}}(E)$ for all $E \in \mathcal{I}$.*

(ii) *For every $V \subseteq X$ the following are equivalent:*
  (a) $V \in \overline{\tau}$,
  (b) $V = \bigcap\{\mathrm{int}_\tau(F) : V \subseteq F \in \mathcal{I}^*\}$,
  (c) $\forall x \in V\, \forall E \in \mathcal{I}\, [x \in cl_\tau(E) \Rightarrow x \notin cl_\tau(E \setminus V)]$.

PROOF: (i) Let $F \in \mathcal{I}^*$. It is clear that $\mathrm{int}_\tau(F) \subseteq \mathrm{int}_{\overline{\tau}}(F)$. On the other hand, fix $x \in \mathrm{int}_{\overline{\tau}}(F)$. Since $\mathrm{int}_{\overline{\tau}}(F) \in [x, F] \cap \overline{\tau}$, then there is $W \in [x, F] \cap \tau$. Thus $x \in \mathrm{int}_\tau(F)$ and we are done.

(ii) We show first that (a) and (b) are equivalent. Let $V \in \overline{\tau}$. By (i) it is clear that $V \subseteq \mathrm{int}_\tau(F)$ for all $F \in \mathcal{I}^*$ with $V \subseteq F$. On the other hand, if $x \notin V$, then $x \notin \mathrm{int}_\tau(X \setminus \{x\})$ and $X \setminus \{x\} \in \mathcal{I}^*$. Thus (b) holds. Conversely, suppose that (b) holds. Then the collection $\{\mathrm{int}_\tau(F) : V \subseteq F \in \mathcal{I}^*\}$ is a net (with respect to reverse inclusion) in τ which converges to V in the $(\mathsf{Fin}, \mathcal{I}^*)$-topology. Thus $V \in \overline{\tau}$.

It follows immediately from (i) that (a) implies (c). We will show that (c) implies (b). Let $V \subseteq F \in \mathcal{I}^*$. It suffices to show that $V \subseteq \mathrm{int}_\tau(F)$. Let $x \in V$ and suppose that $x \notin \mathrm{int}_\tau(F)$. Then $x \in cl_\tau(X \setminus F)$. Since $X \setminus F \in \mathcal{I}$ and $(X \setminus F) \setminus V = X \setminus F$, then by (c) we have that $x \notin cl_\tau(X \setminus F)$ which is a contradiction. □

Example 3.7. Consider the ideal $\mathcal{I}$ of subsets of $\mathbb{R}$ generated by the range of all strictly decreasing sequences. Let τ_l be the topology of the Sorgenfrey line, i.e. the topology generated by the intervals of the form $[a, b)$. Then the $\mathcal{I}$-closure of the usual topology of $\mathbb{R}$ is τ_l.

Proposition 3.8. *The $\mathcal{I}$-closure of a topology τ is the discrete topology iff every set in $\mathcal{I}$ is τ-closed discrete. In particular, for $\mathcal{I} = \mathsf{Fin}$, the Fin-closure of τ is the discrete topology iff τ is T_1.*

PROOF: Let $\overline{\tau}$ be the $\mathcal{I}$-closure of τ. Suppose $\overline{\tau}$ is the discrete topology. By 3.6, $cl_\tau(E) = cl_{\overline{\tau}}(E) = E$ for all $E \in \mathcal{I}$. Therefore each $E \in \mathcal{I}$ is τ-closed discrete. The other claims are easily verified. $\square$

4. Topologies generated by an ideal

Then notion of a topology (weakly) generated by an ideal was given in the introduction. Given a $Y \subseteq X$, we denote by $\mathcal{I}_Y$ the ideal $\mathcal{I} \cap \mathcal{P}(Y)$ over Y. We will say that the topology of X is *hereditarily weakly generated by* $\mathcal{I}$ if, for every $Y \subseteq X$, the subspace topology of Y is weakly generated by $\mathcal{I}_Y$. The following result, whose proof is straightforward, shows the analogy with sequential and Fréchet-Urysohn spaces.

Proposition 4.1. *Let τ and $\mathcal{I}$ be respectively a topology and an ideal over X.*

(i) *If τ is generated by $\mathcal{I}$ and $Y \subseteq X$, then the subspace topology of Y is generated by $\mathcal{I}_Y$.*
(ii) *If τ is weakly generated by $\mathcal{I}$ and $Y \subseteq X$ is τ-closed, then the subspace topology of Y is weakly generated by $\mathcal{I}_Y$.*
(iii) *τ is hereditarily weakly generated by $\mathcal{I}$ iff τ is generated by $\mathcal{I}$.*

Consider the following operator:

$$C_{\mathcal{I},\tau}(A) = \bigcup\{\overline{E} : E \subseteq A \text{ with } E \in \mathcal{I}\}. \tag{1}$$

It is easily seen that $C_{\mathcal{I},\tau}$ is a Čech operator. Let $C^{\infty}_{\mathcal{I},\tau}$ be the corresponding Kuratowski closure operator. Unless there is a danger of confusion about which ideal $\mathcal{I}$ is being used, we will denote $C^{\alpha}_{\mathcal{I},\tau}$ and $C^{\infty}_{\mathcal{I},\tau}$ just by C^{α} and C^{∞}. Notice that $C^{\infty}(A) \subseteq \overline{A}$ for all $A \subseteq X$ and $C^{\infty}(E) = C(E) = \overline{E}$ for all $E \in \mathcal{I}$. Thus we have that τ is generated by $\mathcal{I}$ if $C(A) = \overline{A}$ for all $A \subseteq X$, and τ is weakly generated by $\mathcal{I}$ if whenever $C(A) \subseteq A$, then A is τ-closed.

Proposition 4.2. *Let $\mathcal{I}$ and τ be respectively an ideal and a topology over X. Then*

(i) *τ is generated by $\mathcal{I}$ iff $C = cl_\tau$;*
(ii) *τ is weakly generated by $\mathcal{I}$ iff $C^{\infty} = cl_\tau$;*
(iii) *suppose τ is weakly generated by $\mathcal{I}$ and $t(X) \leq \kappa$; then $cl_\tau = C^{\kappa^+}$.*

Proof: (i) follows immediately from the definitions. (ii) Suppose τ is weakly generated by $\mathcal{I}$. By the definition of C^∞, we have $A \subseteq C^\infty(A) \subseteq \overline{A}$. It is also clear that $C(C^\infty(A)) = C^\infty(A)$. Since τ is weakly generated, $C^\infty(A)$ is τ-closed, thus $C^\infty(A) = \overline{A}$. Conversely, suppose $C^\infty = cl_\tau$ and let $A \subseteq X$ be such that $C(A) \subseteq A$. Then $C^\infty(A) \subseteq A$ and thus $\overline{A} \subseteq A$. (iii) It suffices to show that $C(C^{\kappa^+}(A)) = C^{\kappa^+}(A)$ for every $A \subseteq X$. In fact, let $x \in C(C^{\kappa^+}(A))$. Then there is $E \subseteq C^{\kappa^+}(A)$ such that $x \in \overline{E}$. Since $t(X) \leq \kappa$, there is $F \subseteq E$ of size at most κ such that $x \in \overline{F}$. Therefore, there is $\alpha < \kappa^+$ such that $F \subseteq C^\alpha(A)$. Thus $x \in C^{\alpha+1}(A)$. □

Theorem 4.3. *Let $\mathcal{I}$ be an ideal over X, τ a topology over X and $\overline{\tau}$ the $\mathcal{I}$-closure of τ. Then*

(i) *τ is weakly generated by $\mathcal{I}$ iff τ is $\mathcal{I}$-closed,*
(ii) *C^∞ is the closure operator of $\overline{\tau}$.*

Proof: (i) From 3.6(ii) we have that for every $A \subseteq X$

$$A \text{ is } \overline{\tau}\text{-closed iff } cl_\tau(E) \subseteq A \text{ for all } E \subseteq A \text{ in } \mathcal{I} \tag{2}$$

which simply says that (i) holds. To see (ii), notice first that from 3.6(i) it follows that $C_{\mathcal{I},\tau} = C_{\mathcal{I},\overline{\tau}}$. From this, (i) and 4.2 the results follows. □

Example 4.4. Let τ be a Hausdorff topology over X and $\mathcal{I}$ be the ideal generated by the collection of τ-discrete subsets of X. In this case we talk about spaces which are (weakly) discretely generated. They have been studied in [1], [2], [7]. For instance, if X belongs to the collection of spaces {Sequential, scattered, compact with countable tightness, monotonically normal, regular with a nested local base at every point, radial space}, then X is discretely generated. Moreover, if X is compact or pseudoradial then τ is weakly discretely generated.

There are spaces which are not determined by any reasonable ideal as we show below. We have already said that every topology is obviously generated by $\mathcal{P}(X)$. A similar situation occurs when $\mathcal{I}$ contains the complement of a closed-discrete set. To see this we introduce the following notion. We will say that an ideal $\mathcal{I}$ is *trivial with respect to* τ if for all $x \in X$ there is a τ-open set O_x such that $x \in \operatorname{int}(\overline{O_x})$, $O_x \in \mathcal{I}$ and $\overline{O_x} \setminus O_x$ is discrete.

Proposition 4.5. *Let τ be a topology over X and $\mathcal{I}$ an ideal. If $\mathcal{I}$ is trivial with respect to τ, then τ is generated by $\mathcal{I}$.*

Proof: Let $A \subseteq X$ and $x \in \overline{A} \setminus A$. Let O_x be an open set as in the definition of a trivial ideal. Let $V_x = \operatorname{int}(\overline{O_x})$. Since $x \in \overline{A \cap V_x}$, $A \cap V_x = (A \cap O_x) \cup A \cap (V_x \setminus O_x)$ and $A \cap (V_x \setminus O_x)$ is closed discrete, then $x \in \overline{A \cap O_x}$ and we are done. □

Example 4.6. A space is *maximal* (in the sense of [6]) if X has no isolated points and every topology extending that of X has an isolated point (i.e. the topology of X is maximal in the family of dense-in-itself topologies). It was shown in [6] that a maximal space is extremally disconnected and satisfies that every nowhere dense set is closed discrete and every set with empty interior is nowhere dense. Since in a maximal space every discrete set is closed, such spaces are not weakly discretely generated ([7]). Now we will show that the same happens with respect to any non trivial ideal. Suppose that τ, the topology of a maximal space X, is weakly generated by an ideal $\mathcal{I}$. Let $x \in X$, as X has no isolated points, then $x \in \overline{X \setminus \{x\}}$. Therefore there is $E \in \mathcal{I}$ with $x \in \overline{E} \setminus E$. Hence E is not closed-discrete and thus it has non empty interior. Let $O_x = \mathrm{int}(E)$ and $V_x = \overline{O_x}$. Since X is extremally disconnected, V_x is open. As $E \setminus \mathrm{int}(E)$ is closed-discrete, $x \in V_x$. Therefore $\mathcal{I}$ is trivial with respect to τ.

Moreover, in [6] a countable regular maximal space M was constructed. Since M can be embedded in $2^{\mathfrak{c}}$, then $2^{\mathfrak{c}}$ is an example of a compact space which is not discretely generated ([7]). In view of these results (see Example 4.4), a natural question is whether every compact Hausdorff space is generated by some non trivial ideal.

Remark 4.7. Another way of approximating the closure of a set is by almost closed sets. A set F is *almost closed* if $\overline{F} \setminus F$ is a singleton. A space is Whyburn if for every $x \in \overline{A} \setminus A$ there is an almost closed set $F \subseteq A$ such that $x \in \overline{F}$ ([3], [16]). In general, the topology of a Whyburn space is not necessarily (weakly) generated by a non trivial ideal. For instance, any maximal space is Whyburn ([3]).

4.1 An equivalence relation associated to an ideal

To each ideal $\mathcal{I}$ over X we will associate the following equivalence relation $\sim_{\mathcal{I}}$ on the lattice of all topologies over X:

$$\tau \sim_{\mathcal{I}} \rho \ \text{ iff } \ C_{\mathcal{I},\tau} = C_{\mathcal{I},\rho}.$$

Proposition 4.8. *Let $\mathcal{I}$ be an ideal over X, τ and ρ topologies over X. The following are equivalent:*

(i) $\tau \sim_{\mathcal{I}} \rho$,
(ii) $cl_\tau(E) = cl_\rho(E)$ *for all* $E \in \mathcal{I}$,.
(iii) τ *and* ρ *have equal* $\mathcal{I}$*-closure.*

Proof: We will denote by $\overline{\tau}$ the $\mathcal{I}$-closure of τ. Suppose (i), then it is obvious from 4.3(ii) that (iii) holds. Suppose (iii), notice that $cl_{\overline{\tau}}(E) = C_{\mathcal{I},\tau}(E) = cl_\tau(E)$ for all $E \in \mathcal{I}$, now (ii) follows. Finally, it is obvious that (ii) implies (i). $\square$

Theorem 4.9. *Let $\mathcal{I}$ be an ideal over X, τ and ρ topologies over X. Let τ_0 be the topology generated by the sets $X \setminus cl_\tau(E)$ for $E \in \mathcal{I}$. Then*

(i) *the $\mathcal{I}$-closure of a topology is the largest element in its equivalence class,*
(ii) $\tau_0 \sim_{\mathcal{I}} \tau$,
(iii) *if $\rho \sim_{\mathcal{I}} \tau$, then $\tau_0 \subseteq \rho$. So τ_0 is the smallest element in the equivalence class of τ.*

Proof: (i) follows from 4.8. (ii) Notice that the collection of all sets $cl_\tau(E)$ for $E \in \mathcal{I}$ is closed under finite unions and therefore is a basis of closed sets for τ_0. It is clear that $\tau_0 \subseteq \tau \subseteq \overline{\tau}$ and therefore $cl_{\overline{\tau}}(A) \subseteq cl_{\tau_0}(A)$ for every $A \subseteq X$. On the other hand, for $E \in \mathcal{I}$, $cl_{\overline{\tau}}(E)$ is τ_0-closed, hence $cl_{\tau_0}(E) \subseteq cl_{\overline{\tau}}(E)$. Therefore $\tau_0 \sim_{\mathcal{I}} \tau$. (iii) Suppose $\rho \sim_{\mathcal{I}} \tau$. By 4.8, $cl_\rho(E) = cl_\tau(E)$ for every $E \in \mathcal{I}$. Therefore every τ_0-closed set is ρ-closed. □

Example 4.10. Let $\tau_{\mathbb{R}}$ denote the usual topology on $\mathbb{R}$. We will analyze some examples of ideals on $\mathbb{R}$ and the corresponding equivalence class of $\tau_{\mathbb{R}}$. (i) If $\mathcal{I}$ is Fin, then the equivalence class of $\tau_{\mathbb{R}}$ consists of all T_1 topologies (by 3.8). (ii) If $\mathcal{I}$ is the ideal of countable subsets of $\mathbb{R}$, then clearly $\tau_{\mathbb{R}}$ is the only member of its class. (iii) If $\mathcal{I}$ is the ideal generated by all decreasing sequences in $\mathbb{R}$, we have already shown that τ_l, the topology of the Sorgenfrey line, is the $\mathcal{I}$-closure of $\tau_{\mathbb{R}}$ and therefore τ_l is the largest element of the class.

Example 4.11. Let (X, τ) be a scattered space and $\mathcal{I}$ be the ideal generated by the τ-discrete subsets of X. Then τ is generated by $\mathcal{I}$ ([7]) and the $\mathcal{I}$-equivalence class of τ only contains τ. In fact, in a scattered space every closed set is equal to the closure of a discrete set, therefore τ_0 (as in Theorem 4.9) is equal to τ.

5. The lattice of $\mathcal{I}$-closed topologies

Let $T(\mathcal{I})$ denote the collection of all topologies weakly generated by $\mathcal{I}$ or equivalently, by Theorem 4.3, the collection of $\mathcal{I}$-closed topologies. $T(\mathcal{I})$ ordered by $\subseteq$ is a lattice. In fact, let τ, ρ be $\mathcal{I}$-closed topologies, since $\tau \cap \rho$ is $\mathcal{I}$-closed, then it is the meet of τ and ρ in $T(\mathcal{I})$. The join of τ and ρ in $T(\mathcal{I})$ is defined as the $\mathcal{I}$-closure of the usual join of τ and ρ. In this section we will show that this lattice is isomorphic to a lattice of pre-orders over $\mathcal{I}$. These pre-orders will be denoted by $\sqsubseteq$ and their main properties are listed below. We write $x \sqsubseteq E$ instead of $\{x\} \sqsubseteq E$.

(P1) $\sqsubseteq$ is a pre-order (i.e. transitive and reflexive) relation over $\mathcal{I}$ extending the subset relation.

(P2) If $E \sqsubseteq \emptyset$, then $E = \emptyset$.

(P3) If $E \sqsubseteq F \cup G$, then there are sets E_1 and E_2 such that $E = E_1 \cup E_2$, $E_1 \sqsubseteq F$ and $E_2 \sqsubseteq G$.

(P4) If $E, F \in \mathcal{I}$ and $x \sqsubseteq F$ for all $x \in E$, then $E \sqsubseteq F$.

For $A \subseteq X$, define

$$D_{\sqsubseteq}(A) = \{x : \ x \sqsubseteq E \subseteq A \ \text{ for some } \ E \in \mathcal{I}\}. \tag{3}$$

(P5) For all $A \subseteq X$ and all $E, H \in \mathcal{I}$, if $E \subseteq D(A)$ and $H \sqsubseteq E$, then $H \subseteq D(A)$.

We will write D in place of $D_{\sqsubseteq}$ whenever there is no possible confusion about which pre-order $\sqsubseteq$ is being used. It is clear that P1, P2 and P3 implies that D is a Čech closure operator. We will refer to D as the *associate operator of* $\sqsubseteq$ and denote by $\tau(\sqsubseteq)$ the topology associated with the closure operator D^{∞}. Notice that (P1) together with (P4) imply that $D^2(E) = D(E)$ for all $E \in \mathcal{I}$. Thus $D(E)$ is the $\tau(\sqsubseteq)$-closure of E.

A binary relation $\sqsubseteq$ is called a *specialization relation over* $\mathcal{I}$ if it satisfies the conditions P1, P2 and P3. Now we show that the topologies weakly generated by an ideal are exactly the topologies of the form $\tau(\sqsubseteq)$ with $\sqsubseteq$ a specialization relation.

Given a topology τ we define a pre-order over $\mathcal{P}(X)$ as follows

$$A \sqsubseteq_\tau B \ \text{ iff } \ A \subseteq cl_\tau(B). \tag{4}$$

For a given ideal $\mathcal{I}$ we will denote by $\sqsubseteq_\tau^{\mathcal{I}}$ the restriction of $\sqsubseteq_\tau$ to $\mathcal{I}$. It is routine to check that $\sqsubseteq_\tau^{\mathcal{I}}$ is a specialization relation over $\mathcal{I}$ that also satisfies P4.

Theorem 5.1. *Let $\mathcal{I}$ and τ be respectively an ideal and a topology over X. The following are equivalent.*

(i) *τ is weakly generated by $\mathcal{I}$.*
(ii) *There is a specialization relation $\sqsubseteq$ over $\mathcal{I}$ such that $\tau = \tau(\sqsubseteq)$.*
(iii) *There is a unique specialization relation $\sqsubseteq$ over $\mathcal{I}$ satisfying* P4 *such that $\tau = \tau(\sqsubseteq)$.*

Proof: We show that (i) implies (iii). Suppose that τ is weakly generated by $\mathcal{I}$. To simplify the notation, let $\sqsubseteq$ be $\sqsubseteq_\tau^{\mathcal{I}}$ and D be the associated operator of $\sqsubseteq_\tau^{\mathcal{I}}$. We denote by C the operator $C_{\mathcal{I},\tau}$ defined by (1). We have already noticed that $\sqsubseteq$ is a specialization relation over $\mathcal{I}$ that satisfies P4. Now we show that $\tau = \tau(\sqsubseteq)$. It is clear that $D(E) = D^2(E) = cl_\tau(E)$ for all $E \in \mathcal{I}$. Thus $D(A) = \bigcup\{D(E) : E \subseteq A \text{ with } E \in \mathcal{I}\}$. Hence $D = C$ and we are done (by 4.2).

The property that makes $\sqsubseteq_\tau^{\mathcal{I}}$ unique is P4. In fact, suppose $\sqsubseteq^*$ is a specialization relation over $\mathcal{I}$ that also represents τ, i.e. such that $\tau = \tau(\sqsubseteq^*)$. Let D_* be the operator associated to $\sqsubseteq^*$. Let $F \sqsubseteq^* E$ in $\mathcal{I}$, then $F \subseteq D_*(E) \subseteq cl_\tau(E)$, thus $F \sqsubseteq_\tau^{\mathcal{I}} E$. It remains to see that $\sqsubseteq_\tau^{\mathcal{I}}$ is contained in $\sqsubseteq^*$. In fact, P4 (applied to $\sqsubseteq^*$) implies that $D_*^{\infty}(F) = D_*(F)$ for every $F \in \mathcal{I}$. Since $\sqsubseteq^*$ represents τ, then by definition $D_*^{\infty}(F) = cl_\tau(F)$. Thus if $E \subseteq cl_\tau(F)$, hence $E \sqsubseteq^* F$.

Now we show that (ii) implies (i). Let $\sqsubseteq$ be a relation over $\mathcal{I}$ satisfying P1, P2 and P3. As before, we will write τ, D and C instead of $\tau(\sqsubseteq)$, $D_{\sqsubseteq}$ and $C_{\mathcal{I},\tau}$. We need to show that $C^\infty = D^\infty$. Let $A \subseteq X$. Then by the definition of C and D, we have that $C^\infty(A) \subseteq cl_\tau(A) = D^\infty(A)$. For the other direction it suffices to show by induction that $D^\alpha(A) \subseteq C^\alpha(A)$ for each ordinal α. First observe that if $x \sqsubseteq E$ and $E \in \mathcal{I}$, then $x \in D(E)$. Thus we have

$$D(A) \subseteq \bigcup\{D(E) : E \subseteq A \ \text{ with } \ E \in \mathcal{I}\}$$
$$\subseteq \bigcup\{cl_\tau(E) : E \subseteq A \ \text{ with } \ E \in \mathcal{I}\} = C(A).$$

For the inductive step, first observe that $D^{\alpha+1}(A) \subseteq D(C^\alpha(A))$. Let $x \in D(C^\alpha(A))$, thus there is $E \subseteq C^\alpha(A)$ in $\mathcal{I}$ with $x \sqsubseteq E$. Then $x \in D(E) \subseteq cl_\tau(E)$ and thus $x \in C(C^\alpha(A))$. Hence $D^{\alpha+1}(A) \subseteq C^{\alpha+1}(A)$. The case of a limit ordinal is obvious. □

It is straightforward to verify that if τ is generated by $\mathcal{I}$, then $\sqsubseteq^{\mathcal{I}}_\tau$ satisfies P5. Thus we have the following.

Corollary 5.2. *Let $\mathcal{I}$ be an ideal over X. A topology τ is generated by $\mathcal{I}$ iff there is a specialization relation $\sqsubseteq$ over $\mathcal{I}$ satisfying* P5 *and such that $\tau = \tau(\sqsubseteq)$.*

Let $S(\mathcal{I})$ denote the collection of specialization pre-orders over $\mathcal{I}$ that satisfies P4. We order $S(\mathcal{I})$ by reverse inclusion.

Theorem 5.3. *The map $\tau \mapsto \sqsubseteq^{\mathcal{I}}_\tau$ is a lattice isomorphism from $T(\mathcal{I})$ onto $S(\mathcal{I})$.*

Proof: We claim that given $\sqsubseteq_1, \sqsubseteq_2$ in $S(\mathcal{I})$, then

$$\sqsubseteq_1 \subseteq \sqsubseteq_2 \quad \text{iff} \quad \tau(\sqsubseteq_2) \ \subseteq \ \tau(\sqsubseteq_1). \tag{5}$$

To simplify notation we will write τ_i instead of $\tau(\sqsubseteq_i)$ and cl_i for the corresponding closure operators. From Theorem 5.1 we know that $\sqsubseteq_i = \sqsubseteq^{\mathcal{I}}_{\tau_i}$. Let D_i be the operator associated to $\sqsubseteq_i$. Suppose first that $\sqsubseteq_1 \subseteq \sqsubseteq_2$. Then $D_1(A) \subseteq D_2(A)$ for all $A \subseteq X$. From this it follows easily that $cl_1(A) = D_1^\infty(A) \subseteq D_2^\infty(A) = cl_2(A)$ for all $A \subseteq X$ and thus $\tau_2 \subseteq \tau_1$. Now suppose that $\tau_2 \subseteq \tau_1$. Then it is obvious that $\sqsubseteq^{\mathcal{I}}_{\tau_1} \subseteq \sqsubseteq^{\mathcal{I}}_{\tau_2}$ and we are done.

The result now follows from Theorem 5.1 and (5). □

Example 5.4. The results of this section are well known for the particular case of $\mathcal{I} = \mathsf{Fin}$ but are usually presented in the following form ([11, II 1.8]). To each topology τ over X, the following binary relation is associated:

$$x \leq_\tau y \ \text{ if } \ x \in cl_\tau(\{y\}). \tag{6}$$

Then $\leq_\tau$ is transitive and reflexive, so it is a pre-order on X. Moreover, the map $\tau \mapsto \leq_\tau$ is a lattice isomorphism. Alexandroff topologies are completely characterized by $\leq_\tau$, namely, τ is Alexandroff iff

$$(7) \qquad cl_\tau(A) = \bigcup_{x\in A} cl_\tau(\{x\}) = \bigcup_{x\in A} \{y \in X : y \leq_\tau x\}$$

for every $A \subseteq X$. Moreover, given a pre-order $\leq$ over X, the topology generated by the sets $N_x = \{y \in X : \ x \leq y\}$ for $x \in X$ is an Alexandroff topology and its induced pre-order is precisely $\leq$. Notice that from (7), it is clear that any Alexandroff topology is generated by Fin. It is easy to verify that $\tau \sim_{\mathsf{Fin}} \rho$ iff $\leq_\tau = \leq_\rho$. The pre-order over Fin is the following: $K \sqsubseteq L$ if for all $x \in K$ there is $y \in L$ such that $x \leq_\tau y$.

Remark 5.5. We do not have internal definitions for the lattice operations on $S(\mathcal{I})$. However, given two relations $\sqsubseteq_i$, $i = 1, 2$, in $S(\mathcal{I})$, let $\sqsubseteq$ be the transitive closure of their union. Then $\sqsubseteq$ is a relation that satisfies P1, P2 and P3 and moreover it is routine to verify that its associated topology $\tau(\sqsubseteq)$ is the meet of $\tau(\sqsubseteq_1)$ and $\tau(\sqsubseteq_2)$. Therefore, if $\sqsubseteq$ satisfies P4 (and thus belongs to $S(\mathcal{I})$), then it is the meet of $\sqsubseteq_1$ and $\sqsubseteq_2$.

On the other hand, the natural operation for the join is to take the intersection of the preorders. It follows from (5) that the join of $\sqsubseteq_1$ and $\sqsubseteq_2$ is contained in $\sqsubseteq_1 \cap \sqsubseteq_2$ and it is routine to check that $\sqsubseteq_1 \cap \sqsubseteq_2$ satisfies P1 and P2. But it is not clear whether $\sqsubseteq_1 \cap \sqsubseteq_2$ satisfies P3.

The following example shows that the meet of two topologies generated by $\mathcal{I}$ is not necessarily generated by $\mathcal{I}$.

Example 5.6. Consider the ideal $\mathcal{J}$ on $\mathbb{N}\times\mathbb{N}$ generated by the sets $B_n = \{n\}\times\mathbb{N}$ for $n \in \mathbb{N}$, the vertical sections of $\mathbb{N} \times \mathbb{N}$. Let X be $\mathbb{N} \times \mathbb{N} \cup \{\infty\}$ and consider the topology τ on X where ∞ is the only non isolated point and its nbhd filter is the dual filter of $\mathcal{J}$. Let $\mathcal{I}$ be the ideal on X generated by those subsets $A \subseteq \mathbb{N} \times \mathbb{N}$ such that $A \cap B_n$ is finite for all $n \in \mathbb{N}$ together with the set $\{\infty\}$. It is easy to verify that τ is generated by $\mathcal{I}$. In particular, every horizontal line $\mathbb{N} \times \{n\}$ is a sequence τ-convergent to ∞.

Let $\prec$ be the strict partial order over X given by

$$(n, 0) \prec (0, n)$$

for all $0 < n \in \mathbb{N}$. Let ρ be the Alexandroff topology over X associated to $\preceq$. Then ρ is generated by $\mathcal{I}$ (in fact, by 5.4 it is generated by Fin).

We will show that $\eta = \tau \wedge \rho$ is not generated by $\mathcal{I}$. Notice first that $cl_\tau(B) \cup cl_\rho(B) \subseteq cl_\eta(B)$ for all $B \subseteq X$. Let A_0 be $\mathbb{N} \times \{0\}$. Since $cl_\rho(B_0) = B_0 \cup A_0$ and $cl_\tau(A_0) = A_0 \cup \{\infty\}$, then $\infty \in cl_\eta(B_0)$. However, if $E \in \mathcal{I}$, then $B_0 \cap E$ is finite and therefore η is not generated by $\mathcal{I}$.

Remark 5.7. We do not know if the join of two topologies generated by $\mathcal{I}$ is also generated by $\mathcal{I}$. However, this is true if one of them is Alexandroff. Let ρ be a topology generated by $\mathcal{I}$ and τ be an Alexandroff topology. Let η be the usual join of ρ and τ. We claim that η is generated by $\mathcal{I}$ and thus it is the join of τ and ρ in $T(\mathcal{I})$. Let $A \subseteq X$ and $x \in cl_\eta(A)$. Let N_x be the τ-minimal nbhd of x (see 5.4). Then $x \in cl_\eta(A \cap N_x)$ and hence $x \in cl_\rho(A \cap N_x)$. Since ρ is generated by $\mathcal{I}$, there is $E \subseteq N_x \cap A$ with $E \in \mathcal{I}$, such that $x \in cl_\rho(E)$. Since any η-open set containing x must contain a set of the form $N_x \cap W$ with $x \in W \in \rho$, we have $x \in cl_\eta(E)$.

On the other hand, the usual join of two $\mathcal{I}$-closed topologies is not necessarily $\mathcal{I}$-closed. Suppose τ is weakly generated by $\mathcal{I}$ but not generated by $\mathcal{I}$. By 4.3(i) τ is $\mathcal{I}$-closed. Let $A \subseteq X$ and $x_0 \in X$ be such that $x_0 \in cl_\tau(A)$ but there is no $E \subseteq A$ in $\mathcal{I}$ with $x_0 \in cl_\tau(E)$. Consider an order over X given by $x_0 \prec y$ for all $y \in A$. Let ρ be the Alexandroff topology associated to $\prec$. Since $\mathsf{Fin} \subseteq \mathcal{I}$, then ρ is generated by $\mathcal{I}$ and therefore $\mathcal{I}$-closed. Let η be the usual join of τ and ρ. Then x_0 is in $cl_\eta(A)$ but $x_0 \notin C^\infty_{\eta,\mathcal{I}}(A)$.

6. Convergence associated to ideals

In this section we introduce a notion of a convergence that is analogous to the sequential convergence. Recall that a sequential convergence on a set X is a collection $\mathbb{L} \subseteq X^{\mathbb{N}} \times X$. The standard axioms are the following ([10]):

($\mathcal{L}_0$) If $(S,x),(S,y) \in \mathbb{L}$, then $x = y$.

($\mathcal{L}_1$) $(S,x) \in \mathbb{L}$ for every constant sequence S such that $S(n) = x$ for all n.

($\mathcal{L}_2$) If $(S,x) \in \mathbb{L}$, then $(T,x) \in \mathbb{L}$ for every subsequence T of S.

($\mathcal{L}_3$) Let $S \in X^{\mathbb{N}}$ and $x \in X$. If for every subsequence T of S, there is a subsequence T' of T such that $(T',x) \in \mathbb{L}$, then $(S,x) \in \mathbb{L}$.

A sequential convergence is usually defined as a collection $\mathbb{L}$ satisfying $\mathcal{L}_i$ for $i \in \{0,1,2\}$. It is known that a convergence $\mathbb{L}$ satisfies $\mathcal{L}_i$ for $i \in \{1,2,3\}$ iff $\mathbb{L}$ is the collection $\mathbb{L}(\tau)$ of all τ-convergent sequences for some topology τ. To a given convergence $\mathbb{L}$, a Čech closure operator $C_{\mathbb{L}}$ is associated, defined by $C_{\mathbb{L}}(A) = \{x : \exists\,(S,x) \in \mathbb{L}$ with the range of S in $A\}$. The topology associated to $C_{\mathbb{L}}$ is denoted by $\tau(\mathbb{L})$ and it is the largest topology respect to which S converges to x for every $(S,x) \in \mathbb{L}$. A given sequential convergence $\mathbb{L}$ can be enlarged to another convergence $\mathbb{L}^*$ (called the Urysohn modification of $\mathbb{L}$) as follows: Put $(S,x) \in \mathbb{L}^*$ if for every subsequence T of S there is a subsequence T' of T such that $(T',x) \in \mathbb{L}$. It is known that $(S,x) \in \mathbb{L}^*$ iff S converges to x with respect to $\tau(\mathbb{L})$. Two sequential convergences $\mathbb{L}$ and $\mathbb{K}$ are said to be equivalent if $C_{\mathbb{L}} = C_{\mathbb{K}}$. Then $\mathbb{L}^*$ is the largest sequential convergence in the equivalence class of $\mathbb{L}$. Moreover, $\mathbb{L}^*$ is the unique sequential convergence equivalent to $\mathbb{L}$ that satisfies $\mathcal{L}_3$.

Consider the following sequential convergence associated to a T_2 topology τ (it suffices that a convergent sequence has a unique limit) and an ideal $\mathcal{I}$:

$$(S,x) \in \mathbb{L}(\tau,\mathcal{I}) \text{ if } S \text{ } \tau\text{-converges to } x \text{ and the range of } S \text{ is in } \mathcal{I}.$$

Proposition 6.1. *Suppose τ is T_2 and let $\bar{\tau}$ be the $\mathcal{I}$-closure of τ. Then the Urysohn modification of $\mathbb{L}(\tau,\mathcal{I})$ is $\mathbb{L}(\bar{\tau})$. That is to say $S \xrightarrow{\overline{\tau}} x$ iff every subsequence of S has a subsequence τ-convergent to x with range in $\mathcal{I}$.*

Proof: Suppose $S \xrightarrow{\overline{\tau}} x$. Let A be the range of S, we can assume that $x \notin A$. Since τ is T_2, $A \cup \{x\}$ is $\overline{\tau}$-closed. Therefore $x \in cl_{\overline{\tau}}(A) = C_{\mathcal{I},\tau}(A)$ and thus there is $E \subseteq A$ in $\mathcal{I}$ such that $x \in cl_\tau(E)$. Hence there is a subsequence T of S such that range of T is a subset of E and clearly T converges to x. For the other direction, notice that every sequence τ-convergent to x and with range in $\mathcal{I}$ is necessarily $\overline{\tau}$-convergent. □

6.1 $\mathcal{I}$-convergences

We introduce the notion of an $\mathcal{I}$-convergence. To each subset $\mathbb{A}$ of $\mathcal{I} \times X$ we associated an operator $C_\mathbb{A}$ as follows:

$$x \in C_\mathbb{A}(B) \text{ iff } (E,x) \in \mathbb{A} \text{ for some } E \subseteq B.$$

Consider the following axioms:

($\mathcal{A}_1$) $(\emptyset, x) \notin \mathbb{A}$ for all $x \in X$.

($\mathcal{A}_2$) $(E,x) \in \mathbb{A}$ for all $x \in E \in \mathcal{I}$.

($\mathcal{A}_3$) If $(E \cup F, x) \in \mathbb{A}$, then $(E,x) \in \mathbb{A}$ or $(F,x) \in \mathbb{A}$.

($\mathcal{A}_4$) $C^2_\mathbb{A}(E) = C_\mathbb{A}(E)$ for all $E \in \mathcal{I}$.

($\mathcal{A}_5$) If $E \subseteq F \in \mathcal{I}$ and $(E,x) \in \mathbb{A}$, then $(F,x) \in \mathbb{A}$.

It is obvious that $C_\mathbb{A}$ is monotone. If $\mathbb{A}$ satisfies $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, then $C_\mathbb{A}$ is a Čech closure operator. Axiom $\mathcal{A}_5$ is equivalent to the following: $C_\mathbb{A}(E) = \{x : (E,x) \in \mathbb{A}\}$ for all $E \in \mathcal{I}$.

We say that $\mathbb{A}$ is an *$\mathcal{I}$-convergence* if it satisfies $\mathcal{A}_i$ for $i \in \{1,2,3,4,5\}$ and $\mathbb{A}$ is a *convergence* when $\mathbb{A}$ is an $\mathcal{I}$-convergence for some ideal $\mathcal{I}$. If necessary, we will denote by $\mathcal{I}(\mathbb{A})$ the ideal $\mathcal{I}$ where the convergence $\mathbb{A}$ is defined.

Example 6.2. (a) Given a Čech closure operator C, define $\mathbb{A}_C$ by letting $(A,x) \in \mathbb{A}_C$ if $A \in \mathcal{I}$ and $x \in C(A)$. It is clear that $\mathbb{A}_C$ satisfies $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$.

(b) Let $\mathcal{I}$ be an ideal and τ a topology. Define $(E,x) \in \mathbb{A}$ if $E \in \mathcal{I}$ and $x \in cl_\tau(E)$. Then $\mathbb{A}$ satisfies $\mathcal{A}_i$ for $i \in \{1,2,3,4,5\}$. It is called the convergence associated to τ and $\mathcal{I}$. Notice that $C_\mathbb{A} = C_{\mathcal{I},\tau}$.

(c) Let $\mathbb{A}$ be a convergence and $\mathcal{J}$ an ideal. Define $\mathbb{B}$ by letting $(B,x)\in\mathbb{B}$ if $x\in C_{\mathbb{A}}(B)$ and $B\in\mathcal{J}$. Then $\mathbb{B}$ satisfies $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$. Notice that if $\mathcal{I}(\mathbb{A})\subseteq\mathcal{J}$, then $\mathbb{A}\subseteq\mathbb{B}$.

(d) Let $\sqsubseteq$ be a specialization relation over $\mathcal{I}$. Define $(E,x)\in\mathbb{A}$ iff $x\sqsubseteq E$. Then $\mathbb{A}$ satisfies $\mathcal{A}_i$ for $i\in\{1,2,3,5\}$. If $\sqsubseteq$ satisfies P4, then $\mathbb{A}$ satisfies $\mathcal{A}_4$. Conversely, given an $\mathcal{I}$-convergence $\mathbb{A}$, define $E\sqsubseteq F$ if $(F,x)\in\mathbb{A}$ for all $x\in E$. Then $\sqsubseteq$ is a specialization relation which also satisfies P4.

Let $\tau(\mathbb{A})$ denote the topology given by $C^{\infty}_{\mathbb{A}}$. It is clear that

$$A \text{ is } \tau(\mathbb{A})\text{-closed iff } x\in A \text{ for all } (E,x)\in\mathbb{A} \text{ with } E\subseteq A.$$

From Example 6.2(d) above and Theorem 5.1 we easily get the following

Theorem 6.3. *A topology τ is weakly generated by $\mathcal{I}$ iff there is an $\mathcal{I}$-convergence $\mathbb{A}$ such that $\tau=\tau(\mathbb{A})$.*

Let $\mathbb{A}$ and $\mathbb{B}$ be convergences (we are not assuming $\mathcal{I}(\mathbb{A})=\mathcal{I}(\mathbb{B})$). We will say that $\mathbb{A}$ is equivalent to $\mathbb{B}$, denoted $\mathbb{A}\sim\mathbb{B}$, if $C_{\mathbb{A}}=C_{\mathbb{B}}$. It is straightforward to show the following

Lemma 6.4. *Let $\mathbb{A}$ and $\mathbb{B}$ be two convergences. Then $\mathbb{A}\sim\mathbb{B}$ iff $C_{\mathbb{A}}(E)=C_{\mathbb{B}}(E)$ for all $E\in\mathcal{I}(\mathbb{A})\cup\mathcal{I}(\mathbb{B})$.*

To each convergence $\mathbb{A}$ we associate the following collection of subsets of X:

$$\mathcal{I}(\mathbb{A}^*)=\{A\subseteq X:\ \forall B\subseteq A\,[\,C_{\mathbb{A}}(B)=C^2_{\mathbb{A}}(B)]\}. \tag{8}$$

It is easy to verify that $\mathcal{I}(\mathbb{A}^*)$ is an ideal. Now define $\mathbb{A}^*$ by

$$(A,x)\in\mathbb{A}^* \quad\text{iff}\quad A\in\mathcal{I}(\mathbb{A}^*) \quad\text{and}\quad x\in C_{\mathbb{A}}(A).$$

The property that characterizes $\mathbb{A}^*$ is the following:

($\mathcal{A}_6$) Let $A\subseteq X$ be such that $\forall B\subseteq A\,[C_{\mathbb{A}}(B)=C^2_{\mathbb{A}}(B)]$. Then

$$\forall x\,[x\in C_{\mathbb{A}}(A)\Longrightarrow(A,x)\in\mathbb{A}].$$

Theorem 6.5. *Let $\mathbb{A}$ be a convergence. Then*

(i) *$\mathbb{A}^*$ is a convergence,*
(ii) *$\mathbb{A}\sim\mathbb{A}^*$,*
(iii) *if $\mathbb{B}\sim\mathbb{A}$, then $\mathbb{B}\subseteq\mathbb{A}^*$,*
(iv) *$\mathbb{A}^*$ is the unique convergence in the equivalent class of $\mathbb{A}$ which satisfies $\mathcal{A}_6$.*

PROOF: (i) Since $\mathbb{A}$ satisfies $\mathcal{A}_4$, $\mathcal{I}(\mathbb{A}) \subseteq \mathcal{I}(\mathbb{A}^*)$. From Example 6.2(c) we know that $\mathbb{A}^*$ satisfies $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_5$ and also $\mathbb{A} \subseteq \mathbb{A}^*$. Let $A \in \mathcal{I}(\mathbb{A}^*)$. Then from the definition of A^* we get that $x \in C_{\mathbb{A}^*}(A)$ iff $(A, x) \in \mathbb{A}^*$ iff $x \in C_{\mathbb{A}}(A)$. From this it follows that $\mathbb{A}^*$ satisfies $\mathcal{A}_4$.

(ii) From Lemma 6.4 we conclude that $\mathbb{A} \sim \mathbb{A}^*$.

(iii) Suppose $\mathbb{B} \sim \mathbb{A}$. Since $\mathbb{B}$ satisfies $\mathcal{A}_4$, $\mathcal{I}(\mathbb{B}) \subseteq \mathcal{I}(\mathbb{A}^*)$ and thus $\mathbb{B} \subseteq \mathbb{A}^*$.

(iv) It is straightforward. □

Example 6.6. Given a sequential convergence $\mathbb{L}$ we associate to it the ideal $\mathcal{I}(\mathbb{L})$ generated by the range of all sequences in $\mathbb{L}$. We denote a sequence and its range by the same symbol. Put $(E, x) \in \mathbb{A}(\mathbb{L})$ if $E \in \mathcal{I}(\mathbb{L})$ and $x \in C_{\mathbb{L}}(E)$. To simplify notation we write $\mathbb{A}$ instead of $\mathbb{A}(\mathbb{L})$. It is straightforward to verify that $\mathbb{A}$ is an $\mathcal{I}(\mathbb{L})$-convergence. It is clear that $C_{\mathbb{L}}$ is equal to $C_{\mathbb{A}}$ and hence $\tau(\mathbb{L}) = \tau(\mathbb{A})$. Thus $\mathcal{I}(\mathbb{A}^*)$ is an extension of the Urysohn modification $\mathbb{L}^*$ in the sense that $\mathcal{I}(\mathbb{L}^*) \subseteq \mathcal{I}(\mathbb{A}^*)$. But there might be more sets in $\mathcal{I}(\mathbb{A}^*)$; for example, every $\tau(\mathbb{L})$-closed discrete set belongs to it.

The proof of the following proposition is left to the reader.

Proposition 6.7. *Let $\mathcal{I}$ be an ideal over a set X. The following are equivalent.*

1. *There is an $\mathcal{I}$-convergence that satisfies $\mathcal{A}_6$.*
2. *There is an $\mathcal{I}$-convergence $\mathbb{A}$ such that $\mathcal{I}(\mathbb{A}^*) = \mathcal{I}$.*
3. *There is a topology τ such that*

$$\mathcal{I} = \{A \subseteq X : \forall\, B \subseteq A\ C_{\mathcal{I},\tau}(B) = cl_\tau(B)\}.$$

□

We will finish this section with a few examples concerning axiom $\mathcal{A}_6$.

Example 6.8. To simplify the presentation we will say that an ideal $\mathcal{I}$ is Urysohn if there is an $\mathcal{I}$-convergence that satisfies $\mathcal{A}_6$.

1. Fin is not Urysohn when X is infinite. Let $\mathbb{A}$ be any Fin-convergence. We will show that $\mathcal{I}(\mathbb{A}^*) = \mathcal{P}(X)$. In fact, let $A \subseteq X$ and $x \in C^2_{\mathbb{A}}(A)$. Then there is a finite set $E \subseteq C_{\mathbb{A}}(A)$ such that $(E, x) \in \mathbb{A}$. For each $y \in E$, there is $E_y \subseteq A$ finite, such that $(E_y, y) \in \mathbb{A}$. Let F be the union of the E_y's. Then by $\mathcal{A}_5$, $(F, y) \in \mathbb{A}$ for all $y \in E$. Thus $E \subseteq C_{\mathbb{A}}(F)$ and hence $x \in C^2_{\mathbb{A}}(F)$. Then apply $\mathcal{A}_4$ and conclude that $C^2_{\mathbb{A}}(A) = C_{\mathbb{A}}(A)$. Since this holds for every $A \subseteq X$, we are done.
2. Let κ be a cardinal and X be a set with cardinality larger than κ. It can be shown, as in the previous example, that the ideal of all $A \subseteq X$ with $|A| \leq \kappa$ is not Urysohn.
3. Let $X = \aleph_\omega$ and $\mathcal{I}$ be the ideal of all $A \subseteq X$ such that $|A| < \aleph_\omega$. We will show that $\mathcal{I}$ is Urysohn. Let $\mathbb{A}$ be defined as follows: (i) $(E, x) \in \mathbb{A}$ for all

$x \in E \in \mathcal{I}$, (ii) $(E, \aleph_n) \in \mathbb{A}$ if $\aleph_n \leq |E|$ and $E \in \mathcal{I}$ and (iii) $(E, 0) \in \mathbb{A}$ if $\aleph_n \in E$ for infinitely many n's. We leave to the reader to verify that this is indeed an $\mathcal{I}$-convergence that satisfies $\mathcal{A}_6$.

4. For a given an ideal $\mathcal{I}$ and a topology τ let $\hat{\mathcal{I}} = \{A \subseteq X : \forall\, B \subseteq A\ C_{\mathcal{I},\tau}(B) = cl_\tau(B)\}$. It is routine to verify that $\hat{\mathcal{I}}$ is an ideal and $\hat{\hat{\mathcal{I}}} = \hat{\mathcal{I}}$. From 6.7, $\hat{\mathcal{I}}$ is Urysohn. If τ is weakly generated but not generated by $\mathcal{I}$, then $\hat{\mathcal{I}}$ is not equal to $\mathcal{P}(X)$.

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