On rings close to regular and p-injectivity

Roger Yue Chi Ming

Dedicated to Professor Robert Wisbauer on his 65th birthday.

Abstract. The following results are proved for a ring A: (1) If A is a fully right idempotent ring having a classical left quotient ring Q which is right quasi-duo, then Q is a strongly regular ring; (2) A has a classical left quotient ring Q which is a finite direct sum of division rings iff A is a left TC-ring having a reduced maximal right ideal and satisfying the maximum condition on left annihilators; (3) Let A have the following properties: (a) each maximal left ideal of A is either a two-sided ideal of A or an injective left A-module; (b) for every maximal left ideal M of A which is a two-sided ideal, A/M_A is flat. Then, A is either strongly regular or left self-injective regular with non-zero socle; (4) A is strongly regular iff A is a semi-prime left or right quasi-duo ring such that for every essential left ideal L of A which is a two-sided ideal, A/L_A is flat; (5) A prime ring containing a reduced minimal left ideal must be a division ring; (6) A commutative ring is quasi-Frobenius iff it is a YJ-injective ring with maximum condition on annihilators.

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Introduction

Strongly regular rings, introduced in [1], have drawn the attention of various authors. K.-R. Goodearl's classic [11] has motivated numerous papers in the area of von Neumann regular rings. This paper is mainly motivated by [20]. Following [9], we write "A is VNR" if A is a von Neumann regular ring. It is well-known that A is VNR iff every left (right) A-module is flat (M. Harada (1956); M. Auslander (1957)). This result remains true if we replace "flat" by "p-injective" ([30]) or "YJ-injective" ([38]).

Recall that a right A-module M is (a) p-injective if, for every principal right ideal P of A, any right A-homomorphism of P into M extends to one of A into M; (b) YJ-injective if, for every $o \neq b \in A$, there exists a positive integer n such that $b^n \neq o$ and any right A-homomorphism of b^nA into M extends to one of A into M ([22], [34], [38]). As an analogy to the study of flat modules over rings which are not VNR, many authors have considered p-injectivity and YJ-injectivity over rings not necessarily VNR ([3], [4], [9], [12], [14], [16], [17], [21]–[24]). A theorem of I. Kaplansky asserts that a commutative ring A is VNR iff A is a V-ring (i.e. every simple A-module is injective). (This remains true if "injective" is replaced by "YJ-injective".)

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Y, Z denote respectively the Jacobson radical, the right singular ideal and the left singular ideal of A. An ideal of A will always mean a two-sided ideal of A. A is called left (resp. right) quasi-duo if every maximal left (resp. right) ideal of A is an ideal of A (S.H. Brown (1973)). A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element. We say that A is a left p-injective (resp. YJ-injective) ring if ${}_AA$ is p-injective (resp. YJ-injective). P-injectivity and YJ-injectivity on the right side of A are similarly defined. We know that if A is a right YJ-injective ring, then $Y = J$ [34, Proposition 1] (this extends the well-known result for right self-injective rings and is at the origin of our notation). Also, A is strongly regular iff A is a reduced right p-injective ring iff A is a reduced right YJ-injective ring (cf. [27] and [34]). As usual, A is called fully idempotent (resp. (a) fully right idempotent; (b) fully left idempotent) if every ideal (resp. (a) right ideal; (b) left ideal) of A is idempotent.

Recall that

(a) A is VNR if, for every $a \in A$, $a \in aAa$;
(b) A is strongly regular if, for every $a \in A$, $a \in a^2A$;
(c) Given a left A-module M, any subset S of elements of M, write $l_A(S) = \{a \in A \setminus aS = 0\}$ and $r_A(S) = \{a \in A \setminus Sa = 0\}$ respectively for the left and right annihilators of S in A. In case of no possible confusion, simply write $l(S)$ and $r(S)$. A left ideal I of A is called a left annihilator if $I = l(U)$ for some subset U of A;
(d) Given a left submodule N of a left A-module M,
  (i) N is essential in M if $N \cap Q \neq o$ for any non-zero submodule Q of M;
  (ii) N is a complement submodule of M if N has no proper essential extension in M;
(e) For any left A-module M, $Z(M) = \{y \in M \setminus l_A(y)$ is essential in ${}_AA\}$ is the singular submodule of M. Right singular submodules are similarly defined. ${}_AM$ is called singular (resp. non-singular) if $Z(M) = M$ (resp. $Z(M) = 0$). Thus in our notations, $Z = Z({}_AA)$ and $Y = Z(A_A)$. A is called a left (resp. right) non-singular ring if $Z = 0$ (resp. $Y = 0$);
(f) Quasi-Frobenius rings (introduced by T. Nakayama (1939)) are left and right Artinian, left and right self-injective rings. For results on quasi-Frobenius rings, consult [8], [9], [19], [21], [23].

As usual, a ring Q is a classical left quotient ring of A if (i) $A \subseteq Q$; (ii) every non-zero-divisor of A is invertible in Q; (iii) every element of Q is of the form $q = c^{-1}a$, $c, a \in A$, c being a non-zero-divisor.

Proposition 1. *Let A be a fully right idempotent ring having a classical left quotient ring Q which is right quasi-duo. Then Q is a strongly regular ring.*

Proof: Suppose there exists $q \in Q$ such that $qQ + r_Q(q) \neq Q$. If $q = b^{-1}a$,

$b, a \in A$, b being a non-zero-divisor, then $r_A(a)Q \subseteq r_Q(q)$ which implies that $qQ + r_A(a)Q \neq Q$. Let M be a maximal right ideal of Q containing $qQ + r_A(a)Q$. Since A is fully right idempotent, then $a = ad$, $d \in AaA$ which implies that $1 - d \in r_A(a) \subseteq M$. Now, $d \in AaA = Abb^{-1}aA \subseteq AqQ \subseteq M$ (in as much as M is an ideal of Q) which yields $1 \in M$, contradicting $M \neq Q$. This proves that for each $s \in Q$, $sQ + r_Q(s) = Q$. Then $s = s^2p$ for some $p \in Q$ which implies that Q is strongly regular. □

Recall that A is a left TC-ring if every non-zero complement left ideal of A contains a non-zero ideal of A ([36]). We know that

(1) if A contains a reduced maximal right ideal, then A is a reduced ring ([36, Lemma 2]);
(2) if A is semi-prime left TC, then any complement left ideal of A is a left annihilator ([36, Proposition 1]);
(3) if A has a classical left quotient ring Q, then A is reduced iff Q is reduced ([35, Proposition 1.5]).

Combining [10, Theorem 2.38] with the above three results, we get the following special case of Goldie's theorem:

Theorem 2. *The following conditions are equivalent:*

(1) *A has a classical left quotient ring Q which is a finite direct sum of division rings;*
(2) *A is a semi-prime, left* TC*-ring satisfying the maximum condition on left annihilators;*
(3) *A is a left* TC*-ring having a reduced maximal right ideal and satisfying the maximum condition on left annihilators.*

P. Menal–P. Vamos (1989) proved that any ring can be embedded in an FP-injective ring (this is not true if "FP-injective" is replaced by "self-injective") [9, p. 308]. Recall that a left A-module M is FP-injective if, for every finitely presented left A-module F, $\mathrm{Ext}^1_A(F, M) = 0$; ${}_AM$ is p-injective if $\mathrm{Ext}^1_A(A/A_y, M) = o$ for every $y \in A$. Left FP-injective rings are consequently left p-injective. Therefore, any ring may be embedded in a p-injective or YJ-injective ring. This enhances the attention paid to p-injective or YJ-injective rings (cf. [12], [16], [17], [9, p. 125]). We know that the following result follows from [9, Theorem 6.4]: If A is commutative, then every factor ring of A is FP-injective iff every factor ring of A is p-injective. We proceed to give a sufficient condition for a ring to be left p-injective.

Proposition 3. *Let A be a left* TC*-ring containing a finitely generated p-injective maximal left ideal M. Then A is left p-injective.*

Proof: Let $o \neq b \in M$. If $g : Ab \to M$ is the natural injection, since ${}_AM$ is p-injective, there exists $y \in M$ such that $b = g(b) = by \in bM$. This implies that ${}_AA/M$ is flat ([5, p. 458]). Since ${}_AM$ is finitely generated, then A/M is a finitely related flat left A-module which implies that ${}_AA/M$ is projective ([5, p. 459]). Therefore ${}_AA = {}_AM \oplus {}_AU$, where U is a minimal left ideal of A. Let $M = Ae$, $e = e^2 \in A$, $U = Au$, $u = 1 - e$. Since A is left TC, U contains a non-zero ideal of A and hence U must be an ideal of A. First suppose that M is not an ideal of A. Then $MU \neq o$ (otherwise, $MU = o$, $U = Au$, yields $MAu = Au = o$, contradicting $u \neq o$!). Let $Mv \neq o$ for some $v \in U$. If $p : M \to Av$ is the map defined by $p(m) = mv$ for all $m \in M$, then p is an epimorphism since $Mv = U = Av$ and therefore ${}_AM/\ker p \approx {}_AU$ and since ${}_AU$ is projective, ${}_AM \approx {}_A\ker p \oplus {}_AM/\ker p$ which implies that ${}_AU$ is p-injective. Therefore $A = M \oplus U$ is a left p-injective ring. Now, suppose that M is an ideal of A. Then $eA \subseteq Ae$ implies that $eA(1-e) \subseteq Ae(1-e) = o$ while $uA \subseteq U = Au$ implies that $uA(1-u) \subseteq Au(1-u) = o$. Therefore $ea = eae$ and $ua = uau$ for each $a \in A$, which yields $ua = (1-e)au = au - eau = au$ (because $eau = eaeu = o$). Therefore $A_A = M_A \oplus U_A$ and A/M_A is projective. By [32, Lemma 1], ${}_AA/M$ is injective. Finally, ${}_AA = {}_AM \oplus {}_AU$ is p-injective which completes the proof. □

If A contains a reduced finitely generated left ideal I which is YJ-injective, then ${}_AI$ is a direct summand of ${}_AA$. If I is also a maximal left ideal of A, then A is a reduced ring ([36, Lemma 2]). The next remark then holds.

Remark 1. A is strongly regular with non-zero socle iff A contains a reduced finitely generated p-injective maximal left ideal iff A contains a reduced finitely generated YJ-injective maximal left ideal.

The proof of Proposition 3 yields

Theorem 4. *If A is left* TC*-ring containing an injective maximal left ideal, then A is a left self-injective ring.*

Rings containing an injective maximal left ideal need not be self-injective as shown by the following example, which will also motivate the next two propositions.

Example. If A denotes the 2×2 upper triangular matrix ring over a field, then every simple one-sided module is either injective or projective and A contains an injective maximal left ideal. Every maximal one-sided ideal of A is either injective or an ideal of A.

This example shows that a ring whose essential left ideals are idempotent two-sided ideals needs not be VNR (indeed not even semi-prime or left YJ-injective). But we know that if every essential left ideal of A is an ideal of A and A is a fully idempotent ring, then A must be VNR (cf. [37, Theorem 6]). We now prove a nice result which guarantees the von Neumann regularity of certain rings.

Proposition 5. *Let A be a ring such that every maximal left ideal is either an ideal of A or an injective left A-module. Assume that, for each maximal left ideal*

M of A which is an ideal of A, A/M_A is flat. Then A is either strongly regular or left self-injective regular with non-zero socle.

Proof: First suppose that every maximal left ideal of A is an ideal of A. Let $o \neq b \in A$. If $Ab + l(b) \neq A$, let M be a maximal left ideal of A containing $Ab + l(b)$. Then A/M_A is flat by hypothesis. Since $b \in M$, then $b = db$ for some $d \in M$ ([5, p. 458]). Then $1 - d \in l(b) \subseteq M$ implies that $1 \in M$, contradicting $M \neq A$. Therefore $Ab + l(b) = A$ for all $b \in A$. Then, $1 = ub + t$, $u \in A$, $t \in l(b)$ which yields $b = ub^2$. This proves that A strongly regular. Now suppose that there exists a maximal left ideal N of A which is not an ideal of A. Then, by hypothesis, ${}_AN$ is injective which implies that ${}_AA = {}_AN \oplus {}_AU$, where U is a minimal left ideal of A. The proof of Proposition 3 then shows that A is left self-injective. It remains to prove that the left singular ideal Z of A is zero. Suppose the contrary: there exists $o \neq z \in Z$ such that $z^2 = o$ ([33, Lemma 7]). Let K be a maximal left ideal of A containing $l(z)$. Then ${}_AK$ is not injective (because $l(z)$ is an essential left ideal of A) which implies that K must be an ideal of A. Then A/K_A is flat by hypothesis. Now $z \in l(z) \subseteq K$ implies that $z = wz$ for some $w \in K$ ([5, p. 458]). Therefore $1 - w \in l(z) \subseteq K$ implies that $1 \in K$, a contradiction! This proves that $Z = o$ and since A is left self-injective, then A is VNR with non-zero socle. □

Recall that A is biregular if, for every $a \in A$, AaA is generated by a central idempotent. Biregular rings generalize strongly regular rings and simple rings. We give two new characteristic properties of biregular rings.

Proposition 6. *The following conditions are equivalent:*

(1) *A is biregular;*
(2) *for any $a, b \in A$ such that $T = AaA + AbA \neq A$, there exists a non-zero element $c \in A$ such that T is the left and right annihilator of AcA and $T \cap AcA = o$;*
(3) *for any finite number of elements $b_1, \dots, b_n \in A$, $\sum_{i=1}^{n} Ab_iA = l(u) = r(u)$ for some element $u \in A$ and $Z \cap Y \cap J = o$.*

Proof: Assume (1). Let $a, b \in A$ such that $T = AaA + AbA \neq A$. If either $a = b$ or one of a, b is zero, then T is generated by a central non-trivial idempotent and T is therefore the left and right annihilator of AuA, where u is a non-trivial idempotent. In that case, (1) implies (2). Now assume that $a \neq o$, $b \neq o$, $a \neq b$. Then, $AaA = Ae$, where e is a non-trivial central idempotent. Then $T = Ae + AbA = Ae + Ab(1-e)A = Ae + As$, where $As = Ab(1-e)A$, s being a central idempotent. If $w = (1-e)s$, then $ws = w$, $w^2 = w$, $Aw = A(1-e)s \subseteq As = As^2 = Ab(1-e)As = Ab(1-e)sA \subseteq A(1-e)s = Aw$ which yields $Aw = As$. Since $e(e+w) = e$, $w(e+w) = w$, then $Ae \subseteq A(e+w)$ and $Aw \subseteq A(e+w)$ imply that $Ae + Aw \subseteq A(e+w)$, whence $A(e+w) = Ae + Aw$. Then $T = Ae + Aw = A(e+w) = A(e+w)A$, where $e + w$ is a non-trivial central idempotent and hence T is the left and right annihilator of AcA, where $c = 1 - e - w$ is also a non-trivial central idempotent and $T \cap AcA = o$. Thus (1) implies (2). Similarly, (1) implies (3).

Assume (2). Let $o \neq a \in A$ such that $AaA \neq A$. By hypothesis, there exists a non-zero element c of A such that $AaA = l(AcA)$ and $AaA \cap AcA = o$. Now suppose that $I = AaA + AcA \neq A$. By hypothesis, $I = l(AdA) = r(AdA)$, where d is a non-zero element of A and $I \cap AdA = o$. Therefore $AcA \cdot AdA = o$ which implies that $AdA \subseteq r(AcA) = AaA \subseteq I$, whence $AdA = AdA \cap I = o$, a contradiction! This proves that $AaA + AcA = A$ and since $AaA \cap AcA = o$, then $A = AaA \oplus AcA$ which shows that A must be semi-prime and AaA is therefore generated by a central idempotent, proving that (2) implies (1).

Assume (3). Suppose there exists a non-zero ideal T of A such that $T^2 = o$. If $o \neq t \in T$, then $l(AtA)$ is an essential right ideal of A and $r(AtA)$ is an essential left ideal of A. By hypothesis, $AtA = l(u) = r(u)$, $u \in A$. Then $AtA = l(AuA) = r(AuA)$ and $r(AtA) = r(l(AuA)) = AuA$ which implies that AuA is an essential left ideal of A. Similarly, AuA is an essential right ideal of A. Now $AuA = l(s) = r(s)$, $s \in A$, which implies that $AuA = l(AsA) = r(AsA)$, whence $AsA \subseteq Z \cap Y$. Suppose that $AsA \neq o$. Since AuA is essential in ${}_AA$, then $N = AsA \cap AuA$ is a non-zero left ideal of A and $N^2 \subseteq AuAsA = o$ which implies that $N \subseteq J$, the Jacobson radical of A. We get $N \subseteq Z \cap Y \cap J = o$, a contradiction! We have proved that A is a semi-prime ring. Now for any $a \in A$, $AaA = l(AvA) = r(AvA)$, $v \in A$. Since A is semi-prime, $r(AaA) = l(AaA) = l(r(AvA)) = AvA$. Now $AaA + AvA = AaA + r(AaA)$ is an essential left ideal of A and $AaA + r(AaA) = l(w) = r(w)$, $w \in A$. Since $AaAw = o$ and $l(AaA)w = r(AaA)w = o$, $w \in r(l(AaA)) = AaA$. Therefore $(AwA)^2 \subseteq AaA \cdot AwA = o$ and since A is semi-prime, $w = o$. Therefore $AaA + r(AaA) = A$ and since $AaA \cap r(AaA) = o$ (A being semi-prime), then $A = AaA \oplus r(AaA)$ which implies that AaA is generated by a central idempotent. Thus (3) implies (1). $\square$

Lemma 7. *The following conditions are equivalent:*

(1) *A is fully left idempotent;*
(2) *A is a semi-prime ring such that for every essential left ideal L of A which is an ideal of A, A/L_A is flat.*

Proof: Assume (1). Then A is obviously semi-prime. Let L be an essential left ideal of A which is an ideal of A. For any $o \neq t \in L$, $t \in (At)^2$ implies that $t = dt$ for some $d \in AtA \subseteq L$. Therefore $t \in Lt$ for every $t \in L$ which implies that A/L_A is flat ([5, p. 458]). Thus (1) implies (2).

Assume (2). For any $a \in A$, set $I = AaA + l(AaA)$. There exists a complement left ideal K of A such that $L = I \oplus K$ is an essential left ideal of A. Now $AaAK \subseteq K \cap AaA \subseteq K \cap I = o$ implies that $K \subseteq r(AaA)$. Since A is semi-prime, $r(AaA) = l(AaA)$. Then $K \subseteq I$ which yields $K = K \cap I = o$, whence $I = L$ is an essential left ideal of A which, by hypothesis, yields A/I_A is flat. Since $a \in I$, then $a = ua$ for some $u \in I$ ([5, p. 458]). If $u = w + c$, $w \in AaA$, $c \in l(AaA)$, then $a = wa + ca = wa \in (Aa)^2$, which proves that A is fully left idempotent and hence (2) implies (1). □

It is well-known that semi-prime, $P.I.$-rings which are left (and right) p-injective need not be VNR [4, p. 853] (cf. also [11]).

Proposition 8. *If A is a semi-prime $P.I.$-ring such that for every essential left ideal L of A which is an ideal of A, A/L_A is flat, then A is a VNR left and right V-ring.*

(Apply [2, Theorem 1 and Corollary].)

We now turn to characterizations of strongly regular rings.

A is called a ZI-ring (zero insertive) if, for $a, b \in A$, $ab = o$ implies $aAb = o$. Reduced rings are evidently ZI-rings. The next lemma, proved in [25, Corollary 2.4] is explicit in the proof of [34, Lemma 4.1] (cf. also the proof of "(2) implies (3)" in [31, Theorem 2.1]).

Lemma 9. *If A is a left quasi-duo ring with zero Jacobson radical, then A is a reduced ring.*

Theorem 10. *The following conditions are equivalent:*

(1) *A is strongly regular;*
(2) *A is a* ZI*-ring whose simple left modules are flat;*
(3) *A is a semi-prime left or right quasi-duo ring such that for every essential left ideal L of A which is an ideal of A, A/L_A is flat.*

Proof: (1) implies (2) and (3) evidently.

Assume (2). For any $b \in A$, $r(b)$ is an ideal of A. Suppose that $Ab + r(b) \neq A$. Let M be a maximal left ideal of A containing $Ab + r(b)$. Then ${}_AA/M$ is flat and since $b \in M$, $b = bc$ for some $c \in M$. Now $1 - c \in r(b) \subseteq M$ which implies that $1 \in M$, contradicting $M \neq A$. This proves that $Ab + r(b) = A$ and $b = bub$ for some $u \in A$, which yields A VNR. Since A is ZI, then every idempotent is central in A. A is therefore strongly regular and (2) implies (1).

(3) implies (1) by [3, Theorem 3.1], Lemma 7 and Lemma 9. □

We here give a characterization of division rings.

Proposition 11. *The following conditions are equivalent:*

(1) *A is a division ring;*
(2) *there exists a maximal left ideal of A which contains no non-zero ideal of A and there exists a reduced minimal left ideal of A;*
(3) *A is a prime ring containing a reduced minimal left ideal of A.*

Proof: It is clear that (1) implies (2).

Assume (2). Let M be a maximal left ideal of A which contains no non-zero ideal of A. Then ${}_AA/M$ is simple, faithful which means that A is a primitive ring and hence is a prime ring. Therefore (2) implies (3).

Assume (3). Let U be a reduced minimal left ideal of A. Since A is prime, $U = Ae$, $e = e^2 \in A$. Therefore U is a non-zero reduced left ideal which is a left annihilator. By [30, Proposition 6], A is an integral domain. Then $e(1-e) = o$ implies that $e = 1$. Therefore $U = A$ and every non-zero element of A is invertible in A. Thus (3) implies (1).

(Condition (3) improves [20, Remark 1.20(5)].) □

A remark on simple domains follows. [30, Proposition 6] implies

Remark 2. If A is a prime reduced ring whose simple left modules are either YJ-injective or flat, then A is a simple domain.

(Such rings need not be VNR.)

Theorem 12. *The following conditions are equivalent:*

(1) *A is quasi-Frobenius;*
(2) *A is a left and right* YJ*-injective ring satisfying the maximum condition on right annihilators.*

Proof: (1) implies (2) evidently.

Assume (2). Let U be a minimal left ideal of A. If ${}_AU$ is a direct summand of ${}_AA$, then it is clear that U is a left annihilator. Now, suppose that $U^2 = o$. Let $U = Au$, $u \in A$. Then $u^2 = o$ and since A is right YJ-injective, any right A-homomorphism of uA into A extends to an endomorphism of ${}_AA$. For any $v \in l(r(Au))$, since $r(u) = r(l(r(u))) \subseteq r(v)$, define a right A-homomorphism $f : uA \to A$ by $f(ua) = va$ for all $a \in A$. Since A is right YJ-injective, $v = f(u) = yu$ for some $y \in A$ which proves that $Au = l(r(Au))$. In any case, $U = Au$ is a left annihilator. Similarly, every minimal right ideal of A is a right annihilator. Now a right YJ-injective ring with maximum condition on right annihilators is left Artinian by [6, Theorem 3.7]. Since A is a left Artinian ring whose minimal one-sided ideals are annihilators, then A is quasi-Frobenius ([19, Proposition 1]). Therefore (2) implies (1). □

The next corollary answers positively [37, Question 4] and extends [37, Theorem 11].

Corollary 13. *A commutative ring is quasi-Frobenius iff it is a* YJ*-injective ring with maximum condition on annihilators.*

Recall that strongly regular rings are right (and left) V-rings while right V-rings and right perfect rings (in particular, left Artinian rings) are right max rings (in the sense that all non-zero right modules contain a maximal submodule) [20]. It may also be noted that if A is VNR, then (a) every non-zero right ideal of A

contains a maximal right subideal and (b) every non-zero finitely generated right ideal of A and its maximal right subideal are intersections of maximal right ideals of A (cf. [28]).

In most of our papers, we have considered singular modules (in particular, singular ideals), which play an important role in ring theory. For an exhaustive study of singular modules and ideals, consult K. Goodearl's classic [10]. If every singular left A-module is injective (i.e. A is left SI), then A is left hereditary (K. Goodearl) but not necessarily VNR [9, p. 92]. If, in addition, every injective left A-module is flat, then A must be VNR [28, Theorem 5]. But such rings need not be semi-simple Artinian (even if A is commutative). Consequently, left SI-rings need not be left Noetherian. In this direction, recall that if A is a left non-singular ring, then for every injective left A-module M, the singular submodule $Z(M)$ is injective ([26]) which answers negatively a question raised by F. Sandomierski [18] (cf. also Abraham Zaks' comment in MR 40 (1970) #5664 and [8, Theorem 19.46A]). Singular submodules may also be used to give partial answers to Matlis' problem [15] (which is still open) on complete decomposability: indeed, given a completely decomposable left A-module M, (a) if N is a direct summand of M which contains the singular submodule $Z(M)$ of M, then N is completely decomposable; (b) if there exists an injective submodule of M which contains $Z(M)$, then every direct summand of M is completely decomposable ([29]). It follows that Matlis' problem has a positive answer for left SI-rings.

Recently, Chen-Zhou-Zhu showed that YJ-injective rings need not be p-injective in [7].

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Universite Paris VII-Denis Diderot, UFR de Mathematiques-UMR 9994 CNRS,
2, place Jussieu, 75 251 Paris Cedex 05, France

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