Network character and tightness of the compact–open topology

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Abstract. For Tychonoff X and α an infinite cardinal, let $\alpha \operatorname{def} X :=$ the minimum number of α cozero-sets of the Čech-Stone compactification which intersect to X (generalizing $\mathbb{R}$-defect), and let $\operatorname{rt} X := \min_\alpha \max(\alpha, \alpha \operatorname{def} X)$. Give $C(X)$ the compact-open topology. It is shown that $\tau C(X) \leq n\chi C(X) \leq \operatorname{rt} X = \max(L(X), L(X) \operatorname{def} X)$, where: τ is tightness; $n\chi$ is the network character; $L(X)$ is the Lindelöf number. For example, it follows that, for X Čech-complete, $\tau C(X) = L(X)$. The (apparently new) cardinal functions $n\chi C$ and rt are compared with several others.

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1. Introduction

Our notation and terminology, will usually follow [E] or [McN]. All topological spaces will be Tychonoff. In the following, Y is a space and $y \in Y$.

The *tightness* at y of Y is

$$\tau(y, Y) := \omega + \min\left\{m \mid y \in \overline{A} \Rightarrow \exists\, A(y) \subseteq A,\ |A(y)| \leq m,\ y \in \overline{A(y)}\right\}.$$

The *tightness* of Y is $\tau Y := \omega + \sup\left\{\tau(y, Y) \mid y \in Y\right\}$.

A *local network* for y in Y is an $\mathcal{N} \subseteq \mathcal{P}(Y)$ (the power set) for which: whenever U is a nbd (neighborhood) of y, there is $\mathcal{N}(U) \subseteq \mathcal{N}$ with $\bigcup \mathcal{N}(U) \subseteq U$ and $\bigcup \mathcal{N}(U)$ is a nbd of y. The *network character* of y in Y is

$$n\chi(y, Y) := \omega + \min\left\{|\mathcal{N}| \mid \mathcal{N} \text{ is a local network for } y \text{ in } Y\right\}.$$

Then, $n\chi Y := \sup\left\{n\chi(y, Y) \mid y \in Y\right\}$. (These may be new definitions, albeit somewhat obvious.)

Proposition 1.1. *$\tau(y, Y) \leq n\chi(y, Y)$, and $\tau Y \leq n\chi Y$.*

Proof: Let $\mathcal{N}$ be a local network at y and let $y \in \overline{A}$. For each $N \in \mathcal{N}$ for which $N \cap A \neq \emptyset$, choose $a_N \in N \cap A$ and let $A(y) = \left\{a_N \mid N \in \mathcal{N},\ N \cap A \neq \emptyset\right\}$. Then

$y \in \overline{A(y)}$: if U is a nbd of y, there is $\mathcal{N}(U) \subseteq \mathcal{N}$ with $\bigcup \mathcal{N}(U) \subseteq U$ and $\bigcup \mathcal{N}(U)$ a nbd of y. Since $y \in \overline{A}$, $\left(\bigcup \mathcal{N}(U)\right) \cap A \neq \emptyset$, so $N \cap A \neq \emptyset$ for some $N \in \mathcal{N}(U)$, and thus $a_N \in N \cap A(y) \subseteq U \cap A(y)$.

The inequalities follow. □

Proposition 1.1 gives the first inequality in the abstract. Before discussing the second inequality and the equality, we digress briefly with another simple calculation.

We refer to [E] and [McN] for the standard definitions of character χ, network and network weight nw, and weight w. Here and elsewhere, 2^α for complicated α is written $\exp(\alpha)$.

Proposition 1.2. *$n\chi(y,Y) \leq \chi(y,Y) \leq \exp\left(n\chi(y,Y)\right)$, and $n\chi Y \leq \chi Y \leq \exp(n\chi Y)$. Likewise, $nwY \leq wY \leq \exp(nwY)$.*

Proof: To say that $\mathcal{N}$ is a local network at y is to say that $\left\{\bigcup \mathcal{N}' \mid \mathcal{N}' \subseteq \mathcal{N}\right\}$ contains a local base at y, and likewise for network *versus* base. The inequalities follow. □

If Y is homogeneous, then clearly for each $y \in Y$, $\tau Y = \tau(y,Y)$ and $n\chi Y = n\chi(y,Y)$. So, if $(G,+)$ is a topological group, then $\tau G = \tau(0,G)$ and $n\chi G = n\chi(0,G)$. So, for $C(X)$ with the compact-open topology, $\tau C(X) = \tau(0, C(X))$ and $n\chi C(X) = n\chi(0, C(X))$.

Let A be Tychonoff and $f \in C(A)$. The cozero-set of f is $\operatorname{coz} f := \{a \in A \mid f(a) \neq 0\}$, and $\operatorname{coz} A = \{\operatorname{coz} f \mid f \in C(A)\}$.

α always denotes an infinite cardinal. If $\mathcal{W} \subseteq \operatorname{coz} A$ and $|\mathcal{W}| \leq \alpha$, then $\bigcup \mathcal{W}$ is called an α cozero-set of A; and $\alpha \operatorname{coz} A$ is the collection of all α cozero-sets of A. (For $\alpha = \omega$, $\alpha \operatorname{coz} A = \operatorname{coz} A$ ([GJ]).)

We are interested in $\alpha \operatorname{coz} \beta X$, where βX is the Čech-Stone compactification of Tychonoff X. Here are two seemingly new

Definitions 1.3. The *α-defect* of X in βX is

$$\alpha \operatorname{def} X := \min\left\{|\mathcal{U}| \mid \mathcal{U} \subseteq \alpha \operatorname{coz} \beta X,\ \bigcap \mathcal{U} = X\right\},$$

understanding $\alpha \operatorname{def} X = \infty$ when no such $\mathcal{U}$ exists and construing $\infty > \gamma$ for any cardinal γ.

The *cozero-nesting index* of X in βX is

$$\operatorname{rt} X := \min\{\max(\alpha, \alpha \operatorname{def} X) \mid \alpha \text{ an infinite cardinal}\}.$$

The *Lindelöf number* of X is

$$L(X) := \omega + \min\{\gamma \mid \text{ each open cover of } X \text{ has a subcover of cardinal } \leq \gamma\}.$$

Theorem 1.4. $\operatorname{rt} X = \max\big(L(X), L(X) \operatorname{def} X\big)$.

PROOF:

(i) $\gamma \leq \delta \Rightarrow \gamma \operatorname{def} X \geq \delta \operatorname{def} X$ (since there are more δ cozero-sets).

(ii) Any open set in βX contains an $L(X)$ cozero-set containing X (since in βX, the cozero-sets form a base).

(iii) $\alpha \geq L(X) \Rightarrow \alpha \operatorname{def} X = L(X) \operatorname{def} X$ (by (i) and (ii)).

So, to prove Theorem 1.4, it is to show

(iv) $\alpha < L(X) \Rightarrow \max(\alpha, \alpha \operatorname{def} X) \geq \max\big(L(X), L(X) \operatorname{def} X\big)$.

This is harder. We note that a theorem of Henriksen, Isbell, and Johnson has a proof expounded in 8.3, 9.7, 9.8 of [CN] which generalizes from ω to α, as:

Call $\mathcal{A} \subseteq \mathcal{P}(Y)$ (the power set of Y) an α-ring if $\mathcal{B} \subseteq \mathcal{A}$, $|\mathcal{B}| \leq \alpha \Rightarrow \bigcup \mathcal{B}$ and $\bigcap \mathcal{B}$ belong to $\mathcal{A}$. Note that for any $\mathcal{C} \subseteq \mathcal{P}(Y)$, there is the least α-ring in $\mathcal{P}(Y)$ containing $\mathcal{C}$; call it $\alpha\mathcal{C}$. Then, the proofs from [CN] generalize to show the following.

(v) Let Y be compact, and let $\mathcal{K}(Y)$ denote the family of compact subsets of Y. Then, $X \in \alpha\mathcal{K}(Y) \Rightarrow L(X) \leq \alpha$.

(vi) $\alpha \operatorname{def} X \leq \alpha \Rightarrow L(X) \leq \alpha$ (by(v)).

(Another proof of (vi) is available using the inequality $n\chi C(X) \leq \operatorname{rt} X$ (Theorem 1.5 below); see Corollary 3.5 below.)

We now prove (iv) from (vi). Suppose both $\alpha < L(X)$ and $\max(\alpha, \alpha \operatorname{def} X) < \max\big(L(X), L(X) \operatorname{def} X\big)$. If $\alpha \operatorname{def} X \leq \alpha$, then (vi) gives $L(X) \leq \alpha$ — contradiction. If $\alpha \leq \alpha \operatorname{def} X$, then $\alpha \operatorname{def} X < L(X)$ (since $\alpha \operatorname{def} X \geq L(X) \operatorname{def} X$ by (i)). Let $m = \alpha \operatorname{def} X$. So $m \geq \alpha$, thus by (i) $m \operatorname{def} X \leq \alpha \operatorname{def} X = m$. By (vi) using m, $L(X) \leq m = \alpha \operatorname{def} X$ — contradiction. □

The following is the second inequality in the abstract, which is our main Theorem.

Theorem 1.5. *For any Tychonoff space X, $n\chi C(X) \leq \operatorname{rt} X$.*

What is to be proved is that $n\chi C(X) \leq \max(\alpha, \alpha \operatorname{def} X)\ \forall\, \alpha$, which we do in the next section. Before that, some remarks to fix ideas:

Remarks 1.6.

(1) Evidently, $\alpha \operatorname{def} X < \infty$ iff $\forall\, p \in \beta X - X\ \exists\, Z$ with $\beta X - Z \in \alpha \operatorname{coz} \beta X$ and $Z \cap X = \emptyset$. This condition is satisfied iff X is α^+-compact in the sense of Herrlich, by a theorem of Herrlich and Hong, cited as 2.1 of [CT]. These spaces have been considered also by Arhangel'skii and Uspenski, as noted in [McN, p. 72].

(2) Consequently (or otherwise), X is realcompact iff $\omega\operatorname{def} X < \infty$ ([GJ]). When X is realcompact, $\omega\operatorname{def} X$ is Mrowka's $\mathbb{R}$-defect ([M2]), which motivated our term $\alpha\operatorname{def}$. For realcompact X, the $\mathbb{R}$-defect of X is the least m such that X closed-embeds into $\mathbb{R}^{wX+m}$ (wX the weight), which explains the term "$\mathbb{R}$-defect". There is a similar justification for $\alpha\operatorname{def} X$: In [H], Hušek finds a space $P(\alpha^+)$ for which X is α^+-compact iff X closed-embeds into some power of $P(\alpha^+)$. It is then not hard to show that, when X is α^+-compact, $\alpha\operatorname{def} X$ is the least m such that X closed-embeds into $P(\alpha^+)^{wX+m}$.

(3) Exactly what is a particular value $\alpha\operatorname{def} X$ seems quite involved with Set Theory (and beyond our intentions in the present paper). To illustrate for just $\omega\operatorname{def} D(m)$ ($D(m)$ the discrete space of power m): (a) In [E, p. 135], we find $m = \omega\operatorname{def} D(m)$ if either $m = \mathfrak{c}$ (Engelking, or Engelking-Mrowka), or $m <$ the first weakly inaccessible cardinal (Mycielski); (b) [M1] $\omega\operatorname{def} D(m) \leq 2^m$ iff $m <$ the first Ulam measurable cardinal; (c) In [M1], the class $M = \{m \mid \omega\operatorname{def} D(m) \leq m\}$ is defined, and the author says he does not know if every Ulam non-measurable cardinal belongs to M.

(4) Suppose $X \subseteq Y$. The notions $\alpha\operatorname{def}(X, Y)$ and $\operatorname{rt}(X, Y)$ are easily defined. We note: (a) If Y is a compactification of X, then $\alpha\operatorname{def} X \leq \alpha\operatorname{def}(X, Y)$. (b) $\alpha\operatorname{def}(X, Y) < \infty\ \forall Y \supseteq X$ iff $LX \leq \alpha$. (b) generalizes Mrowka's observation: X is G_δ-closed in every $Y \supseteq X$ iff X is Lindelöf (see [E, p. 244]).

(5) Theorem 1.5 has evolved from 4.11 of [BH], which asserts countable tightness for each of a class of topologies defined on certain 1-groups, for the purpose of describing their epimorphisms.

2. Proof of Theorem 1.5

For any space Y, $\mathcal{K}(Y)$ denotes the family of compact subsets of Y.

For $K \in \mathcal{K}(\beta X)$ (repeat: βX), and $\epsilon \in (0, 1)$, let

$$N(K, \epsilon) \equiv \big\{ f \in C(X) \mid |f| \leq \epsilon \ \text{ on } \ K \big\}.$$

Here, the meaning of "$|f| \leq \epsilon$ on K" is "$x \in K \Rightarrow |\beta f(x)| \leq \epsilon$", where $\beta f : \beta X \to [-\infty, +\infty]$ is the Čech-Stone extension of f.

Recall that, for $C(X)$ with the compact-open topology, for $K \in \mathcal{K}(X)$ (repeat: X), $N(K, \epsilon)$ is a nbd of 0 (probably not open) and $\{N(K, \epsilon) \mid K \in \mathcal{K}(X),\ \epsilon \in (0, 1)\}$ is a local base at 0. But for $K \in \mathcal{K}(\beta X)$, $N(K, \epsilon)$ can easily fail to be a nbd of 0 (but, we shall show, some of these constitute a local network).

For any set $S, \mathcal{P}(S)$ denotes the power set and $\mathcal{P}_{\text{fin}}(S)$ the collection of non-void finite subsets of S.

Lemma 2.1. *Let $W \in \alpha\operatorname{coz}\beta X$. Then, for an index set J of cardinal α, there is $\big\{C_j \mid j \in J\big\} \subseteq \mathcal{K}(\beta X)$, which is closed under finite intersection and up-directed by inclusion, with $W = \bigcup_J \operatorname{int} C_j$.*

Proof: There is A of cardinal α with $W = \bigcup_A \operatorname{coz} f_a$, for $f_a \in C(\beta X, [0,1])$, and we can suppose $\{f_a \mid a \in A\}$ is closed under formation of finite suprema. Let $Y(a,n) = f_a^{-1}(\frac{1}{n}, 1]$, so that $\operatorname{coz} f_a = \bigcup_{\mathbb{N}} \operatorname{int} Y(a,n)$. For $F \in \mathcal{P}_{\text{fin}}(A \times \mathbb{N}) \equiv J$, let $C_F = \bigcap_{\mathbb{F}} Y(a,n)$. □

Notation 2.2. Suppose $\mathcal{W} = \{W_i \mid i \in I\} \subseteq \alpha \operatorname{coz} \beta X$, and for each i, $W_i = \bigcup \{C_{ij} \mid j \in J\}$ $(|J| = \alpha)$ as in Lemma 2.1.

For $F \in \mathcal{P}_{\text{fin}}(I)$, J^F is the set of functions $p : F \to J$.

For $F \in \mathcal{P}_{\text{fin}}(I)$, $p \in J^F$, and $n \in \mathbb{N}$, we set $C_p = \bigcap_F C_{i\,p(i)} \in \mathcal{K}(\beta X)$, and $N(C_p, \frac{1}{n}) = \{f \in C(X) \mid |f| \leq \frac{1}{n} \text{ on } C_p\}$.

For any $P \subseteq \bigcup \{J^F \mid F \in \mathcal{P}_{\text{fin}}(I)\}$, we set $L(P) = \bigcap \{C_p \mid p \in P\} \in \mathcal{K}(\beta X)$.

In Proposition 2.3–2.5, $\mathcal{W} = \{W_i \mid i \in I\}$, etc., are as in Notation 2.2. Corollary 2.6 immediately implies Theorem 1.5. The proofs will follow Corollary 2.6.

Proposition 2.3.
(a) $\bigcup_P N\big(C_p, \frac{1}{n}\big) \subseteq N\big(L(P), \frac{1}{n}\big)$.
(b) $N\big(L(P), \frac{1}{2n}\big) \subseteq \bigcup_P N\big(C_p, \frac{1}{n}\big)$ *if* $\{C_p \mid p \in P\}$ *is closed under finite intersection.*

Proposition 2.4. *Let* $K \in \mathcal{K}(X)$, *and set*

$$P_K = \Big\{p \mid p \in \bigcup \Big\{J^F \mid F \in \mathcal{P}_{\text{fin}}(I)\Big\}, K \subseteq C_p\Big\}$$

Then, $\{C_p \mid p \in P_K\}$ *is closed under finite intersection and* $L(P_K) \supseteq K$, *and consequently,*

$$N\Big(L(P_K), \frac{1}{2n}\Big) \subseteq \bigcup_{P_K} N\Big(C_p, \frac{1}{n}\Big) \subseteq N\Big(K, \frac{1}{n}\Big).$$

Proposition 2.5. *Suppose that* $\bigcap \mathcal{W} = X$.
(a) *If* $\bigcup \{\operatorname{dom} p \mid p \in P\} = I$, *then* $L(P) \in \mathcal{K}(X)$, *and consequently, each* $N\big(L(P), \epsilon\big)$ *is a nbd of* 0.
(b) *For any* $K \in \mathcal{K}(X)$, *the* P_K *in Proposition* 2.4 *has* $\bigcup \{\operatorname{dom} p \mid p \in P_K\} = I$.
(c) $\mathcal{N}(\mathcal{W}) \equiv \{N\big(C_p, \frac{1}{n}\big) \mid p \in \bigcup \{J^F \mid F \in \mathcal{P}_{\text{fin}}(I)\},\ n \in \mathbb{N}\}$ *is a local network at* 0.

Corollary 2.6. $n\chi C(X) \leq \max(\alpha, \alpha \operatorname{def} X)$ *for any* α.

Proof of 2.6: When $\alpha \operatorname{def} X = \infty$, this is true. Otherwise, there is $\mathcal{W} = \{W_i\}_I$ with $\bigcap \mathcal{W} = X$ and $|I| = \alpha \operatorname{def} X$, and the $\mathcal{N}(\mathcal{W})$ in Proposition 2.5(c) has cardinal $\leq \max(\alpha, \alpha \operatorname{def} X)$. □

PROOF OF PROPOSITION 2.3:

(a) $C_p \supseteq L(P)\ \forall p \in P$.

(b) Suppose $f \notin N\big(C_p, \frac{1}{n}\big)\ \forall p \in P$. Then,

$$E(p) \equiv \Big\{x \in \beta X \mid |f(x)| \geq \frac{3}{4n}\Big\} \cap C_p \neq \emptyset \quad \forall p \in P.$$

But $\big\{E(p) \mid p \in P\big\}$ has the finite intersection property, since $\forall p, q \in P\ \exists r \in P$ with $E(p) \cap E(q) = E(r) \neq \emptyset$. Thus, by compactness, there is $x_0 \in \bigcap_P E(p)$. So $x_0 \in \bigcap_P C_p \equiv L(P)$, and by continuity, $|f(x_0)| \geq \frac{3}{4n}$. Therefore $f \notin N\big(L(P), \frac{1}{2n}\big)$. □

PROOF OF PROPOSITION 2.4: Let $p, q \in P_K$, say $p \in J^F$ and $q \in J^G$. Let $H = F \cup G$, and define $r \in J^H$ as, $r \mid F - G = p \mid F - G$, $r \mid G - F = q \mid G - F$, and for $i \in F \cap G$: Recall that $W_i = \bigcup_J C_{ij}$, with $\{C_{ij}\}_J$ closed under finite intersection (2.1). So choose $r(i) \in J$ for which $C_{i\,p(i)} \cap C_{i\,q(i)} = C_{i\,r(i)}$. Now $r \in J^H$ is defined, and one checks that $C_p \cap C_q = C_r$.

We now have the first inclusion in Proposition 2.4 *via* Proposition 2.3(b). The second inclusion follows from Proposition 2.3(a) and $L(P_K) \supseteq K$ (the latter being obvious). □

PROOF OF PROPOSITION 2.5:

(a) Since $\mathcal{W} = \{W_i\}_I$ has $\bigcap \mathcal{W} = X$, $\big[x \in \beta X - X \Rightarrow \exists\, i$ with $x \notin W_i = \bigcup_J C_{ij}\big]$, so $x \notin C_{ij}\ \forall j$. Choose $p \in P \cap J^F$ with $i \in F$. Then $x \notin C_{i\,p(i)}$, so $x \notin C_p \supseteq L(P)$. Thus, $L(P) \subseteq X$.

(b) Let $K \in \mathcal{K}(X)$, and let $i_0 \in I$. We have $K \subseteq X = \bigcap_I W_i$, so that $K \subseteq W_{i_0} = \bigcup_J \operatorname{int} C_{i_0 j}$ (from Lemma 2.1). Since K is compact, there are $j_1, \ldots, j_b$ with $K \subseteq \bigcup_{s=1}^{b} \operatorname{int} C_{i_0 j_s}$. Since $\big\{C_{i_0 j}\big\}_J$ is up-directed, there is j_0 with $K \subseteq C_{i_0 j_0}$. Now just define $p \in J^{\{i_0\}}$ as $p(i_0) = j_0$, so that $C_p = C_{i_0 j_0} \supseteq K$.

(c) Let $N\big(K, \frac{1}{n}\big)$ be a basic nbd of 0, $K \in \mathcal{K}(X)$. By (a) and (b), $N\big(L(P_k), \frac{1}{2n}\big)$ is a nbd of 0. Now the inclusion in Proposition 2.4 show that $\mathcal{N}(\mathcal{W})$ is a local network at 0. □

The proof of Corollary 2.6 (Theorem 1.5) is concluded.

3. Comparison of cardinal functions

We have shown that $\tau C(X) \leq n\chi C(X) \leq \operatorname{rt}(X)$ for all X. This is a relation among the cardinal functions $\tau C, n\chi C, \operatorname{rt}$, of which the second and third seem to be new. Now we discuss some relations among cardinal functions which impinge on the above inequalities. The other cardinal functions which we consider are: weight w; network weight nw; compact network weight knw; compact Lindelöf number kL; character χ; compact Arens number ka. We refer to [McN] for definitions of the preceding. Also: compact-covering number μ, from [CB] (there denoted κ). For f any cardinal function (including constant functions α), $m_f(X) \equiv \max\bigl(f(X), f(X) \operatorname{def} X\bigr)$ (numbers in the definition of $\operatorname{rt} X$).

The following chart shows relations among cardinal functions, more-or-less in the manner of [E]: An arrow from f to g means $\bigl[\forall\, X\bigl(g(X) \le f(X)\bigr)\ \&\ \exists\, X\bigl(g(X) < f(X)\bigr)\bigr]$. No arrow between f and g means $\bigl[\exists\, X\bigl(f(X) < g(X)\bigr)\ \&\ \exists\, Y\bigl(f(Y) > g(Y)\bigr)\bigr]$. Further, the arrows and $=$'s are labeled: The label $[a \cdot b \cdot c]$ is a reference to [McN]; $(a \cdot b)$ to the present paper. The arrow labeled Mc indicates that the proof in [Mc] that $kL(X) = \omega \Rightarrow L(X) = \omega$ generalizes. No label means "it is obvious".

Chart 3.1.

$$
\begin{tikzcd}
\exp(\mathrm{knw}) \arrow[d, "(1.2)"'] & m_\mu \arrow[dd] \arrow[dddrr] & m_\alpha \arrow[ddl] & & \\
w \arrow[d, "{[4.1.1]}"'] & & & & \exp(n\chi C) \arrow[dl, "(1.2)"] \\
{[4.1.2]}\quad nwC = \mathrm{knw} \arrow[dr] & \mathrm{rt} = m_L \quad (1.4) \arrow[d, "(1.5)"] & & \chi C = \mathrm{ka} \quad [4.4.1] \arrow[dll] \arrow[d] & \\
& n\chi C \arrow[d, "(1.1)"] & & \mu \arrow[dll] & \\
& {[4.7.1]}\quad \tau C = kL \arrow[d, "\mathrm{Mc}"] & & & \\
& L & & &
\end{tikzcd}
$$

In the chart, the $=$'s are major theorems whose ultimate forms seem to have been achieved in [McN], but which originated (as recounted in [McN]) from work of Michael - [4.1.2]; Arens - [4.4.1]; McCoy, Arhangel'skii, Gerlits/Nagy - [4.7.1]. Regarding these theorems, two comments: (1) For example "$\tau C(X) = kL(X)$" does not put to rest the question "What is $\tau C(X)$?", because the value of $kL(X)$ is frequently not obvious. (2) The fact that these theorems (and similar theorems in [McN]) exist raises the questions: What is f such that $n\chi C(X) = f(X)\ \forall\, X$? What is g such that $gC(X) = \operatorname{rt}(X)\ \forall\, X$? We have not addressed these.

In attempting to be brief, we now shall only point out how certain hypotheses on the spaces collapse parts of the chart (which yields some computations of $\mathrm{rt}(X)$ in familiar terms), and exhibit certain not-so-obvious examples that certain arrows are "proper" and certain cardinal functions are incomparable.

The proofs of the following are not difficult, and omitted.

Proposition 3.2. *X is Čech-complete iff $\exists\,\alpha(\alpha \operatorname{def} X \leq \omega)$ iff $L(X) \operatorname{def} X \leq \omega$; that implies $L(X) = \operatorname{rt} X$.*

X is locally compact iff $\exists\,\alpha(\alpha \operatorname{def} X) = 1$ iff $L(X) \operatorname{def} X = 1$; that implies $L(X) = \operatorname{ka} X = m_{\mu}(X)$.

If X is metrizable, then $L(X) = wX$.

Corollary 3.3. *These are equivalent.* (a) $\operatorname{rt} X = \omega$, (b) $\omega \operatorname{def} X \leq \omega$, (c) *$X$ is Čech-complete and Lindelöf. Thus, for the rationals $\mathbb{Q}$, $\omega < \operatorname{rt} \mathbb{Q} = \omega \operatorname{def} X \leq 2^{\omega}$.*

PROOF: (a) $\Rightarrow$ (c). $\operatorname{rt} X = \omega$ implies $\omega = \operatorname{rt} X = \max(\omega, \omega \operatorname{def} X)$. So, using Proposition 3.2, $\omega \operatorname{def} X \leq \omega$, thus X is Čech-complete, and therefore $\omega = \operatorname{rt} X = m_L(X)$, so X is Lindelöf. (c) $\Rightarrow$ (b) by Proposition 3.2 using $L(X) = \omega$. (b) $\Rightarrow$ (a) since $\operatorname{rt} X \leq \max(\omega, \omega \operatorname{def} X)$.

$\omega < \operatorname{rt} \mathbb{Q}$ since $\mathbb{Q}$ is not Čech-complete, and $\operatorname{rt} \mathbb{Q} = \omega \operatorname{def} \mathbb{Q}$ by Theorem 1.4. Obviously, $\omega \operatorname{def} \big(\mathbb{Q}, [-\infty, +\infty]\big) \leq 2^{\omega}$ (see Remarks 1.6(4)), and this implies $\omega \operatorname{def} \mathbb{Q} \leq 2^{\omega}$. $\square$

N.b. The value of $\omega \operatorname{def} \mathbb{Q}$ depends on the set theory, as explained by Hechler, as recounted in [M2]. Further, see Examples 3.4(g) below.

Examples 3.4. We locate spaces X with the indicated properties. In all cases except (g) the smaller number is ω.

- **(a)** $\tau C(X) < n\chi C(X)$. Take $X = \lambda D$ for $|D| > 2^{\omega}$, λD being discrete D with a point adjoined whose nbds have countable complement. [Mc] shows $\tau C(X) = \omega$. Now, by Proposition 1.2 (with $Y = C(X)$), $\chi C(X) \leq \exp\big(n\chi C(X)\big)$, while $\chi C(X) = \operatorname{ka} X = |D|$, since compact sets are finite in λD. Since $|D| > 2^{\omega}$, it follows that $n\chi C(X) > \omega$.
- **(b)** $n\chi C(X) < \operatorname{rt} X$. Take $X = \mathbb{Q}$. So $n\chi C(X) \leq wX = \omega$ (see the chart), and $\omega < \operatorname{rt} X$ by Corollary 3.3.
- **(c)** $n\chi C(X) = \operatorname{rt} X < \chi C(X)$. Take $X =$ the irrationals. So $\operatorname{rt} X = \omega$, by Corollary 3.3, while $\omega < \chi C(X)$ since X is not hemicompact. (In the chart, "$\chi C(Y) = \operatorname{ka} C(Y) = \omega$" means Y is hemicompact.)
- **(d)** $n\chi C(X) < n\omega C(X)$. [McN, 5.7.6] says that for X locally compact, $nwC(X) = \omega$ iff X is Polish. So, using Proposition 3.2, take X locally compact Lindelöf, not metrizable.

(e) rt and ka are incomparable. (e_1) $\mathrm{rt}\, X < \mathrm{ka}\, X$. See (c). ($e_2$) $\mathrm{rt}\, X > \mathrm{ka}\, X$. Let X be Arens' space, [E, 1.6.20]. X is hemicompact, so $\mathrm{ka}\, X = \omega$, but X is not a k-space ([E, p. 278]), so X is not Čech-complete ([E, p. 198]), so $\mathrm{rt}\, X > \omega$ (Corollary 3.3 above).

(f) rt is not comparable with knw, or w: (f_1) $\mathrm{knw}\, X = wX < \mathrm{rt}\, X$ for $X = \mathbb{Q}$ (Corollary 3.3). (f_2) $\mathrm{rt}\, X < \mathrm{knw}\, X \leq wX$ for any X which is locally compact Lindelöf and not metrizable. See (d). Here $\mathrm{rt}\, X = \omega$ by Proposition 3.2.

(g) One may ask about the status of the inequalities above when the lower number is demanded to be uncountable. The only thing we have to say regards (f): Corollary 3.3 says $\omega = wX < \mathrm{rt}\, X = \omega\,\mathrm{def}\, X \leq 2^\omega$ for $X = \mathbb{Q}$. [M2] shows that $\big[\mathrm{ZFC} + (m = n^+ = 2^n <$ the first Ulam measurable cardinal$)\big] \Rightarrow \exists\, X(m = wX < \omega\,\mathrm{def}\, X)$. For these X, we have not addressed the question "What is $\mathrm{rt}\, X$?".

But, for any uncountable m, $\exists\, X(\omega \leq \mathrm{rt}\, X \leq \omega\,\mathrm{def}\, X \leq m < wX)$. Let Y be realcompact with $L(Y) > m$, so $wY > m$ and $\omega\,\mathrm{def}\, Y > m$. (See the chart.) Then there is a family $\{C_\alpha \mid \alpha < m\}$ of cozero-sets of βY which contain Y, with $\big[\alpha < \beta \Rightarrow C_\alpha \nsubseteq C_\beta\big]$. For $X = \bigcap_\alpha C_\alpha$, $\omega\,\mathrm{def}\, X \leq m$ by the construction, $\omega \leq \mathrm{rt}\, X$ by Corollary 3.3 and $m < wX$ since $X \supseteq Y$.

We conclude with another proof of the crucial item (vi) in the proof of Theorem 1.4 — indeed, of a stronger statement.

Corollary 3.5. *$\alpha\,\mathrm{def}\, X \leq \alpha \Rightarrow kL(X) \leq \alpha$.*

Proof: $\alpha\,\mathrm{def}\, X \leq \alpha$ means $m_\alpha X = \alpha$, so by Chart 3.1, $\alpha = m_\alpha X \geq \mathrm{rt}\, X \geq kL(X)$. (This uses Proposition 1.1, Theorem 1.5, and the items at the bottom of the chart labeled [4.7.1] and Mc.) □

References

[BH] Ball R., Hager A., *Epi-topology and epi-convergence in archimedean lattice-ordered groups with unit*, submitted.

[CB] Comfort W., Balaglou G., *Compact-covering numbers*, Fund. Math. **131** (1988), 69–82.

[CN] Comfort W., Negrepontis S., *Continuous pseudometrics*, Lecture Notes in Pure and Appl. Math. 14, Dekker, New York, 1975.

[CT] Comfort W., Retta T., *Generalized perfect maps and a theorem of I. Juhász*, Lecture Notes in Pure and Appl. Math. 95, Dekker, New York, 1985, pp. 79–102.

[E] Engelking R., *General Topology*, Heldermann, Berlin, 1989.

[GJ] Gillman L., Jerison M., *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.

[H] Hušek M., *The class of k-compact spaces is simple*, Math. Z. **110** (1969), 123–126.

[Mc] McCoy R., *Function spaces which are k-spaces*, Topology Proc. **5** (1980), 139–154.

[McN] McCoy R., Ntantu I., *Topological Properties of Spaces of Continuous Functions*, Lecture Notes in Mathematics 1315, Springer, Berlin, 1988.

[M1] Mrowka S., *On E-compact spaces. II*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **14** (1966), 597–605.

[M2] Mrowka S., *$\mathcal{R}$-compact spaces with weight* $X < \mathrm{Exp}_{\mathcal{R}}\, X$, Proc. Amer. Math. Soc. **128** (2000), 3701–3709.

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