Non-singular precovers over polynomial rings

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Abstract. One of the results in my previous paper *On torsionfree classes which are not precover classes*, preprint, Corollary 3, states that for every hereditary torsion theory τ for the category R-mod with $\tau \geq \sigma$, σ being Goldie's torsion theory, the class of all τ-torsionfree modules forms a (pre)cover class if and only if τ is of finite type. The purpose of this note is to show that all members of the countable set $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\sigma(R[x_1, \dots, x_n]), n < \omega\}$ of rings have the property that the class of all non-singular left modules forms a (pre)cover class if and only if this holds for an arbitrary member of this set.

Keywords: hereditary torsion theory, torsion theory of finite type, Goldie's torsion theory, non-singular module, non-singular ring, precover class, cover class

Classification: 16S90, 18E40, 16D80

In what follows, R stands for an associative ring with the identity element and R-mod denotes the category of all unitary left R-modules. The basic properties of rings and modules can be found in [1].

A class $\mathcal{G}$ of modules is called *abstract*, if it is closed under isomorphic copies. If $\mathcal{G}$ is an abstract class of modules, then a homomorphism $\varphi : G \to M$ with $G \in \mathcal{G}$ is called a $\mathcal{G}$*-precover* of the module M, if for each homomorphism $f : F \to M$ with $F \in \mathcal{G}$ there exists a homomorphism $g : F \to G$ such that $\varphi g = f$. A $\mathcal{G}$-precover φ of M is said to be a $\mathcal{G}$*-cover*, if every endomorphism f of G with $\varphi f = \varphi$ is an automorphism of the module G. An abstract class $\mathcal{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathcal{G}$-precover ($\mathcal{G}$-cover). A more detailed study of precovers and covers can be found in [13].

Recall that a *hereditary torsion theory* $\tau_R = (\mathcal{T}_\tau, \mathcal{F}_\tau)$, or simply $\tau = (\mathcal{T}, \mathcal{F})$, for the category R-mod consists of two abstract classes $\mathcal{T}$ and $\mathcal{F}$, the τ*-torsion class* and the τ*-torsionfree class*, respectively, such that $\mathrm{Hom}(T, F) = 0$ whenever $T \in \mathcal{T}$ and $F \in \mathcal{F}$, the class $\mathcal{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathcal{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. It is easy to see that every module M contains the unique largest τ-torsion submodule

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(isomorphic to T), which is called the *τ-torsion part of the module M* and it is usually denoted by $\tau(M)$. For two hereditary torsion theories τ and τ' the symbol $\tau \leq \tau'$ means that $\mathcal{T}_\tau \subseteq \mathcal{T}_{\tau'}$ and consequently $\mathcal{F}_{\tau'} \subseteq \mathcal{F}_\tau$. Associated to each hereditary torsion theory τ is the *Gabriel filter* $\mathcal{L}_\tau$ (or simply $\mathcal{L}$) of left ideals of R consisting of all the left ideals $I \leq R$ such that $R/I \in \mathcal{T}$. Recall that τ is said to be of *finite type*, if $\mathcal{L}$ contains a cofinal subset of finitely generated left ideals. A module Q is called *τ-injective*, if it is injective with respect to all short exact sequences $0 \to A \to B \to C \to 0$, where $C \in \mathcal{T}$. Following [10] we say, that a τ-torsionfree module is *τ-exact*, if any its τ-torsionfree homomorphic image is τ-injective.

For a module M, a submodule K is called *essential* in M if $K \cap L \neq 0$ for each non-zero submodule L of M and the *singular submodule* $Z(M)$ consists of all elements $a \in M$, the *annihilator left ideal* $(0 : a)_R = \{r \in R \mid ra = 0\}$, or simply $(0 : a)$, of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \{M \in R\text{-mod} \mid Z(M/Z(M)) = M/Z(M)\}$ and $\mathcal{F} = \{M \in R\text{-mod} \mid Z(M) = 0\}$. Note, that throughout this paper the letter σ will always denote Goldie's torsion theory and that the modules from the class $\mathcal{F}_\sigma$ are usually called *non-singular modules*. A ring R is said to be *non-singular* if it is non-singular as a left R-module. For more details on torsion theories we refer to [9] or [8].

Recently, in [4, Corollary 3], it has been proved that for each hereditary torsion theory τ with $\tau \geq \sigma$ the class of all τ-torsionfree modules is a precover class if and only if it is a cover class and these conditions are satisfied exactly when the torsion theory τ is of finite type. In this note we are going to show that these conditions are equivalent for Goldie's torsion theory for all members of the countable set $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\ \sigma(R[x_1, \dots, x_n]), n < \omega\}$ of rings whenever they are equivalent for an arbitrary member of this set. Moreover, for each element $S \in \mathfrak{M}$ and each hereditary torsion theory τ_S for the category S-mod such that $\tau_S \geq \sigma_S$ the class of all τ_S-torsionfree modules is a precover class whenever Goldie's torsion theory σ_R for the category R-mod is of finite type.

We start our investigations with some relations between the left ideals of the ring R belonging to the Gabriel filter $\mathcal{L}_\sigma$ corresponding to Goldie's torsion theory and the essential left ideals of the non-singular factor-ring $\bar{R} = R/\sigma(R)$ (it should be mentioned that for the non-singular ring R the Gabriel filter $\mathcal{L}_\sigma$ of Goldie's torsion theory σ consists of essential left ideals of R, only). Our main aim is to show that Goldie's torsion theory is of finite type in the category R-mod if and only if the same is true in the category $\bar{R}$-mod, where $\bar{R} = R/\sigma(R)$.

Lemma 1. *If every essential left ideal of the ring R essentially contains a finitely generated left ideal, then every left ideal of R essentially contains a finitely generated left ideal.*

PROOF: Let $0 \neq I \leq R$ be an arbitrary non-essential left ideal of the ring R and let J be a left ideal of R maximal with respect to $I \cap J = 0$. Then $I \oplus J$ is essential in R and consequently the hypothesis yields the existence of a finitely generated left ideal $K = \sum_{i=1}^{n} Ra_i \subseteq I \oplus J$ which is essential in $I \oplus J$ and hence in R. Now $a_i = b_i + c_i$, $b_i \in I$, $c_i \in J$, $i = 1, \dots, n$, and for an arbitrary element $0 \neq u \in I$ we have $0 \neq ru = \sum_{i=1}^{n} r_i b_i + \sum_{i=1}^{n} r_i c_i \in K$ for suitable elements $r, r_1, \dots, r_n \in R$ and consequently $0 \neq ru = \sum_{i=1}^{n} r_i b_i$, showing that the left ideal $\sum_{i=1}^{n} Rb_i$ is essential in I. □

Lemma 2. *If J is a left ideal of the ring R lying in the Gabriel filter $\mathcal{L}_\sigma$, then $\bar{J} = \frac{J+\sigma(R)}{\sigma(R)}$ is essential in the factor-ring $\bar{R} = R/\sigma(R)$.*

Proof: By the hypothesis there is a left ideal K of R containing J such that $Z(R/J) = K/J$ and $Z(R/K) = R/K$. Now let $\bar{r} \in \bar{R} \setminus \bar{J}$, $\bar{r} = r + \sigma(R)$, be an arbitrary element. Then $(K : r)$ is essential in R and $r \notin \sigma(R)$ yields that $(\sigma(R) : r) \notin \mathcal{L}_\sigma$. Thus $(K : r) \nsubseteq (\sigma(R) : r)$ and this gives the existence of an element $s \in (K : r) \setminus (\sigma(R) : r)$. So, $0 \neq \bar{s}\bar{r} \in \bar{K}$ and $sr = k \in K \setminus \sigma(R)$. Further, $(J : k)$ is essential in R and $(\sigma(R) : k) \notin \mathcal{L}_\sigma$, hence $(J : k) \nsubseteq (\sigma(R) : k)$ and this yields the existence of an element $t \in R$ for which $tsr = tk \in J \setminus \sigma(R)$. Thus $0 \neq \bar{t}\bar{s}\bar{r} \in \bar{J}$, as we wished to show. □

Lemma 3. *If $\bar{J} = J/\sigma(R)$ is an essential left ideal of the factor-ring $\bar{R} = R/\sigma(R)$, then J is essential in R.*

Proof: For an arbitrary element $r \in R \setminus J$ we have $\bar{r} \neq 0$, hence $0 \neq \bar{s}\bar{r} \in \bar{J}$ for some $\bar{s} \in \bar{R}$ by the hypothesis, and consequently $0 \neq sr \in J$, as desired. □

Lemma 4. *If every essential left ideal of the factor-ring $\bar{R} = R/\sigma(R)$ contains an essential finitely generated left ideal, then the same holds for essential left ideals of the ring R.*

Proof: Let J be an essential left ideal of the ring R. By Lemma 2 the left ideal $\bar{J} = \frac{J+\sigma(R)}{\sigma(R)}$ is essential in $\bar{R}$ and so it contains an essential finitely generated left ideal $\bar{K} = \sum_{i=1}^{m} \bar{R}\bar{a}_i$ by the hypothesis. We need now to show that the left ideal $K = \sum_{i=1}^{m} Ra_i$ of the ring R is essential in J, assuming without loss of generality that the elements $a_1, \dots a_m$ lie in J. From Lemma 3 we know that the left ideal $K + \sigma(R)$ of R is essential in $J + \sigma(R)$. So, for each $u \in J \setminus (K + \sigma(R))$ we have $0 \neq \bar{r}\bar{u} \in \bar{K}$ for some $\bar{r} \in \bar{R}$, hence $ru = k + v$, $0 \neq k \in K$, $v \in \sigma(R)$. Now $(0 : v) \in \mathcal{L}_\sigma$, $(0 : k) \notin \mathcal{L}_\sigma$ since $\bar{r}\bar{u} \neq 0$, so $(0 : v) \nsubseteq (0 : k)$ and we can take $s \in (0 : v) \setminus (0 : k)$ giving that $0 \neq sru = sk + sv = sk \in K$, as we wished to show. □

Theorem 5. *Goldie's torsion theory σ for the category R-*mod *is of finite type if and only if Goldie's torsion theory $\bar{\sigma}$ for the category $\bar{R}$-*mod*, where $\bar{R} = R/\sigma(R)$, is of finite type.*

Proof: Without loss of generality we can suppose that $\sigma(R) \neq R$, the case $\sigma(R) = R$ being trivial.

Assume first, that σ is of finite type. Since $Z(\bar{R}) = 0$, the Gabriel filter $\mathcal{L}_{\bar{\sigma}}$ consists of essential left ideals, only. So, if $J/\sigma(R) = \bar{J} \leq' \bar{R}$ is arbitrary, then $J \leq' R$ by Lemma 3 and consequently there is $K \leq R$ such that $K \subseteq J$, $J/K \in \mathcal{T}_\sigma$ and $K = \sum_{i=1}^m Ra_i$ is a finitely generated left ideal of R. If $a_i \in \sigma(R)$ for each $i = 1, \dots, m$, then $K \leq \sigma(R) \cap J \leq J$, which yields $\frac{J}{J \cap \sigma(R)} \cong \frac{J+\sigma(R)}{\sigma(R)} \in \mathcal{T}_\sigma \cap \mathcal{F}_\sigma$, hence $J \subseteq \sigma(R)$ and this contradicts the facts that $J \leq' R$ and $\sigma(R) \neq R$. Thus $\bar{K} = \frac{K+\sigma(R)}{\sigma(R)}$ is non-zero, it is finitely generated and it remains to verify that $\bar{J}/\bar{K}$ is $\bar{\sigma}$-torsion. However, $\bar{J}/\bar{K} = J/\sigma(R)/(K + \sigma(R))/\sigma(R) \cong J/(K + \sigma(R)) \in \mathcal{T}_\sigma$ and $\bar{J}/\bar{K} \in \mathcal{T}_{\bar{\sigma}}$ by Lemma 2 as a homomorphic image of J/K. The converse follows immediately from Lemma 4 and Lemma 1. □

We proceed now to some relations between essential left ideals of the ring R and that of the ring $R[x]$ of polynomials over R. First of all we are going to show that R is non-singular if and only if $R[x]$ is so.

Lemma 6. *Let $0 \neq a \in R$ be an arbitrary element. Then $(0 : a)_{R[x]} = R[x](0 : a)_R = (0 : a)_R[x]$.*

Proof: For the sake of simplicity we shall denote by I the left annihilator ideal $(0 : a)_R$ of R and by J the left annihilator ideal $(0 : a)_{R[x]}$ of $R[x]$. For any $g \in R[x]$ and any $r \in I$ we have $ra = 0$, hence $gra = 0$ and so $gr \in J$, proving the inclusion $R[x]I \subseteq J$. Conversely, let $g = \sum_{j=0}^m b_j x^j \in J$ be an arbitrary element. Then $0 = ga = \sum_{j=0}^m b_j a x^j$ yields $b_j a = 0$ and consequently $b_j \in I$ for each $j = 0, 1, \dots, m$. But this means that $g \in R[x]I$ and we are through, the rest being obvious. □

Lemma 7. *If I is an essential left ideal of the ring R, then $J = I[x] = R[x]I$ is an essential left ideal of the polynomial ring $R[x]$. Especially, if the left annihilator ideal $(0 : a)_R$ of an element $0 \neq a \in R$ is essential in R, then the left annihilator ideal $(0 : a)_{R[x]}$ of a is essential in $R[x]$.*

Proof: Let $g = \sum_{j=0}^m b_j x^j$ be an arbitrary polynomial which does not belong to J. If $b_0 \in I$ then we put $r_0 = 1$, while in the opposite case there is an element $r_0 \in R$ such that $0 \neq r_0 b_0 \in I$. Continuing by the induction let us assume that the elements $r_0, r_1, \dots, r_s \in R$, $0 \leq s < m$, such that $r_s \dots r_1 r_0 b_i \in I$ for all $i = 0, 1, \dots, s$, and that at least one of these elements is non-zero, have been already constructed. If $r_s \dots r_1 r_0 b_{s+1} \in I$, then we put $r_{s+1} = 1$ and we shall find $r_{s+1} \in R$ such that $0 \neq r_{s+1} r_s \dots r_1 r_0 b_{s+1} \in I$ in the opposite case. It is clear now that after $m + 1$ steps we obtain a non-zero multiple rg of g which lies in J. The special statement now immediately follows from Lemma 6. □

Lemma 8. *If I is a left ideal of the ring R such that the left ideal $J = R[x]I$ is essential in $R[x]$, then I is essential in R.*

Proof: Let $0 \neq r \in R$ be an arbitrary element. Then $r \in R[x]$ yields the existence of a polynomial $g = \sum_{j=0}^{m} b_j x^j \in R[x]$ such that $0 \neq gr \in J$. Thus there is a non-zero coefficient $b_i r$ of gr which obviously lies in I and the proof is complete. $\square$

Lemma 9. *Let $f = \sum_{i=0}^{n} a_i x^i$ be a non-zero polynomial of the degree n. If K is a left ideal of the ring R such that the left annihilator ideal $J = (K[x] : f)$ is essential in $R[x]$, then the left annihilator ideal $I = (K : a_n)$ is essential in R.*

Proof: Proving indirectly let us suppose that there exists a non-zero left ideal L of R such that $L \cap I = 0$. Now $L[x]$ is a non-zero left ideal of $R[x]$ and we are going to show that $L[x] \cap J = 0$. Assume, on the contrary, that $g = \sum_{j=0}^{m} b_j x^j$ is a non-zero element of $L[x] \cap J$ of the degree m. Then $gf \in K[x]$ means that the coefficient $b_m a_n$ of the product gf at the power x^{m+n} belongs to K. On the other hand, $0 \neq b_m \in L$ means that $b_m \notin I$, hence $b_m a_n \notin K$, which is a contradiction finishing the proof. $\square$

Theorem 10. *For any ring R the equality $\sigma(R[x]) = \sigma(R)[x]$ holds. Especially, a ring R is non-singular if and only if the polynomial ring $R[x]$ is so.*

Proof: We start with the equality $Z(R[x]) = Z(R)[x]$. If $f = \sum_{i=0}^{n} a_i x^i$ is an element of $Z(R)[x]$, then $(0 : a_i)$ is essential in R for each $i = 0, 1, \dots, n$ and consequently the intersection $I = \bigcap_{i=0}^{n}(0 : a_i)$ is essential in R. By Lemma 7 the left ideal $I[x]$ is essential in $R[x]$ and the obvious inclusion $I[x] \subseteq (0 : f)$ yields that $f \in Z(R[x])$. Thus the inclusion $Z(R)[x] \subseteq Z(R[x])$ holds. Conversely, let $f = \sum_{i=0}^{n} a_i x^i \in Z(R[x])$ be an arbitrary non-zero element of the degree n. Then $(0 : f) \leq' R[x]$ and so $(0 : a_n) \leq' R$ by Lemma 9. Hence $a_n \in Z(R)$ and so $a_n x^n \in Z(R)[x] \subseteq Z(R[x])$. Thus $f - a_n x^n \in Z(R[x])$ and continuing by the induction we finally obtain that $f = \sum_{i=0}^{n} a_i x^i \in Z(R)[x]$, as we wished to show.

Now we are going to finish the proof in the similar way as above. So, let $f = \sum_{i=0}^{n} a_i x^i \in \sigma(R)[x]$ be arbitrary. Then $(Z(R) : a_i) \leq' R$ for each $i = 0, 1, \dots, n$ and consequently $I = \bigcap_{i=0}^{n}(Z(R) : a_i)$ is essential in R. By Lemma 7 the left ideal $I[x]$ is essential in $R[x]$. For an arbitrary element $g = \sum_{j=0}^{m} b_j x^j \in I[x]$ we have $b_j \in I$ and hence $b_j a_i \in Z(R)$ for all relevant indices i and j. Thus $gf \in Z(R)[x]$ and so $g \in (Z(R)[x] : f)$. This means that $I[x] \subseteq (Z(R)[x] : f)$ and consequently $f \in \sigma(R[x])$ and the inclusion $\sigma(R)[x] \subseteq \sigma(R[x])$ is verified. In order to prove the equality let $0 \neq f = \sum_{i=0}^{n} a_i x^i$ be an arbitrary element of $\sigma(R[x])$ of the degree n. Then $(Z(R[x]) : f)$ is essential in $R[x]$ and so the left annihilator ideal $(Z(R) : a_n)$ is essential in R by Lemma 9 in view of the equality $Z(R[x]) = Z(R)[x]$ proved in the first part of the proof. Thus $a_n \in \sigma(R)$ gives that $a_n x^n \in \sigma(R)[x] \subseteq \sigma(R[x])$. From this we infer that $f - a_n x^n \in \sigma(R[x])$ and we can proceed by the induction similarly as in the first part of the proof. Finally we obtain that $f = \sum_{i=0}^{n} a_i x^i \in \sigma(R)[x]$, as we wished to show. The rest is easy. $\square$

Corollary 11. *Let $f = \sum_{i=0}^{n} a_i x^i$ be a non-zero polynomial of the degree n. If $(0 : f)$ is essential in $R[x]$ then $\bigcap_{i=0}^{n}(0 : a_i)$ is essential in R.*

Proof: In the above proof we have shown that $(0 : a_n) \leq' R$ and that $f - a_n x^n \in Z(R)[x]$. Continuing by the induction we shall obtain that $(0 : a_i) \leq' R$ for each $i = 0, 1, \ldots, n$, from which the assertion follows immediately. □

Lemma 12. *If I is a left ideal of the ring R such that the left ideal $J = I[x] = R[x]I$ of $R[x]$ essentially contains a finitely generated left ideal $K \leq R[x]$, then I essentially contains a finitely generated left ideal L of the ring R.*

Proof: By the hypothesis there is a finitely generated left ideal $K = \sum_{i=1}^{m} R[x]f_i$ of $R[x]$ which is essential in J. Now if $L \leq R$ is the left ideal of R generated by all the coefficients of all the polynomials f_i, $i = 1, \ldots, m$, then for each element $r \in I \setminus L$ we have a non-zero multiple $0 \neq gr \in K$ and so $gr = \sum_{i=1}^{m} g_i f_i$, where $g_i \in R[x]$ are suitable polynomials. It is now clear that any non-zero coefficient of gr is a non-zero multiple of r which obviously lies in L and the proof is therefore complete. □

Notation. For a polynomial $f \in R[x]$ we denote by $f[i]$ the coefficient of f at the power x^i. Further, if J is a left ideal of $R[x]$ then we denote by $J[i]$ the set consisting of 0 and of all leading coefficients of all polynomials of degree i which lie in J. In other words, $a \in J[i]$ if and only if either $a = 0$ or if there is $f \in J$ such that $f = \sum_{r=0}^{i} a_r x^r$ and $a_i = a$.

Lemma 13. *Let R be a non-singular ring such that Goldie's torsion theory σ is of finite type. If J is an essential left ideal of the polynomial ring $R[x]$, then there is an index $k < \omega$ such that $J[l]$ is essential in R for each $l \geq k$.*

Proof: It is clear that $J[i]$ is a left ideal of R and the obvious equality $(xf)[i+1] = f[i]$ for each $f \in R[x]$ yields the inclusion $J[i] \subseteq J[i+1]$ for each $i < \omega$. It remains now to show that the assumption that all the $J[i]$'s are not essential in R leads to a contradiction. First of all we are going to verify that the non-descending union $\tilde{J} = \bigcup_{i<\omega} J[i]$ is essential in R. Clearly, if $r \in R \setminus \tilde{J}$ is an arbitrary element, then $r \notin J[i]$ for each $i < \omega$. Especially, $r \notin J$ and so there is an element $g \in R[x]$ with $0 \neq gr \in J$. Now if $m \geq 0$ is the degree of the polynomial gr, then for $g = \sum_{j=0}^{n} b_j x^j$ we see that $0 \neq b_m r \in J[m] \subseteq \tilde{J}$, as we wished to show. Further, if all $J[i]$'s are not essential in R, then there is infinitely many indices $l < \omega$ such that $J[l]$ is not essential in $J[l+1]$. Clearly, in the opposite case we see that there is $k < \omega$ such that $J[l]$ is essential in $J[l+1]$ for each $l \geq k$. Now if $0 \neq r \in R$ is an arbitrary element then there is an element $s \in R$ with $0 \neq sr \in \tilde{J}$, hence $0 \neq sr \in J[l]$ for some $l \geq k$ and consequently $0 \neq tsr \in J[k]$ for a suitable element $t \in R$, which means that $J[k]$ is essential in R. Thus to finish the proof let $k_1 < k_2 < \ldots$ be an infinite sequence of integers such that $J[k_i]$ is not essential in $J[k_i + 1]$ for each $i < \omega$. Then there is a left ideal $L_i \leq R$

such that $0 \neq L_i \subseteq J[k_i + 1]$ and $J[k_i] \cap L_i = 0$ for each $i < \omega$. Obviously, the ideals L_i are σ-torsionfree left ideals of R and they form the infinite direct sum $\oplus_{i<\omega} L_i$ in R, which contradicts [12, Theorem 2.1] stating that σ is of finite type if and only if the ring R contains no infinite direct sum of σ-torsionfree left ideals. □

Lemma 14. *If every essential left ideal of a non-singular ring R essentially contains a finitely generated left ideal, then every essential left ideal of $R[x]$ essentially contains a finitely generated left ideal.*

Proof: Let J be an essential left ideal of the polynomial ring $R[x]$. By Lemma 13 there is an index $k < \omega$ such that $J[k]$ is essential in R. With respect to Lemma 1 each left ideal $J[i]$, $i \leq k$, contains an essential finitely generated left ideal $K_i = \sum_{l=1}^{s_i} Ra_{il}$. Now for each a_{il} there is a polynomial $f_{il} \in J$ of the degree i and with the leading coefficient a_{il}. Now we put $K = \sum_{i=0}^{k} \sum_{l=1}^{s_i} R[x]f_{il}$ and we are going to show that K is essential in J. So, let $f = \sum_{i=0}^{s} a_i x^i$ be an element of J of the degree $s \geq 0$. If we set $K_t = K_k$ for each $t \geq k$, then for a suitable element $r_s \in (K_s : a_s) \setminus (0 : a_s)$ there is a polynomial $g_s \in K$ such that $r_s f + g_s$ is of the degree less than s and $r_s f$ is non-zero. Continuing by the induction let us assume that for some $0 < j \leq s$ the elements $r_j \in R$ and $g_j \in K$ have been already constructed in such a way that $r_j a_s \neq 0$ and $r_j f + g_j$ is of the degree less than j. If $r_j f + g_j \in K$ then we are through. In the opposite case we take $\tilde{r}_{j-1} \in R$ such that $\tilde{r}_{j-1} r_j a_s \neq 0$ and $\tilde{r}_{j-1} b_{j-1} \in K_{j-1}$, b_{j-1} being the coefficient at the power x^{j-1} in the polynomial $r_j f + g_j$. Setting $r_{j-1} = \tilde{r}_{j-1} r_j$ we see that $r_{j-1} f \neq 0$ and $r_{j-1} f + \tilde{r}_{j-1} g_j + \tilde{g}_j$ is of the degree less than $j - 1$ for a suitable polynomial $\tilde{g}_j \in K$. Now we set $g_{j-1} = \tilde{r}_{j-1} g_j + \tilde{g}_j$ and it is obvious that after a finite number of steps we shall come to $r_0 f + g_0$ with $r_0 f \neq 0$ and $r_0 f, g_0 \in K$, as we wished to show. □

Remark 15. In our previous paper [4, Corollary 3] it has been especially proved that for any hereditary torsion theory $\tau \geq \sigma$ the following conditions are equivalent: (i) τ is of finite type; (ii) the class of all τ-torsionfree modules is a precover (cover) class; (iii) the class of all τ-torsionfree τ-injective modules is a precover (cover) class; (iv) the class of all τ-exact modules is a precover (cover) class. In the light of this result we shall formulate the following Main Theorem of this note and its consequences for the precover classes, only.

Theorem 16. *The following conditions are equivalent for a ring R:*

(i) *the class of all non-singular left R-modules is a precover class;*
(ii) *the class of all non-singular left $R/\sigma(R)$-modules is a precover class;*
(iii) *the class of all non-singular left $R[x]$-modules is a precover class;*
(iv) *the class of all non-singular left $R[x]/\sigma(R[x])$-modules is a precover class.*

Proof: In view of the above Remark it is obvious that the conditions (i) and (ii) as well as (iii) and (iv) are equivalent by Theorem 5. By Theorem 10 we know that $\sigma(R[x]) = \sigma(R)[x]$ and consequently it is easy to see that $R[x]/\sigma(R[x]) \cong (R/\sigma(R))[x]$. Then (iv) follows from (ii) by Lemma 14, while the converse follows immediately from Lemma 12. □

Corollary 17. *Let R be an arbitrary ring and let $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\sigma(R[x_1, \dots, x_n]), n < \omega\}$ be a countable set of rings. If for some $S \in \mathfrak{M}$ the class of all non-singular left S-modules forms a precover class, then the same holds for each member of the set $\mathfrak{M}$.*

PROOF: It follows immediately from Theorem 16, Remark 15 and the induction principle. □

Lemma 18. *Let R be an arbitrary ring. If Goldie's torsion theory σ_R for the category R-mod is of finite type, then every hereditary torsion theory τ_R for the category R-mod such that $\tau_R \geq \sigma_R$ is of finite type, too.*

PROOF: Goldie's torsion theory $\sigma_{\bar{R}}$ for the category $\bar{R}$-mod of all $R/\sigma(R)$-modules is of finite type by Theorem 5. If $I \in \mathcal{L}_{\tau_R}$ is an arbitrary element then Lemma 4 together with Lemma 1 yields the existence of a finitely generated left ideal $K \leq R$ which is essential in I. Since $\mathcal{T}_{\sigma_R} \subseteq \mathcal{T}_{\tau_R}$, the ideal K lies in $\mathcal{L}_{\tau_R}$ and we are through. □

Corollary 19. *Let R be an arbitrary ring and let $\mathfrak{M} = \{R, R/\sigma(R), R[x_1, \dots, x_n], R[x_1, \dots, x_n]/\sigma(R[x_1, \dots, x_n]), n < \omega\}$ be a countable set of rings. If Goldie's torsion theory σ_R for the category R-mod is of finite type, then for each element $S \in \mathfrak{M}$ and each hereditary torsion theory τ_S for the category S-mod such that $\tau_S \geq \sigma_S$, the class of all τ_S-torsionfree modules is a precover class.*

PROOF: By Theorem 16 and Remark 15 for each element $S \in \mathfrak{M}$ Goldie's torsion theory σ_S is of finite type and the proof is therefore complete, τ_S being of finite type by the preceding lemma. □

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