f-derivations on rings and modules

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Abstract. If τ is a hereditary torsion theory on $\mathbf{Mod}_R$ and $Q_\tau : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is the localization functor, then we show that every f-derivation $d : M \to N$ has a unique extension to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$ when τ is a differential torsion theory on $\mathbf{Mod}_R$. Dually, it is shown that if τ is cohereditary and $C_\tau : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is the colocalization functor, then every f-derivation $d : M \to N$ can be lifted uniquely to an f_τ-derivation $d_\tau : C_\tau(M) \to C_\tau(N)$.

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The purpose of this paper is to study certain derivations on rings and modules and their relation to the concept of a differential torsion theory. Throughout, R will denote an associative ring with identity, all modules will be unitary right R-modules and $\mathbf{Mod}_R$ will denote the category of unitary right R-modules. Also if N is a submodule of an R-module M, then for any $x \in M$, $(N : x)$ will denote the right ideal of R given by $\{a \in R \mid xa \in N\}$. Finally if $f : M \to N$ is an R-linear mapping and X is a submodule of M, then $f|_X$ will denoted f restricted to X.

An additive mapping $\delta : R \to R$ is a *derivation* on R if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. If δ is a derivation on R and $f : M \to N$ is an R-linear mapping, then an additive mapping $d : M \to N$ is a (δ, f)-*derivation* if $d(xa) = d(x)a + f(x)\delta(a)$ for all $x \in M$ and all $a \in R$. We now assume that δ is a fixed but arbitrarily chosen derivation on R. With this in mind, we will refer to a (δ, f)-derivation simply as an f-*derivation* with δ understood. If $f : M \to M$, then $d : M \to M$ is an f-derivation on M and if $f = \mathrm{id}_M$, then d is a *derivation* on M. Note that f-derivations always exist, since if we let $f = 0$, then d is simply an R-linear mapping. Note also that if $d_1, d_2 : M \to N$ are f-derivations, then there is an R-linear mapping $\varphi : M \to N$ such that $d_2 = d_1 + \varphi$ and, conversely, if $d_1 : M \to N$ is an f-derivation and $\varphi : M \to N$ is an R-linear mapping, then $d_1 + \varphi$ is an f-derivation. In the first case, simply let $\varphi = d_2 - d_1$ and, in the case of the converse, direct computation shows that $d_1 + \varphi$ is an f-derivation. Moreover if $\oplus_{\alpha\in\Delta} R_\alpha$ is a free R-module, where $R_\alpha = R$ for each $\alpha \in \Delta$, then a derivation δ on R gives a derivation $d : \oplus_{\alpha\in\Delta} R_\alpha \to \oplus_{\alpha\in\Delta} R_\alpha$ defined by $d((a_\alpha)) = (\delta(a_\alpha))$ for each $(a_\alpha) \in \oplus_{\alpha\in\Delta} R_\alpha$.

Golan proved in [9] that if $d : M \to M$ is a derivation on M, then d can be extended to a derivation $d_\tau : Q_\tau(M) \to Q_\tau(M)$ provided that $d(t_\tau(M)) \subseteq t_\tau(M)$, where $Q_\tau(M)$ denotes the module of quotients of M. It was thus left to determine the type of torsion theory over which it is always possible to extend derivations defined on M to derivations defined on $Q_\tau(M)$ for all R-modules M. In this paper we solve a more general problem by showing that an f-derivation $d : M \to N$ has a unique extension to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$ when τ is a differential torsion theory. This generalizes the results of Golan and extends results in [4].

1. Differential torsion theory

A torsion theory τ on $\mathbf{Mod}_R$ is a pair (T, F) of classes of R-modules such that the following conditions hold.

1. $\mathsf{T} \cap \mathsf{F} = 0$.
2. If $M \to N \to 0$ is an exact sequence in $\mathbf{Mod}_R$ and $M \in \mathsf{T}$, then $N \in \mathsf{T}$.
3. If $0 \to M \to N$ is an exact sequence in $\mathbf{Mod}_R$ and $N \in \mathsf{F}$, then $M \in \mathsf{F}$.
4. For each R-module M, there is a short exact sequence $0 \to T \to M \to F \to 0$ in $\mathbf{Mod}_R$ with $T \in \mathsf{T}$ and $F \in \mathsf{F}$.

It follows that the class T is closed under factor modules, direct sums and extensions and that F is closed under submodules, direct products and extensions. Modules in T will be called *τ-torsion* and those in F are called *τ-torsion free*. If N is a submodule of M such that M/N is τ-torsion, then N will be referred to as a *τ-dense submodule* of M. Each R-module has a largest and necessarily unique τ-torsion submodule given by $t_\tau(M) = \Sigma_{N \in \mathsf{S}} N$, where S is the set of τ-torsion submodules of M. A torsion theory will be called *hereditary* if T is closed under submodules and *cohereditary* if F is closed under factor modules. Standard results and terminology on torsion theory can be found in [5] and [10] while general information on rings and modules can be found in [2].

A nonempty collection $\mathcal{F}$ of right ideals of R is said to be a (Gabriel) *filter* [7] if the following two conditions hold.

1. If $K \in \mathcal{F}$, then $(K : a) \in \mathcal{F}$ for each $a \in R$.
2. If I is a right ideal of R and $K \in \mathcal{F}$ is such that $(I : a) \in \mathcal{F}$ for each $a \in K$, then $I \in \mathcal{F}$.

It can be shown that each filter of right ideals of R also satisfies the following three conditions.

3. If $J \in \mathcal{F}$ and K is a right ideal of R such that $J \subseteq K$, then $K \in \mathcal{F}$.
4. If $J, K \in \mathcal{F}$, then $J \cap K \in \mathcal{F}$.
5. If $J, K \in \mathcal{F}$, then $JK \in \mathcal{F}$.

If $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, then $\mathcal{F}_\tau = \{K \mid K$ is a right ideal of R and $R/K \in \mathsf{T}\}$ is a filter. An element x of an R-module M is said to be a *τ-torsion element* of M if there is a $K \in \mathcal{F}_\tau$ such that $xK = 0$.

The set of all τ-torsion elements of M is the τ-torsion submodule $t_\tau(M)$ of M mentioned earlier. Moreover, an R-module M is τ-torsion if $t_\tau(M) = M$ and τ-torsion free if $t_\tau(M) = 0$. Conversely, if $\mathcal{F}$ is a filter of right ideals of R and $t(M) = \{x \in M \mid xK = 0 \text{ for some } K \in \mathcal{F}\}$, then $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, where $\mathsf{T} = \{M \mid t(M) = M\}$ and $\mathsf{F} = \{M \mid t(M) = 0\}$. It follows that there is a one-to-one correspondence between the hereditary torsion theories on $\mathbf{Mod}_R$ and the filters of right ideals of R.

If $\mathcal{F}$ is a filter of right ideals of R, then $\mathcal{F}$ will be called a *differential filter* if for each $K \in \mathcal{F}$, there is an $I \in \mathcal{F}$ such that $\delta(I) \subseteq K$. If τ is a hereditary torsion theory on $\mathbf{Mod}_R$ and $\mathcal{F}_\tau$ is a differential filter, then τ is said to be a *differential torsion theory*. We note in passing that if $\mathcal{F}$ is a differential filter and $K \in \mathcal{F}$ and $I \in \mathcal{F}$ is such that $\delta(I) \subseteq K$, then I can be selected to be such that $I \subseteq K$. Clearly, if $I \in \mathcal{F}$ is such that $\delta(I) \subseteq K$ and we let $I' = I \cap K$, then $I' \in \mathcal{F}$, $I' \subseteq K$ and $\delta(I') \subseteq K$.

The following examples show that differential torsion theories do indeed exist.

Example 1.1. If R is a commutative ring, then every filter $\mathcal{F}$ of right ideals of R is a differential filter. Indeed if $I \in \mathcal{F}$, then $I^2 \in \mathcal{F}$, so if $a, b \in I$, then $\delta(ab) = \delta(a)b + a\delta(b) \in I$. It follows that $\delta(I^2) \subseteq I$. So the hereditary torsion theory determined by $\mathcal{F}$ is a differential torsion theory.

Example 1.2. Jans has shown in [11] that if $\tau = (T, F)$ is a hereditary torsion theory on $\mathbf{Mod}_R$ such that T is closed under direct products, then there is a necessarily idempotent ideal $I \in \mathcal{F}_\tau$ such that $I \subseteq K$ for each $K \in \mathcal{F}_\tau$. If $ab \in I^2 = I$, then $\delta(ab) = \delta(a)b + a\delta(b) \in I$ and from this we can conclude that $\delta(I) \subseteq K$. Thus τ is a differential torsion theory.

Example 1.3. If R is left perfect, then Alin and Armendariz [1] and Dlab [6] have independently proved that if $\tau = (\mathsf{T}, \mathsf{F})$ is a hereditary torsion theory on $\mathbf{Mod}_R$, then T is closed under direct products. Thus, we see from the previous example that when R is left perfect every hereditary torsion theory on $\mathbf{Mod}_R$ is a differential torsion theory.

Example 1.4. Let S be a multiplicatively closed set of elements of R that is a *right denominator set* ([12]). Then S satisfies:

1. If $(a, s) \in R \times S$, then there is a $(b, t) \in R \times S$ such that $at = sb$.
2. If $sa = 0$ with $s \in S$ and $a \in R$, then $at = 0$ for some $t \in S$.

The set $\mathcal{F} = \{K \mid K \text{ is a right ideal of } R \text{ and } K \cap S \neq \varnothing\}$ is a filter of right ideals of R. If $K \in \mathcal{F}$, let $s \in K \cap S$. Since $(\delta(s), s) \in R \times S$, there is a $(b, t) \in R \times S$ such that $\delta(s)t = sb$. Now $\delta(st) = \delta(s)t + s\delta(t) = sb + s\delta(t) \in sR \subseteq K$, so if $a \in R$, then $\delta(sta) = \delta(st)a + st\delta(a) \in K$. Hence $\delta(stR) \subseteq K$. Therefore $\mathcal{F}$ is a differential filter, so the torsion theory determined by $\mathcal{F}$ is a differential torsion theory.

Lemma 1.5. *Let τ be a hereditary torsion theory on $\mathbf{Mod}_R$. If τ is a differential torsion theory and $d : M \to N$ is an f-derivation, then $d(t_\tau(M)) \subseteq t_\tau(N)$.*

PROOF: Let $d : M \to N$ be an f-derivation and suppose that $x \in t_\tau(M)$. Then $f(x) \in t_\tau(N)$, so $(0 : f(x)) \in \mathcal{F}_\tau$. Hence if τ is a differential torsion theory, then there is an $I \in \mathcal{F}_\tau$ such that $\delta(I) \subseteq (0 : f(x)) \in \mathcal{F}_\tau$. If $a \in K = I \cap (0 : x) \in \mathcal{F}_\tau$, then $xa = f(x)\delta(a) = 0$. Thus, $0 = d(xa) = d(x)a + f(x)\delta(a) = d(x)a$ and so $d(x)K = 0$. Hence $d(x) \in t_\tau(N)$, so $d(t_\tau(M)) \subseteq t_\tau(N)$. □

2. f-derivations and modules of quotients

If τ is a torsion theory on $\mathbf{Mod}_R$, then an R-module $Q_\tau(M)$ together with an R-homomorphism $\varphi_M : M \to Q_\tau(M)$ is said to be a *localization* of M at τ provided that $\ker \varphi_M$ and $\operatorname{coker} \varphi_M$ are τ-torsion and $Q_\tau(M)$ is τ-injective and τ-torsion free. An R-module M is said to be *τ-injective* if $\operatorname{Hom}_R(-, M)$ preserves short exact sequences $0 \to N_1 \to N \to N_2 \to 0$ in $\mathbf{Mod}_R$, where N_2 is a τ-torsion R-module. The module $Q_\tau(M)$, called the *module of quotients* of M, is unique up to isomorphism whenever it can be shown to exist. Ohtake [14] has shown that a localization $\varphi_M : M \to Q_\tau(M)$ exists for every R-module M if and only if the torsion theory is hereditary. It is well known that if τ is hereditary, then we can set $Q_\tau(M) = E_\tau(M/t_\tau(M))$, where $E_\tau(M/t_\tau(M))$ is the *τ-injective envelope* of $M/t_\tau(M)$ ([5], [10]). In this case, if $\eta_M : M \to M/t_\tau(M)$ is the natural mapping and $\mu_M : M/t_\tau(M) \to Q_\tau(M)$ is the canonical injection, then $\varphi_M = \mu_M \eta_M$. For the remainder of this section τ will denote a hereditary torsion theory on $\mathbf{Mod}_R$.

If $d : M \to N$ is an f-derivation and if there is an R-linear mapping $f_\tau : Q_\tau(M) \to Q_\tau(N)$ and an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$ such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "f"'] & Q_\tau(M) \arrow[d, "f_\tau"] \\ N \arrow[r, "\varphi_N"] & Q_\tau(N) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "d"'] & Q_\tau(M) \arrow[d, "d_\tau"] \\ N \arrow[r, "\varphi_N"] & Q_\tau(N) \end{tikzcd}$$

are commutative, then we say that the f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$ is an *extension* of $d : M \to N$ or more simply that d_τ extends d. If both f_τ and d_τ are unique, then d_τ is said to be a *unique extension* of d.

We need the following well-known proposition and corollaries. Brief proofs are provided for the sake of completeness.

Proposition 2.1. *Suppose that N is a τ-torsion free τ-injective R-module. If L is a τ-dense submodule of M and $f : L \to N$ is an R-linear mapping, then there exists a unique R-linear mapping $g : M \to N$ that agrees with f on L.*

PROOF: The fact that N is τ-injective shows that such a map g exists, so we need only show uniqueness. If $g' : M \to N$ is R-linear and also agrees with f on L, then $h : M/L \to N$ defined by $h(x+L) = (g-g')(x)$ for all $x + L \in M/L$ is a well-defined R-linear mapping. Since M/L is τ-torsion and N is τ-torsion free, $h = 0$. Hence $g = g'$. □

Corollary 2.2. *Suppose that N is a τ-torsion free τ-injective R-module. If $K \in \mathcal{F}_\tau$ and $f : K \to N$ is an R-linear mapping, then there is a unique $x \in N$ such that $f(a) = xa$ for all $a \in K$.*

PROOF: There is a unique R-linear mapping $g : R \to N$ that agrees with f on K, so let $x = g(1)$. □

Corollary 2.3. *If $f : M \to N$ is an R-linear mapping, then there is a unique R-linear mapping $f_\tau : Q_\tau(M) \to Q_\tau(N)$ such that the diagram*

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "f"'] & Q_\tau(M) \arrow[d, "f_\tau"] \\ N \arrow[r, "\varphi_N"] & Q_\tau(N) \end{tikzcd}$$

is commutative.

PROOF: Since $f(t_\tau(M)) \subseteq t_\tau(N)$, we have an induced R-linear mapping $f^* : M/t_\tau(M) \to N/t_\tau(N)$ and since $\mu_M(M/t_\tau(M))$ is τ-dense in $Q_\tau(M)$, the proposition shows there is a unique R-linear mapping f_τ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "f"'] & M/t_\tau(M) \arrow[r, "\mu_M"] \arrow[d, "f^*"'] & Q_\tau(M) \arrow[d, "f_\tau"'] \\ N \arrow[r, "\eta_N"] & M/t_\tau(N) \arrow[r, "\mu_M"] & Q_\tau(N) \end{tikzcd}$$

is commutative. □

The proof of the following proposition is similar to but more general than the proof given for the first theorem presented in [9].

Proposition 2.4. *Suppose that $f : M \to N$ is an R-linear mapping, where M is τ-torsion free. If $d : M \to N$ is an f-derivation, then d can be extended to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$.*

PROOF: First, by Corollary 2.3, there is a unique R-linear map $f_\tau : Q_\tau(M) \to Q_\tau(N)$ that extends $f : M \to N$. Since M is τ-torsion free, $\varphi_M : M \to Q_\tau(M)$ is an embedding, so we can identify M with $\varphi_M(M)$ and consider M to be a submodule of $Q_\tau(M)$. Thus M is τ-dense in $Q_\tau(M)$, so for $x \in Q_\tau(M)$ there is a $K \in \mathcal{F}_\tau$ such that $xK \subseteq M$. This gives an additive mapping $h_x : K \to Q_\tau(N)$ defined by

$$h_x(a) = \varphi_N d(xa) - f_\tau(x)\delta(a) \qquad (\#)$$

which is R-linear since

$$\begin{aligned}
h_x(ar) &= \varphi_N d(xar) - f_\tau(x)\delta(ar) \\
&= \varphi_N d(xa)r + \varphi_N f(xa)\delta(r) - f_\tau(x)\delta(a)r - f_\tau(x)a\delta(r) \\
&= \varphi_N d(xa)r + f_\tau(x)a\delta(r) - f_\tau(x)\delta(a)r - f_\tau(x)a\delta(r) \\
&= [\varphi_N d(xa) - f_\tau(x)\delta(a)]r \\
&= h_x(a)r.
\end{aligned}$$

Therefore, by Corollary 2.2, there is a unique $y \in Q_\tau(N)$ such that $h_x(a) = ya$ for all $a \in K$. Note that y is independent of the choice of K. Indeed, suppose that K' is also such that $xK' \subseteq M$. If $h'_x : K' \to Q_\tau(N)$ is defined as in (#), then h_x and h'_x agree on $K \cap K' \in \mathcal{F}_\tau$. Due to Proposition 2.1, $h_x|_{K\cap K'} = h'_x|_{K\cap K'}$ has a unique extension to an R-linear map $h : R \mapsto Q_\tau(N)$. Since h also uniquely extends h_x and h'_x to R, we have $y = y'$.

If $d_\tau : Q_\tau(M) \to Q_\tau(N)$ is defined by $d_\tau(x) = y$, then $h_x(a) = d_\tau(x)a$ for all $a \in K$. We claim that d_τ is an f_τ-derivation. To see this, suppose that $x, x' \in Q_\tau(M)$. Then there are K, $K' \in \mathcal{F}_\tau$ such that $xK \subseteq M$ and $x'K' \subseteq M$. But $K \cap K' \in \mathcal{F}_\tau$ and $(x + x')(K \cap K') \subseteq M$, so we have mappings $h_x : K \to Q_\tau(N)$, $h_{x'} : K' \to Q_\tau(N)$ and $h_{x+x'} : K \cap K' \to Q_\tau(N)$, each defined as in (#). Thus for $a \in K \cap K'$ we see that

$$\begin{aligned}
h_{x+x'}(a) &= \varphi_N d((x + x')a) - f_\tau(x + x')\delta(a) \\
&= \varphi_N d(xa) - f_\tau(x)\delta(a) + \varphi_N d(x'a) - f_\tau(x')\delta(a) \\
&= h_x(a) + h_{x'}(a) \text{ and this implies that} \\
d_\tau(x + x')a &= d_\tau(x)a + d_\tau(x')a \quad \text{for all} \quad a \in K \cap K'.
\end{aligned}$$

Hence $[d_\tau(x + x') - d_\tau(x) - d_\tau(x')](K \cap K') = 0$ which gives $d_\tau(x + x') - d_\tau(x) - d_\tau(x') \in t_\tau(Q_\tau(N)) = 0$. Therefore $d_\tau(x + x') = d_\tau(x) + d_\tau(x')$ and so d_τ is additive. Similarly, if $x \in Q_\tau(M)$ and $r \in R$, then there is a $K \in \mathcal{F}_\tau$ such that $xK \subseteq M$. Moreover $(K : r) \in \mathcal{F}_\tau$. Let $h_x : K \to Q_\tau(N)$ and $h_{xr} : (K : r) \to Q_\tau(N)$ be defined as in (#). If $a \in K \cap (K : r) \in \mathcal{F}_\tau$, then

$$\begin{aligned}
h_{xr}(a) - h_x(ra) &= \varphi_N d(xra) - f_\tau(xr)\delta(a) - \varphi_N d(xra) + f_\tau(x)\delta(ra) \\
&= -f_\tau(x)r\delta(a) + f_\tau(x)\delta(r)a + f_\tau(x)r\delta(a) \\
&= f_\tau(x)\delta(r)a, \quad \text{so} \\
d_\tau(xr)a - d_\tau(x)ra &= f_\tau(x)\delta(r)a \quad \text{for all} \quad a \in K \cap (K : r).
\end{aligned}$$

Therefore $[d_\tau(xr) - d_\tau(x)r - f_\tau(x)\delta(r)](K \cap (K : r)) = 0$ which means that $d_\tau(xr) - d_\tau(x)r - f_\tau(x)\delta(r) \in t_\tau(Q_\tau(N)) = 0$. Thus $d_\tau(xr) = d_\tau(x)r + f_\tau(x)\delta(r)$ and so d_τ is an f_τ-derivation.

Finally, we claim that d_τ extends d. If $x \in M$, then $xR \subseteq M$ and $R \in \mathcal{F}_\tau$. If $h_x : R \to Q_\tau(N)$, then $d_\tau(x)a = h_x(a) = \varphi_N d(xa) - f_\tau(x)\delta(a)$ for all $a \in R$. In particular, if $a = 1$, then $\delta(1) = 0$, so we have $d_\tau(x) = \varphi_N d(x)$. Since we have identified x with $\varphi_M(x)$ under the injective mapping φ_M, we can replace x by $\varphi_M(x)$ to get $d_\tau \varphi_M(x) = \varphi_N d(x)$. Thus d_τ extends d, as asserted. □

We now come to the main result of this section.

Proposition 2.5. *If τ is a differential torsion theory τ on $\mathbf{Mod}_R$, then an f-derivation $d : M \to N$ can be extended uniquely to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$.*

Proof: Let $d : M \to N$ be an f-derivation, suppose that τ is a differential torsion theory on $\mathbf{Mod}_R$ and let $f^* : M/t_\tau(M) \to N/t_\tau(N)$ be the R-linear mapping induced by f. Calling on Lemma 1.5 we see that $d(t_\tau(M)) \subseteq t_\tau(N)$, so $d^* : M/t_\tau(M) \to N/t_\tau(N)$ defined by $d^*(x + t_\tau(M)) = d(x) + t_\tau(N)$ is an f^*-derivation. Moreover $M/t_\tau(M)$ is τ-torsion free, so Proposition 2.4 shows that d^* extends to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(N)$. Since the diagrams

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "f"'] & M/t_\tau(M) \arrow[r, "\mu_M"] \arrow[d, "f^*"'] & Q_\tau(M) \arrow[d, "f_\tau"'] \\ N \arrow[r, "\eta_N"] & M/t_\tau(N) \arrow[r, "\mu_M"] & Q_\tau(N) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "d"'] & M/t_\tau(M) \arrow[r, "\mu_M"] \arrow[d, "d^*"'] & Q_\tau(M) \arrow[d, "d_\tau"'] \\ N \arrow[r, "\eta_N"] & M/t_\tau(N) \arrow[r, "\mu_M"] & Q_\tau(N) \end{tikzcd}$$

are commutative, it follows f_τ extends f uniquely and that d_τ extends d. To show uniqueness of d_τ, suppose that $x \in Q_\tau(M)$ and that $\bar{d}_\tau : Q_\tau(M) \to Q_\tau(N)$ is also an f_τ-derivation that extends d. Then $(d_\tau - \bar{d}_\tau)\varphi_M(M) = \varphi_N(d - d)(M) = 0$ gives $(d_\tau - \bar{d}_\tau)(x(\varphi(M) : x)) = 0$. But $d_\tau - \bar{d}_\tau$ is an R-linear mapping, so we have $(d_\tau - \bar{d}_\tau)(x)(\varphi(M) : x) = 0$. Hence $(d_\tau - \bar{d}_\tau)(x) \in t_\tau(Q_\tau(N)) = 0$ and consequently $d_\tau = \bar{d}_\tau$. □

Corollary 2.6. *If τ is a differential torsion theory τ on $\mathbf{Mod}_R$, then an f-derivation $d : M \to M$ can be extended uniquely to an f_τ-derivation $d_\tau : Q_\tau(M) \to Q_\tau(M)$.*

Corollary 2.7. *If τ is a differential torsion theory τ on $\mathbf{Mod}_R$, then a derivation $d : M \to M$ can be extended uniquely to derivation $d_\tau : Q_\tau(M) \to Q_\tau(M)$.*

If $g : L \to M$ is R-linear and $d : M \to N$ is an f-derivation, then a direct computation shows that $dg : L \to N$ is an fg-derivation. Similarly, if $d : L \to M$ is an f-derivation and $g : M \to N$ is R-linear, then $gd : L \to N$ is a gf-derivation. We also have the following proposition.

Proposition 2.8. *If τ is a differential torsion theory on $\mathbf{Mod}_R$, let $L \xrightarrow{f} M \xrightarrow{f'} N$ be a sequence of R-module homomorphisms such that $f'f = 0$ and suppose that*

$d : L \to M$ and $d' : M \to N$ are f-derivations and f'-derivations, respectively. Then there exists an R-linear mapping $g : L \to N$ such that $d'd : L \to M$ is a g-derivation. Moreover, the sequence $Q_\tau(L) \xrightarrow{f_\tau} Q_\tau(M) \xrightarrow{f'_\tau} Q_\tau(N)$ is such that $f'_\tau f_\tau = 0$ and $d'_\tau d_\tau : Q_\tau(L) \to Q_\tau(N)$ is a g_τ-derivation that extends $d'd : L \to N$.

PROOF: Since f, f', d and d' are additive mappings, it is obvious that $g = f'd + d'f$ is additive, so suppose that $x \in M$ and $a \in R$. Then

$$\begin{aligned} g(xa) &= (f'd + d'f)(xa) = f'd(xa) + d'f(xa) \\ &= f'[d(x)a + f(x)\delta(a)] + d'[f(x)a] \\ &= f'd(x)a + f'f(x)\delta(a) + d'f(x)a + f'f(x)\delta(a) \\ &= [f'd + d'f](x)a = g(x)a, \end{aligned}$$

so g is R-linear. Note next that

$$\begin{aligned} d'd(xa) &= d'[d(x)a + f(x)\delta(a)] \\ &= d'd(x)a + f'd(x)\delta(a) + d'f(x)\delta(a) + f'f(x)\delta^2(a) \\ &= d'd(x)a + [f'd + d'f](x)\delta(a) = d'd(x)a + g(x)\delta(a), \end{aligned}$$

so since $d'd$ is clearly additive, we see that $d'd$ is a g-derivation.

Finally, if we can show that $f'_\tau f_\tau = 0$, then the fact that $d'_\tau d_\tau$ is an extension of $d'd$ will follow from what was demonstrated above and the fact that Corollary 2.3 and Proposition 2.5 give $g_\tau = f'_\tau d_\tau + d'_\tau f_\tau$. If $x \in Q_\tau(L)$, then there is a $K \in \mathcal{F}_\tau$ such that $xK \subseteq \varphi_L(L)$. If $xk \in xK$, let $y \in L$ be such that $\varphi_L(y) = xk$, then $f'f(y) = 0$. Now the diagram

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "\varphi_L"'] & M \arrow[r, "f'"] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"'] \\ Q_\tau(L) \arrow[r, "f_\tau"] & Q_\tau(M) \arrow[r, "f'_\tau"] & Q_\tau(M) \end{tikzcd}$$

is commutative, so $f_\tau(xk) = f_\tau\varphi_L(y) = \varphi_M f(y)$. Therefore $f'_\tau f_\tau(x)k = f'_\tau f_\tau(xk) = f'_\tau\varphi_M f(y) = \varphi_N f'f(y) = 0$. Hence $f'_\tau f_\tau(x)K = 0$ and so $f'_\tau f_\tau(x) \in t_\tau(Q_\tau(N)) = 0$. Thus, $f'_\tau f_\tau = 0$. $\square$

3. f-derivations and modules of coquotients

In this section we develop results for colocalizations of modules that are similar to but dual to the results of the previous section. Colocalizations have been investigated under various approaches by several authors, for example see [3], [8] and [13].

An R-module $C_\tau(M)$ together with an R-linear mapping $\psi_M : C_\tau(M) \to M$ is said to be a *colocalization* of M at τ provided that $\ker \psi_M$ and $\operatorname{coker} \psi_M$ are τ-torsion free and $C_\tau(M)$ is τ-torsion and τ-projective. If $\psi_M : C_\tau(M) \to M$ is a colocalization of M at τ, then $C_\tau(M)$ is called a *module of coquotients* of M. An R-module M is *τ-projective* if $\operatorname{Hom}_R(M, -)$ preserves short exact sequences $0 \to N_1 \to N \to N_2 \to 0$ in $\mathbf{Mod}_R$, where N_1 is a τ-torsion free R-module. Ohtake proved in [14] that a torsion theory τ is cohereditary if and only if every R-module M has a colocalization at τ. If $\psi_M : C_\tau(M) \to M$ is a colocalization of M at τ, then there is an R-epimorphism $\pi_M : C_\tau(M) \to t_\tau(M)$ such that if $\mu_M : t_\tau(M) \to M$ is the canonical injection, then $\psi_M = \mu_M \pi_M$. Furthermore, a module of coquotients is unique up to isomorphism whenever it can be shown to exist.

If $\psi_M : C_\tau(M) \to M$ and $\psi_N : C_\tau(N) \to N$ are colocalizations of M and N at τ, respectively, and $d : M \to N$ is an f-derivation, then we will say that an f_τ-derivation $d_\tau : C_\tau(M) \to C_\tau(N)$ *lifts* d, provided that the diagrams

$$\begin{tikzcd} C_\tau(M) \arrow[r, "\psi_M"] \arrow[d, "f_\tau"'] & M \arrow[d, "f"] \\ C_\tau(N) \arrow[r, "\psi_N"] & N \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} C_\tau(M) \arrow[r, "\psi_M"] \arrow[d, "d_\tau"'] & M \arrow[d, "d"] \\ C_\tau(N) \arrow[r, "\psi_N"] & N \end{tikzcd}$$

are commutative. If f_τ and d_τ are both unique, then d_τ is said to *lift d uniquely*.

When $\tau = (\mathsf{T}, \mathsf{F})$ is cohereditary, the class F of τ is both a torsion and a torsion free class, and the class F generates a hereditary torsion theory $\sigma = (\mathsf{F}, \mathsf{D})$ on $\mathbf{Mod}_R$, where $\mathsf{D} = \{N \mid \operatorname{Hom}_R(M, N) = 0 \text{ for all } M \in \mathsf{F}\}$. The pair (τ, σ) is often referred to as a TTF theory. Jans has shown in [11] that there is a one-to-one correspondence between TTF theories and idempotent ideals I of R. If (τ, σ) is a TTF theory with corresponding idempotent ideal I, then in this setting, $t_\tau(R) = I$ and $t_\tau(M) = MI$ for each R-module M.

Sato has shown in [15] that if (τ, σ) is a TTF theory with corresponding idempotent ideal I, then $I \otimes_R I \xrightarrow{\pi} I \xrightarrow{\mu} R$ is a colocalization of R, where the map $\pi : I \otimes_R I \to I$ is given by $\Sigma_{i=1}^n (a_i \otimes b_i) \mapsto \Sigma_{i=1}^n a_i b_i$. Furthermore $I \otimes_R I$ is a ring, possibly without an identity, and an (R, R)-bimodule. Sato also shows in [15] that $M \otimes_R I \otimes_R I \xrightarrow{\pi} MI \xrightarrow{\mu} M$ is a colocalization of M at τ. In this case, the map $\pi : M \otimes_R I \otimes_R I \to MI$ is such that $\Sigma_{i-1}^n (x_i \otimes a_i \otimes b_i) \mapsto \Sigma_{i=1}^n x_i a_i b_i$. Since I is an idempotent ideal, $\delta(I) \subseteq I$ and $d(MI) \subseteq NI$ for each f-derivation $d : M \to N$. Hence, δ and d restricted to I and MI produces a derivation on $\delta : I \to I$ and an f-derivation $d : MI \to NI$ which we also denote by δ and d.

We need the following lemma to prove the main result of this section.

Lemma 3.1. *Let $f : M \to N$ be an R-linear mapping and suppose that $d : M \to N$ is an f-derivation and that I is an idempotent ideal of R. Then* *the map* $\rho' : M \times I \times I \to N \otimes_R I \otimes_R I$ *given by*

$$\rho'((x, a, b)) = d(x) \otimes a \otimes b + f(x) \otimes \delta(a) \otimes b + f(x) \otimes a \otimes \delta(b)$$

is R*-balanced. That is,* ρ' *is additive in each variable and such that* $\rho'((xr, a, b)) = \rho'((x, ra, b))$ *and* $\rho'((x, ar, b)) = \rho'((x, a, rb))$ *for all* $(x, a, b) \in M \times I \times I$ *and all* $r \in R$.

PROOF: Since f, d and δ are additive, it is easy to see that ρ' is additive in each variable. We show $\rho'((xr, a, b)) = \rho'((x, ra, b))$ with a similar proof holding for $\rho'((x, ar, b)) = \rho'((x, a, rb))$. If $(x, a, b) \in M \times I \times I$ and $r \in R$, then

$$\begin{aligned}
\rho'((xr, a, b)) &= d(xr) \otimes a \otimes b + f(xr) \otimes \delta(a) \otimes b + f(xr) \otimes a \otimes \delta(b) \\
&= d(x)r \otimes a \otimes b + f(x)\delta(r) \otimes a \otimes b + f(x)r \otimes \delta(a) \otimes b \\
&\quad + f(x)r \otimes a \otimes \delta(b) \\
&= d(x) \otimes ra \otimes b + f(x) \otimes [\delta(r)a + r\delta(a)] \otimes b + f(x) \otimes ra \otimes \delta(b) \\
&= d(x) \otimes ra \otimes b + f(x) \otimes \delta(ra) \otimes b + f(x) \otimes ra \otimes \delta(b) \\
&= \rho'((x, ra, b))
\end{aligned}$$

which completes the proof. □

Proposition 3.2. *If* τ *is a cohereditary torsion theory on* $\mathbf{Mod}_R$*, then each* f*-derivation* $d : M \to N$ *lifts uniquely to an* f_τ*-derivation* $d_\tau : C_\tau(M) \to C_\tau(N)$.

PROOF: If τ is a cohereditary torsion theory, let I be the idempotent ideal corresponding to the TTF theory (τ, σ). If $d : M \to N$ is an f-derivation, then we have a commutative diagram

$$\begin{tikzcd}
M \times I \times I \arrow[r, "\rho"] \arrow[dr, "\rho'"'] & M \otimes_R I \otimes_R I \arrow[d, "d_\tau"] \\
& N \otimes_R I \otimes_R I
\end{tikzcd}$$

where $\rho : M \times I \times I \to M \otimes_R I \otimes_R I$ is the canonical R-balanced map given by $\rho((x, a, b)) = x \otimes a \otimes b$, ρ' is the R-balanced map of Lemma 3.1 and d_τ is the unique group homomorphism produced by the tensor product $M \otimes_R I \otimes_R I$. Now consider the diagram

$$\begin{tikzcd}
M \otimes_R I \otimes_R I \arrow[r, "\pi_M"] \arrow[d, "d_\tau"'] & MI \arrow[r, "\mu_M"] \arrow[d, "d"] & M \arrow[d, "d"] \\
N \otimes_R I \otimes_R I \arrow[r, "\pi_N"] & NI \arrow[r, "\mu_N"] & N.
\end{tikzcd}$$

Since $\psi_M = \mu_M \pi_M$, where $\pi_M : M \otimes_R I \otimes_R I \to M$ is such that $\pi_M(\Sigma_{i=1}^n (x_i \otimes a_i \otimes b_i)) = \Sigma_{i=1}^n x_i a_i b_i$ and $\mu_M : MI \to M$ is the canonical injection, we see that $\psi_M(\Sigma_{i=1}^n (x_i \otimes a_i \otimes b_i)) = \Sigma_{i=1}^n x_i a_i b_i$ for each $\Sigma_{i=1}^n (x_i \otimes a_i \otimes b_i)$ in $M \otimes_R I \otimes_R I$ with a similar observation holding for ψ_N. So if $x \otimes a \otimes b$ is a generator of $M \otimes_R I \otimes_R I$, then

$$\begin{aligned}
\psi_N d_\tau(x \otimes a \otimes b) &= \psi_N \rho'((x,a,b)) \\
&= \psi_N[d(x) \otimes a \otimes b) + f(x) \otimes \delta(a) \otimes b + f(x) \otimes a \otimes \delta(b)] \\
&= d(x)ab + f(x)\delta(a)b + f(x)a\delta(b) \\
&= d(x)ab + f(x)[\delta(a)b + a\delta(b)] \\
&= d(x)ab + f(x)\delta(ab) \\
&= d(xab) \\
&= d\psi_M(x \otimes a \otimes b).
\end{aligned}$$

Since $\psi_N d_\tau$ and $d\psi_M$ are additive functions, this proves that $\psi_N d_\tau = d\psi_M$, so the diagram

$$\begin{tikzcd}
M\otimes_R I \otimes_R I \arrow[r, "\psi_M"] \arrow[d, "d_\tau"'] & M \arrow[d, "d"] \\
N\otimes_R I \otimes_R I \arrow[r, "\psi_N"] & N
\end{tikzcd}$$

is commutative. Finally, if $\rho : M \times I \times I \to M \otimes_R I \otimes_R I$ is the canonical R-balanced map and if $\bar{f} : M \times I \times I \to N \otimes_R I \otimes_R I$ is the R-balanced map given by $\bar{f}((x,a,b)) = f(x) \otimes a \otimes b$, then we have a commutative diagram

$$\begin{tikzcd}
M \times I \times I \arrow[r, "\rho"] \arrow[dr, "\bar{f}"'] & M\otimes_R I \otimes_R I \arrow[d, "f_\tau"] \\
& N\otimes_R I \otimes_R I
\end{tikzcd}$$

so there is a unique group homomorphism $f_\tau : M \otimes_R I \otimes_R I \to N \otimes_R I \otimes_R I$ such that $f_\tau(x \otimes a \otimes b) = f(x) \otimes a \otimes b$ for each generator $x \otimes a \otimes b$ of $M \otimes_R I \otimes_R I$. If $M \otimes_R I \otimes_R I$ and $N \otimes_R I \otimes_R I$ are viewed as R-modules, then the map f_τ is clearly R-linear and if $r \in R$, then

$$\begin{aligned}
d_\tau((x \otimes a \otimes b)r) &= d_\tau(x \otimes a \otimes br) \\
&= d(x) \otimes a \otimes br + f(x) \otimes \delta(a) \otimes br + f(x) \otimes a \otimes \delta(br) \\
&= d(x) \otimes a \otimes br + f(x) \otimes \delta(a) \otimes br + f(x) \otimes a \otimes \delta(b)r \\
&\quad + f(x) \otimes a \otimes b\delta(r) \\
&= [d(x) \otimes a \otimes b + f(x) \otimes \delta(a) \otimes b + f(x) \otimes a \otimes \delta(b)]r \\
&\quad + (f(x) \otimes a \otimes b)\delta(r) \\
&= d_\tau(x \otimes a \otimes b)r + f_\tau(x \otimes a \otimes b)\delta(r).
\end{aligned}$$

Therefore, since d_τ is additive, we see that d_τ is an f_τ-derivation that lifts d. In view of how the maps ρ' and $\bar{f}$ are defined and due to the fact that f_τ and d_τ are unique group homomorphisms, we also see that d_τ lifts d uniquely. $\square$

We also have the following proposition whose proof is similar but dual to that of Proposition 2.8.

Proposition 3.3. *If τ is a cohereditary torsion theory on $\mathbf{Mod}_R$, let $L \xrightarrow{f} M \xrightarrow{f'} N$ be a sequence of R-modules and R-module homomorphisms such that $f'f = 0$ and suppose that $d : L \to M$ and $d' : M \to N$ are f-derivations and f'-derivations, respectively. Then there exists an R-linear mapping $g : L \to N$ such that $d'd : L \to N$ is a g-derivation. Moreover, the sequence $C_\tau(L) \xrightarrow{f_\tau} C_\tau(M) \xrightarrow{f'_\tau} C_\tau(N)$ is such that $f'_\tau f_\tau = 0$ and $d'_\tau d_\tau : C_\tau(L) \to C_\tau(N)$ is a g_τ-derivation that lifts $d'd : L \to N$.*

References

[1] Alin J.S., Armendariz E.P., *TTF classes over perfect rings*, J. Austral. Math. Soc. **11** (1970), 499–503.

[2] Anderson F.W., Fuller K.R., *Rings and Categories of Modules*, Springer, Berlin, 1973.

[3] Beachy J.A., *Cotorsion radical and projective modules*, Bull. Austral Math. Soc. **5** (1971), 241–253.

[4] Bland P.E., *Differential torsion theory*, J. Pure Appl. Algebra **204** (2006), no. 1, 1–8.

[5] Bland P.E., *Topics in Torsion Theory*, Mathematical Research 103, Wiley-VCH, Berlin, 1998.

[6] Dlab V., *A characterization of perfect rings*, Pacific J. Math. **33** (1970), 79–88.

[7] Gabriel P., *Des catégories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[8] Bronn S.D., *Cotorsion theories*, Pacific J. Math. **48** (1973), 355–363.

[9] Golan J.S., *Extension of derivations to modules of quotients*, Comm. Algebra **9** (1981), no. 3, 275–281.

[10] Golan J.S., *Torsion Theories*, Pitman Monographs and Surveys in Pure and Applied Mathematics 29, Longman Scientific and Technical, Harlow, 1986.

[11] Jans J.P., *Some aspects of torsion*, Pacific J. Math. **15** (1965), 1249–1259.

[12] Lam T.Y., *Lecture on Modules and Rings*, Graduate Texts in Mathematics 189, Springer, New York, 1999.

[13] McMaster R.J., *Cotorsion theories and colocalization*, Canad. J. Math. **27** (1971), 618–628.

[14] Ohtake K., *Colocalization and localization*, J. Pure Appl. Algebra **11** (1977), 217–241.

[15] Sato M., *The concrete description of colocalization*, Proc. Japan Acad. **52** (1976), 501–504.

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