On the "zero-two" law for positive contractions in the Banach-Kantorovich lattice $L^p(\nabla, \mu)$

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Abstract. In the present paper we prove the "zero-two" law for positive contractions in the Banach-Kantorovich lattices $L^p(\nabla, \mu)$, constructed by a measure μ with values in the ring of all measurable functions.

Keywords: Banach-Kantorovich lattice, "zero-two" law, positive contraction

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1. Introduction

In [W] some properties of the convergence of Banach-valued martingales were described and their connections with the geometrical properties of Banach spaces were established too. In accordance with the development of the theory of Banach-Kantorovich spaces (see [KVP], [K1], [K2], [G1], [G2]) there arises naturally the necessity to study some ergodic properties of positive contractions and martingales defined on such spaces. In [CG] an analog of the individual ergodic theorem for positive contractions of $L^p(\nabla, \mu)$ - Banach-Kantorovich space has been established. In [Ga3] the convergence of martingales on such spaces was proved.

Let (X, Σ, μ) be a measure space and let $L^p(X, \mu)$ $(1 \leq p \leq \infty)$ be the usual real L^p-space. A linear operator $T : L^p(X, \mu) \to L^p(X, \mu)$ is called a *positive contraction* if for every $x \in L^p(X, \mu)$, $x \geq 0$, we have $Tx \geq 0$ and $\|T\|_p \leq \mathbb{1}$, where $\|T\|_p = \sup_{x:\|x\|_p=\mathbb{1}} \|Tx\|_p$.

In [OS] Ornstein and Sucheston proved that for any positive contraction T on an L^1-space, either $\|T^n - T^{n+1}\|_1 = 2$ for all n or $\lim_{n\to\infty} \|T^n - T^{n+1}\|_1 = 0$. An extension of this result to positive operators on L^∞-spaces was given by Foguel [F]. In [Z1], [Z2] Zahoropol generalized these results, called *"zero-two" laws*, and his result can be formulated as follows:

Theorem A. *Let T be a positive contraction of $L^p(X, \mu)$, $p > 1, p \neq 2$. If $\big\||T^{m+1} - T^m|\big\|_p < 2$ for some $m \in \mathbb{N} \cup \{0\}$, then*

$$\lim_{n\to\infty} \|T^{n+1} - T^n\|_p = 0.$$

In [KT] this result was generalized to Köthe spaces.

In the present paper we are going to prove the "zero-two" law for positive contractions of the Banach-Kantorovich lattices $L^p(\nabla,\mu)$, constructed by means of a measure μ with values in the ring of all measurable functions.

2. Preliminaries

Let (Ω,Σ,λ) be a measurable space with finite measure λ, $L_0(\Omega)$ be the algebra of all measurable functions on Ω (here the functions equal a.e. are identified) and let $\nabla(\Omega)$ be the Boolean algebra of all idempotents in $L_0(\Omega)$. By ∇ we denote an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$.

A mapping $\mu:\nabla\to L_0(\Omega)$ is called an $L_0(\Omega)$-*valued measure* if the following conditions are satisfied:

1) $\mu(e)\geq 0$ for all $e\in\nabla$;

2) if $e\wedge g=0, e,g\in\nabla$, then $\mu(e\vee g)=\mu(e)+\mu(g)$;

3) if $e_n\downarrow 0,\ e_n\in\nabla,\ n\in\mathbb{N}$, then $\mu(e_n)\downarrow 0$.

An $L_0(\Omega)$-valued measure μ is called *strictly positive* if $\mu(e)=0$, $e\in\nabla$ implies $e=0$.

In the sequel we will consider a strictly positive $L_0(\Omega)$-valued measure μ with the property $\mu(ge)=g\mu(e)$ for all $e\in\nabla$ and $g\in\nabla(\Omega)$.

By $X(\nabla)$ we denote an extremal completely disconnected compact, corresponding to a Boolean algebra ∇. The algebra of all continuous functions on $X(\nabla)$, which take the values $\pm\infty$ on nowhere dense sets in $X(\nabla)$, is denoted by $L_0(\nabla)$ ([S]). It is clear that $L_0(\Omega)$ is a subalgebra of $L_0(\nabla)$.

Following [B], [S] the well known scheme of the construction of L^p-spaces, a space $L^p(\nabla,\mu)$ can be defined in the following way:

$$L^p(\nabla,\mu)=\left\{f\in L_0(\nabla):\int|f|^p d\mu \ \text{ exists}\right\},\quad p\geq 1,$$

where μ is an $L_0(\Omega)$-valued measure on ∇.

Let E be a linear space over the real field $\mathbb{R}$. By $\|\cdot\|$ we denote an $L_0(\Omega)$-valued norm on E. Then the pair $(E,\|\cdot\|)$ is called a *lattice-normed space* (LNS) *over* $L_0(\Omega)$. An LNS E is said to be d-*decomposable* if for every $x\in E$ and the decomposition $\|x\|=f+g$ with f and g disjoint positive elements in $L_0(\Omega)$ there exist $y,z\in E$ such that $x=y+z$ with $\|y\|=f$, $\|z\|=g$.

Suppose that $(E,\|\cdot\|)$ is an LNS over $L_0(\Omega)$. A net $\{x_\alpha\}$ of elements of E is said to be (bo)-*converging* to $x\in E$ (in this case we write $x=(bo)\text{-}\lim x_\alpha$), if the net $\{\|x_\alpha-x\|\}$ (o)-converges to zero in $L_0(\Omega)$ (written as $(o)\text{-}\lim\|x_\alpha-x\|=0$). A net $\{x_\alpha\}_{\alpha\in A}$ is called (bo)-*fundamental* if $(x_\alpha-x_\beta)_{(\alpha,\beta)\in A\times A}$ (bo)-converges to zero.

An LNS in which every (bo)-fundamental net (bo)-converges is called (bo)-*complete*. A *Banach-Kantorovich space* (BKS) *over* $L_0(\Omega)$ is a (bo)-complete

d-decomposable LNS over $L_0(\Omega)$. It is well known ([K1], [K2]) that every BKS E over $L_0(\Omega)$ admits an $L_0(\Omega)$-module structure such that $\|fx\| = |f| \cdot \|x\|$ for every $x \in E$, $f \in L_0(\Omega)$, where $|f|$ is the modulus of a function $f \in L_0(\Omega)$.

It is known ([K1]) that $L^p(\nabla, \mu)$ is a BKS over $L_0(\Omega)$ with respect to the $L_0(\Omega)$-valued norm $|f|_p = \big(\int |f|^p d\mu\big)^{1/p}$. Moreover, $L^p(\nabla, \mu)$ is a module over $L_0(\Omega)$.

Naturally, these $L^p(\nabla, \mu)$ spaces should have many of similar properties like the classical L^p-spaces, constructed by real valued measures. The proofs of such properties can be realized along the following ways.

1. Repeating step by step all the steps of the known arguments of the classical L^p-spaces, taking into account the special properties of $L_0(\Omega)$-valued measures.
2. Using Boolean-valued analysis, which gives a possibility to reduce $L_0(\Omega)$-modulus $L^p(\nabla, \mu)$ to the classical L^p-spaces, in the corresponding set theory.
3. Representating $L^p(\nabla, \mu)$ as a measurable bundle of the classical L^p-spaces.

The first method is not really effective, since it has to repeat all known steps of the arguments modifying them to $L_0(\Omega)$-valued measures. The second one is connected with the use of drawing an enough labour-intensive apparatus of Boolean-valued analysis and its realization requires a huge preparatory work, which connects with establishing intercommunications of ordinary and Boolean-valued methods for the studied objects of the set theory.

A more natural way to investigate the properties of $L^p(\nabla, \mu)$ is to follow the third one, since one has a sufficiently well explored theory of measurable decompositions of Banach lattices ([G1]). Hence, it is an effective tool which gives a good opportunity to obtain various properties of BKS ([Ga1], [Ga2]). Therefore we are going to follow this way, and now recall certain definitions and results of the theory.

Let $(\Omega, \Sigma, \lambda)$ be as above and X be a real Banach space $X(\omega)$ assigned to each point $\omega \in \Omega$. A *section* of X is a function u defined λ-almost everywhere in Ω that takes values $u(\omega) \in X(\omega)$ for all ω in the domain $\mathrm{dom}(u)$ of u. Let L be a set of sections. The pair (X, L) is called a *measurable Banach bundle over* Ω if

(1) $\alpha_1 u_1 + \alpha_2 u_2 \in L$ for every $\alpha_1, \alpha_2 \in \mathbb{R}$ and $u_1, u_2 \in L$, where $\alpha_1 u_1 + \alpha_2 u_2 : \omega \in \mathrm{dom}(u_1) \cap \mathrm{dom}(u_2) \to \alpha_1 u_1(\omega) + \alpha_2 u_2(\omega)$;
(2) the function $\|u\| : \omega \in \mathrm{dom}(u) \to \|u(\omega)\|_{X(\omega)}$ is measurable for every $u \in L$;
(3) the set $\{u(\omega) : u \in L, \omega \in \mathrm{dom}(u)\}$ is dense in $X(\omega)$ for every $\omega \in \Omega$.

A section s is called *step-section* if it has a form

$$s(\omega) = \sum_{i=1}^{n} \chi_{A_i}(\omega) u_i(\omega),$$

for some $u_i \in L$, $A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i,j = 1,\dots,n$, $n \in \mathbb{N}$, where χ_A is the indicator of a set A. A section u is called *measurable* if for every $A \in \Sigma$ with $\lambda(A) < \infty$ there exists a sequence of step-functions $\{s_n\}$ such that $s_n(\omega) \to u(\omega)$ λ-a.e. on A.

Denote by $M(\Omega, X)$ the set all measurable sections, and by $L_0(\Omega, X)$ the factor space of $M(\Omega, X)$ with respect to the equivalence relation of the equality a.e. Clearly, $L_0(\Omega, X)$ is an $L_0(\Omega)$-module. The equivalence class of an element $u \in M(\Omega, X)$ is denoted by $\hat{u}$. The norm of $\hat{u} \in L_0(\Omega, X)$ is defined as a class of equivalence in $L_0(\Omega)$ containing the function $\|u(\omega)\|_{X(\omega)}$, namely $\|\hat{u}\| = (\widehat{\|u(\omega)\|_{X(\omega)}})$. In [G1] it was proved that $L_0(\Omega, X)$ is a BKS over $L_0(\Omega)$. Furthermore, for every BKS E over $L_0(\Omega)$ there exists a measurable Banach bundle (X, L) over Ω such that E is isomorphic to $L_0(\Omega)$.

Put

$$\begin{aligned}\mathcal{L}^\infty(\Omega, X) &= \{u \in M(\Omega, X) : \|u(\omega)\|_{X(\omega)} \in \mathcal{L}^\infty(\Omega)\},\\ L^\infty(\Omega, X) &= \{\hat{u} \in L_0(\Omega, X) : u \in \mathcal{L}^\infty(\Omega, X),\ u \in \hat{u}\},\end{aligned}$$

where $\mathcal{L}^\infty(\Omega)$ is the set all bounded measurable functions on Ω.

In the spaces $\mathcal{L}^\infty(\Omega, X)$ and $L^\infty(\Omega, X)$ one can define real-valued norms $\|u\|_{\mathcal{L}^\infty(\Omega,X)} = \sup_\omega \|u(\omega)\|_{X(\omega)}$ and $\|\hat{u}\|_{L^\infty(\Omega,X)} = \||\hat{u}|\|_{L^\infty(\Omega)}$, respectively.

A BKS $(\mathcal{U}, \|\cdot\|)$ is called a *Banach-Kantorovich lattice* if $\mathcal{U}$ is a vector lattice and the norm $\|\cdot\|$ is monotone, i.e. $|u_1| \leq |u_2|$ implies $\|u_1\| \leq \|u_2\|$. It is known ([K1]) that the cone $\mathcal{U}_+$ of positive elements is (bo)-closed. Note that the space $L^p(\nabla, \mu)$ is a Banach-Kantorovich lattice ([K1]).

Let X be a mapping assisting an L^p-space constructed by a real-valued measure μ_ω, i.e. $L^p(\nabla_\omega, \mu_\omega)$ to each point $\omega \in \Omega$ and let

$$L = \left\{ \sum_{i=1}^{n} \alpha_i e_i : \alpha_i \in \mathbb{R},\quad e_i(\omega) \in \nabla_\omega,\ i = \overline{1,n},\ n \in \mathbb{N} \right\}$$

be a set of sections. In [Ga2], [GaC] it has been established that the pair (X, L) is a measurable bundle of Banach lattices and $L_0(\Omega, X)$ is modulo ordered isomorphic to $L^p(\nabla, \mu)$.

Let ρ be a lifting in $L^\infty(\Omega)$ (see [G1]). Let as before ∇ be an arbitrary complete Boolean subalgebra of $\nabla(\Omega)$ and μ be an $L_0(\Omega)$-valued measure on ∇. The set of all essentially bounded functions w.r.t. μ taken from $L_0(\nabla)$ is denoted by $L^\infty(\nabla, \mu)$.

In [CG] the existence of a mapping $\ell : L^\infty(\nabla, \mu)(\subset L^\infty(\Omega, X)) \to \mathcal{L}^\infty(\Omega, X)$, which satisfies the following conditions, was proved:

(1) $\ell(\hat{u}) \in \hat{u}$ for all $\hat{u}$ such that $\mathrm{dom}(\hat{u}) = \Omega$;
(2) $\|\ell(\hat{u})\|_{L^p(\nabla_\omega, \mu_\omega)} = \rho(|\hat{u}|_p)(\omega)$;

(3) $\ell(\hat{u}+\hat{v}) = \ell(\hat{u}) + \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L^\infty(\nabla, \mu)$;
(4) $\ell(h \cdot \hat{u}) = \rho(h)\ell(\hat{u})$ for every $\hat{u} \in L^\infty(\nabla, \mu)$, $h \in L^\infty(\Omega)$;
(5) $\ell(\hat{u})(\omega) \geq 0$ whenever $\hat{u} \geq 0$;
(6) the set $\{\ell(\hat{u})(\omega) : \hat{u} \in L^\infty(\nabla, \mu)\}$ is dense in $X(\omega)$ for all $\omega \in \Omega$;
(7) $\ell(\hat{u} \vee \hat{v}) = \ell(\hat{u}) \vee \ell(\hat{v})$ for every $\hat{u}, \hat{v} \in L^\infty(\nabla, \mu)$.

The mapping ℓ is called a *vector-valued lifting* on $L^\infty(\nabla, \mu)$ associated with the lifting ρ (cp. [G1]).

Let as before $p \geq 1$ and $L^p(\nabla, \mu)$ be a Banach-Kantorovich lattice, and $L^p(\nabla_\omega, \mu_\omega)$ be the corresponding L^p-spaces constructed by real valued measures. Let $T : L^p(\nabla, \mu) \to L^p(\nabla, \mu)$ be a linear mapping. As usually we will say that T is *positive* if $T\hat{f} \geq 0$ whenever $\hat{f} \geq 0$.

We say that T is an $L_0(\Omega)$*-bounded mapping* if there exists a function $k \in L_0(\Omega)$ such that $|T\hat{f}|_p \leq k|\hat{f}|_p$ for all $\hat{f} \in L^p(\nabla, \mu)$. For such a mapping we can define an element of $L_0(\Omega)$ as follows

$$\|T\| = \sup_{|\hat{f}|_p \leq \mathbb{1}} |T\hat{f}|_p,$$

which is called the $L_0(\Omega)$*-valued norm* of T. If $\|T\| \leq \mathbb{1}$ then T is said to be a *contraction*.

Now we give an example of a nontrivial contraction.

Example. Let $(\Omega, \nabla, \lambda)$ be a measurable space with a finite measure and let ∇_0 be a right Boolean subalgebra of ∇. By λ_0 we denote the restriction of λ onto ∇_0. Now let $E(\cdot|\nabla_0)$ be a conditional expectation from $L_1(\Omega, \nabla, \lambda)$ onto $L_1(\Omega, \nabla_0, \lambda_0)$. It is clear that $\mu(\hat{e}) = E(\hat{e}|\nabla_0)$ is a strictly positive $L_1(\Omega, \nabla_0, \lambda_0)$-valued measure on ∇. Let ∇_1 be another arbitrary right Boolean subalgebra of ∇ such that $\nabla_1 \supset \nabla_0$. By μ_1 we denote the restriction of μ onto ∇_1. According to [K1, Theorem 4.2.9] there exists a conditional expectation $T : L_1(\nabla, \mu) \to L_1(\nabla_1, \mu_1)$ which is positive and maps $L^p(\nabla, \mu)$ onto $L^p(\nabla, \mu)$ for all $p > 1$. Moreover, $|T\hat{f}|_p \leq |\hat{f}|_p$ for every $\hat{f} \in L^p(\nabla, \mu)$ and $T\mathbb{1} = \mathbb{1}$.

In the sequel we will need the following

Theorem 2.1. *Let $T : L^p(\nabla, \mu) \to L^p(\nabla, \mu)$ be a positive linear contraction such that $T\mathbb{1} \leq \mathbb{1}$. Then for every $\omega \in \Omega$ there exists a positive contraction $T_\omega : L^p(\nabla_\omega, \mu_\omega) \to L^p(\nabla_\omega, \mu_\omega)$ such that $T_\omega f(\omega) = (T\hat{f})(\omega)$ λ-a.e. for every $\hat{f} \in L^p(\nabla, \mu)$.*

Proof: The positivity of T implies that $|T\hat{f}| \leq T|\hat{f}| \leq \|\hat{f}\|_\infty \mathbb{1}$ for every $\hat{f} \in L^\infty(\nabla, \mu)$, i.e. the operator T maps $L^\infty(\nabla, \mu)$ to $L^\infty(\nabla, \mu)$ and it is continuous in norm $\|\cdot\|_\infty$, where $\|f\|_\infty = \operatorname{varisup}|f|$. One can see that $|T\hat{f}|_p \in L^\infty(\Omega)$ and $|\hat{f}|_p \in L^\infty(\Omega)$ for $\hat{f} \in L^\infty(\nabla, \mu)$. Now define a linear operator $\varphi(\omega)$ from

$\{\ell(\hat{f})(\omega) : \hat{f} \in L^\infty(\nabla, \mu)\}$ to $L^p(\nabla_\omega, \mu_\omega)$ by

$$\varphi(\omega)(\ell(\hat{f})(\omega)) = \ell(T\hat{f})(\omega),$$

where ℓ is the vector-valued lifting on $L^\infty(\nabla, \mu)$ associated with the lifting ρ. From $|T\hat{f}|_p \leq |\hat{f}|_p$ we obtain

$$\begin{aligned}\|\ell(T\hat{f})(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)} &= \rho(|T\hat{f}|_p)(\omega)\\ &\leq \rho(|\hat{f}|_p)(\omega)\\ &= \|\ell(\hat{f})(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)}\end{aligned}$$

which implies that the operator $\varphi(\omega)$ is correctly defined and bounded. Using the fact that $\{\ell(\hat{f})(\omega) : \hat{f} \in L^\infty(\nabla, \mu)\}$ is dense in $L^p(\nabla_\omega, \mu_\omega)$ we can extend $\varphi(\omega)$ to a continuous linear operator on $L^p(\nabla_\omega, \mu_\omega)$. This extension is denoted by T_ω.

We are going to show that T_ω is positive. Indeed, let $f(\omega) \in L^p(\nabla_\omega, \mu_\omega)$ and $f(\omega) \geq 0$. Then there exists a sequence $\{\hat{f}_n\} \subset L^\infty(\nabla, \mu)$ such that $\ell(\hat{f}_n)(\omega) \to f(\omega)$ in norm of $L^p(\nabla_\omega, \mu_\omega)$. Put $\hat{g}_n = \hat{f}_n \vee 0$; then $\hat{g}_n \geq 0$ and according to the properties of the vector-valued lifting ℓ we infer

$$\ell(\hat{g}_n)(\omega) = \ell(\hat{f}_n)(\omega) \vee 0 \to f(\omega) \vee 0 = f(\omega)$$

in norm of $L^p(\nabla_\omega, \mu_\omega)$. Whence

$$0 \leq \ell(T\hat{g}_n)(\omega) = \varphi(\omega)(\ell(\hat{g}_n)(\omega)) \to T_\omega(f(\omega)),$$

this means $T_\omega f(\omega) \geq 0$. It is clear that $\|T_\omega\|_\infty \leq 1$ and $T_\omega f(\omega) = (T\hat{f})(\omega)$ a.e. for every $\hat{f} \in L^\infty(\nabla, \mu)$, here $\|\cdot\|_\infty$ is the norm of an operator from $L^\infty(\nabla_\omega, \mu_\omega)$ to $L^\infty(\nabla_\omega, \mu_\omega)$.

Now let $\hat{f} \in L^p(\nabla, \mu)$. Since $L^\infty(\nabla, \mu)$ is (bo)-dense in $L^p(\nabla, \mu)$, there is a sequence $\{\hat{f}_n\} \subset L^\infty(\nabla, \mu)$ such that $|\hat{f}_n - \hat{f}|_p \xrightarrow{(o)} 0$. Then $\|f_n(\omega) - f(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)} \to 0$ for almost all ω. The equality $T\hat{f} = |\cdot|_p\text{-}\lim_n T\hat{f}_n$ implies that

$$\|T_\omega f_n(\omega) - (T\hat{f})(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)} = \|(T\hat{f}_n)(\omega) - (T\hat{f})(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)} \to 0 \quad \text{a.e. } \omega,$$

which means that $(T\hat{f})(\omega) = \lim_n T_\omega f_n(\omega)$ a.e. On the other hand, the continuity of T_ω yields that $\lim_n T_\omega f_n(\omega) = T_\omega f(\omega)$ a.e. Hence for every $\hat{f} \in L^p(\nabla, \mu)$ we have $(T\hat{f})(\omega) = T_\omega f(\omega)$ a.e. This completes the proof. $\square$

3. Main results

In this section we will prove an analog of Theorem A formulated in the introduction. Before formulating it we are going to provide certain useful assertions.

Proposition 3.1. *Let $T^{(i)} : L^p(\nabla,\mu) \to L^p(\nabla,\mu)$, $i=1,2$ be positive linear contractions such that $T^{(i)}\mathbb{1} \le \mathbb{1}$. Then*

$$\|T^{(1)} - T^{(2)}\|(\omega) = \|T^{(1)}_\omega - T^{(2)}_\omega\|_{p,\omega}, \quad \text{a.e.}$$

Here as above, $\|\cdot\|_{p,\omega}$ is the norm of an operator from $L^p(\nabla_\omega,\mu_\omega)$ to $L^p(\nabla_\omega,\mu_\omega)$.

Proof: According to Theorem 2.1 we have $T^{(i)}_\omega f(\omega) = (T^{(i)}\hat{f})(\omega)$, $i=1,2$ a.e. for every $\hat{f} \in L^p(\hat{\nabla},\hat{\mu})$. Using this fact we get

$$\begin{aligned}
|(T^{(1)} - T^{(2)})\hat{f}|_p(\omega) &= \|(T^{(1)} - T^{(2)})\hat{f}(\omega)\|_{L^p(\nabla_\omega,\mu_\omega)}\\
&= \|(T^{(1)}_\omega - T^{(2)}_\omega)f(\omega)\|_{L^p(\nabla_\omega,\mu_\omega)}\\
&\le \|T^{(1)}_\omega - T^{(2)}_\omega\|_{p,\omega}\|f(\omega)\|_{L^p(\nabla_\omega,\mu_\omega)}
\end{aligned}$$

which implies

$$\|T^{(1)} - T^{(2)}\|(\omega) \le \|T^{(1)}_\omega - T^{(2)}_\omega\|_{p,\omega}, \quad \text{a.e.} \tag{1}$$

By similar arguments we obtain

$$\begin{aligned}
\|(T^{(1)}_\omega - T^{(2)}_\omega)f(\omega)\|_{L^p(\nabla_\omega,\mu_\omega)} &= |(T^{(1)} - T^{(2)})\hat{f}|_p(\omega)\\
&\le \left(\|T^{(1)} - T^{(2)}\||\hat{f}|_p\right)(\omega)\\
&= \|T^{(1)} - T^{(2)}\|(\omega)|\hat{f}|_p(\omega)\\
&= \|T^{(1)} - T^{(2)}\|(\omega)\|f_\omega\|_{L^p(\nabla_\omega,\mu_\omega)},
\end{aligned}$$

which yields

$$\|T^{(1)} - T^{(2)}\|(\omega) \ge \|T^{(1)}_\omega - T^{(2)}_\omega\|_{p,\omega}. \quad \text{a.e.}$$

The last inequality with (1) implies the required equality. This completes the proof. □

Proposition 3.2. *Let $T^{(i)} : L^p(\nabla,\mu) \to L^p(\nabla,\mu)$, $i=1,2$ be positive linear contractions such that $T^{(i)}\mathbb{1} \le \mathbb{1}$. Then*

$$\left\||T^{(1)}_\omega - T^{(2)}_\omega|\right\|_{p,\omega} \le \left\||T^{(1)} - T^{(2)}|\right\|(\omega), \quad \text{a.e.,}$$

where $|\cdot|$ *is the modulus of an operator.*

PROOF: Using the formula

$$|Ax| \leq |A||x|,$$

where $A : E \to E$ is a linear operator and E is a vector lattice (see [V, p. 231]), we have

$$|(T_\omega^{(1)} - T_\omega^{(2)})g(\omega)| \leq \left(|T^{(1)} - T^{(2)}||\hat{g}|\right)(\omega) \quad \text{a.e.}$$

for every $\hat{g} \in L^p(\nabla, \mu)$.

If $|\hat{g}| \leq |\hat{f}|$, where $\hat{f} \in L^p(\nabla, \mu)$, then $|g(\omega)| \leq |f(\omega)|$. This implies

$$|(T_\omega^{(1)} - T_\omega^{(2)})g(\omega)| \leq \left(|T^{(1)} - T^{(2)}||\hat{f}|\right)(\omega) \quad \text{a.e.}$$

Now by means of the formula

$$|A|x = \sup_{|y| \leq x} |Ay|,$$

where A is as above and $x \geq 0$ (see [V, p. 231]), we infer that

$$|T_\omega^{(1)} - T_\omega^{(2)}||f(\omega)| = \sup_{|g(\omega)| \leq |f(\omega)|} |(T_\omega^{(1)} - T_\omega^{(2)})g(\omega)| \leq \left(|T^{(1)} - T^{(2)}||\hat{f}|\right)(\omega).$$

Then the monotonicity of the norm $\|\cdot\|_{L^p(\nabla_\omega, \mu_\omega)}$ implies

$$\begin{aligned}
\left\|\left(|T_\omega^{(1)} - T_\omega^{(2)}||f|\right)(\omega)\right\|_{L^p(\nabla_\omega, \mu_\omega)} &\leq \left\|\left(|T^{(1)} - T^{(2)}||\hat{f}|\right)(\omega)\right\|_{L^p(\nabla_\omega, \mu_\omega)} \\
&= \left||T^{(1)} - T^{(2)}||\hat{f}|\right|_p (\omega) \\
&\leq \left(\left\||T^{(1)} - T^{(2)}|\right\| |\hat{f}|_p\right)(\omega) \\
&= \left\||T^{(1)} - T^{(2)}|\right\| (\omega)|\hat{f}|_p(\omega) \\
&= \left\||T^{(1)} - T^{(2)}|\right\| (\omega)\|f(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\||T_\omega^{(1)} - T_\omega^{(2)}|\right\|_{p,\omega} &= \sup_{\|f(\omega)\|_{L^p(\nabla_\omega, \mu_\omega)} \leq 1} \left\||T_\omega^{(1)} - T_\omega^{(2)}||f(\omega)|\right\|_{L^p(\nabla_\omega, \mu_\omega)} \\
&\leq \left\||T^{(1)} - T^{(2)}|\right\| (\omega) \quad \text{a.e.}
\end{aligned}$$

□

The next theorem is an analog of theorem in [Z2] for positive contractions of $L^1(\nabla, \mu)$.

Theorem 3.3. *Let $T : L^1(\nabla,\mu) \to L^1(\nabla,\mu)$ be a positive linear contraction such that $T\mathbb{1} \le \mathbb{1}$. If $\|T^{m+1} - T^m\| < 2\mathbb{1}$ for some $m \in \mathbb{N}\cup\{0\}$, then*

$$(o) - \lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

Proof: According to Theorem 2.1 there exist positive contractions $T_\omega : L^1(\nabla_\omega,\mu_\omega) \to L^1(\nabla_\omega,\mu_\omega)$ such that $(T\hat{f})(\omega) = T_\omega(f(\omega))$ a.e. From Proposition 3.1 we get $\|T_\omega^{m+1} - T_\omega^m\|_{p,\omega} = \|T^{m+1} - T^m\|(\omega)$ a.e. The assumption of the theorem implies $\|T_\omega^{m+1} - T_\omega^m\|_{p,\omega} < 2$ a.e. Hence the contractions T_ω satisfy the assumption of Theorem 1.1. ([OS]) a.e., therefore

$$\lim_{n\to\infty} \|T_\omega^{n+1} - T_\omega^n\|_{p,\omega} = 0 \ \text{ a.e.}$$

As $\|T_\omega^{n+1} - T_\omega^n\|_{p,\omega} = \|T^{n+1} - T^n\|(\omega)$ a.e. we obtain that

$$\lim_{n\to\infty} \|T^{n+1} - T^n\|(\omega) = 0 \ \text{ a.e.,}$$

therefore

$$(o) - \lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

The theorem is proved. □

Now we can formulate the following theorem, which is an analog of Theorem A for the Banach-Kantorovich lattice $L^p(\nabla,\mu)$.

Theorem 3.4. *Let $T : L^p(\nabla,\mu) \to L^p(\nabla,\mu)$, $p > 1, p \ne 2$ be a positive linear contraction such that $T\mathbb{1} \le \mathbb{1}$. If $\big\||T^{m+1} - T^m|\big\| < 2\mathbb{1}$ for some $m \in \mathbb{N}\cup\{0\}$, then*

$$(o) - \lim_{n\to\infty} \|T^{n+1} - T^n\| = 0.$$

The proof goes along the same lines as the proof of Theorem 3.3, but here instead of Proposition 3.1, Proposition 3.2 should be used.

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