Covering properties in countable products, II

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Abstract. In this paper, we discuss covering properties in countable products of Čech-scattered spaces and prove the following: (1) If Y is a perfect subparacompact space and $\{X_n : n \in \omega\}$ is a countable collection of subparacompact Čech-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is subparacompact and (2) If $\{X_n : n \in \omega\}$ is a countable collection of metacompact Čech-scattered spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact.

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1. Introduction

A space X is said to be *subparacompact* (*metacompact*) if every open cover of X has a σ-locally finite closed (point finite open) refinement. It is well known that every countably compact, subparacompact (metacompact) space is compact.

Telgársky [Te] introduced the notion of C-scattered spaces and proved that the product of a paracompact (Lindelöf) C-scattered space and a paracompact (Lindelöf) space is paracompact (Lindelöf). Yajima [Y1], Gruenhage and Yajima [GY] proved similar results for subparacompact (metacompact) spaces. Furthermore, the second author ([T1], [T2]) proved the following: (1) if Y is a perfect paracompact (hereditarily Lindelöf, perfect subparacompact) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf, subparacompact) C-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf, subparacompact) and (2) if $\{X_n : n \in \omega\}$ is a countable collection of metacompact C-scattered spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact.

On the other hand, Hohti and Ziqiu [HZ] introduced the notion of Čech-scattered spaces, which is a generalization of C-scattered spaces and studied paracompactness of countable products. Furthermore Aoki, Mori and the second author [AMT] proved that if Y is a perfect paracompact (hereditarily Lindelöf) space and $\{X_n : n \in \omega\}$ is a countable collection of paracompact (Lindelöf) Čech-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf).

It seems to be natural to consider subparacompactness and metacompactness of countable products of Čech-scattered spaces. In this paper, the following will be shown: (1) If Y is a perfect subparacompact space and $\{X_n : n \in \omega\}$ is a countable collection of subparacompact Čech-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is subparacompact and (2) If $\{X_n : n \in \omega\}$ is a countable collection of metacompact Čech-scattered spaces, then the product $\prod_{n\in\omega} X_n$ is metacompact.

All spaces are assumed to be Tychonoff spaces. Let ω denote the set of natural numbers. Let $|A|$ denote the cardinality of a set A. Undefined terminology can be found in Engelking [E].

2. Preliminaries

A space X is said to be *scattered* if every nonempty (closed) subset A has an isolated point in A and X is said to be *C-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a compact neighborhood in A. Then scattered spaces and locally compact spaces are C-scattered. A space X is said to be *Čech-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a Čech-complete neighborhood in A. Thus locally Čech-complete spaces and C-scattered spaces are Čech-scattered. It is well known that the space of irrationals $P = \omega^\omega$ is not C-scattered. However, it is Čech-complete and hence, Čech-scattered.

Let X be a space. For a closed subset A of X, let

$$A^* = \{x \in A : x \text{ has no Čech-complete neighborhood in } A\}.$$

Let $A^{(0)} = A, A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal α. Note that every $A^{(\alpha)}$ is a closed subset of X and X is Čech-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α.

Let X be a Čech-scattered space and $Y \subset X$. If Y is open or closed in X, then Y is also Čech-scattered. Furthermore, if Y is an open subset of X, then $Y^{(\alpha)} = Y \cap X^{(\alpha)}$ for each ordinal α. However, if Y is a closed subset of X, then $Y^{(\alpha)} \subset Y \cap X^{(\alpha)}$ for each ordinal α. So we consider α-th derivatives with respect to X. A subset A of X is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty Čech-complete subset and $A \cap X^{(\alpha(A)+1)} = \emptyset$. Let $\mathrm{Top}(A) = A \cap X^{(\alpha(A))}$. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $\mathrm{rank}(x) = \alpha$. Then there is a neighborhood base $\mathcal{B}_x$ of x in X, consisting of open subsets of X, such that for each $B \in \mathcal{B}_x, \overline{B}$ is topped in X and $\alpha(\overline{B}) = \mathrm{rank}(x)$.

It is clear that if X and Y are Čech-scattered spaces, then the product $X \times Y$ is Čech-scattered.

Lemma 1 (Engelking [E]). *A space X is Čech-complete if and only if there is a sequence $(\mathcal{A}_n)$ of open covers of X satisfying that if $\mathcal{F}$ is a collection of closed* *subsets of X, with the finite intersection property, such that for each $n \in \omega$, there are $F_n \in \mathcal{F}$ and $A_n \in \mathcal{A}_n$ with $F_n \subset A_n$, then the intersection $\bigcap \mathcal{F}$ is nonempty.*

In Lemma 1, the intersection $\bigcap \mathcal{F}$ is countably compact. So, if X is subparacompact (metacompact), then $\bigcap \mathcal{F}$ is compact. The proof of the following lemma is routine and hence, we omit it.

Lemma 2. (1) *If X is a subparacompact Čech-scattered space and Y is a closed subset of X, then every open cover of Y has a σ-locally finite topped, closed refinement.*

(2) *If X is a metacompact Čech-scattered space and Y is a closed subset of X, then for every open cover $\mathcal{U}$ of Y, there is a point finite open cover $\mathcal{V}$ of Y such that for each $V \in \mathcal{V}$, $\overline{V}$ is topped and is contained in some member of $\mathcal{U}$.*

Reduction. In considering covering properties of countable products of Čech-scattered spaces, we may consider $Y \times X^\omega$ or X^ω. Furthermore, we may assume that X has a single top point a, that is, $\mathrm{Top}(X) = \{a\}$. For, let $\{X_n : n \in \omega\}$ be a countable collection of Čech-scattered spaces. Take an $a \notin \bigcup_{n\in\omega} X_n$ and let

$$Y_m = \bigoplus_{n\in\omega} X_n \quad \text{for each} \quad m \in \omega \quad \text{and}$$

$$X = \bigoplus_{m\in\omega} Y_m \cup \{a\}.$$

The topology of X is as follows: every X_n is open and closed in X and the neighborhood base at a is $\{U_m \cup \{a\} : m \in \omega\}$, where $U_m = \bigoplus_{k\geq m} Y_k$ for each $m \in \omega$. Then X is Čech-scattered and if every X_n is subparacompact (metacompact), then X is also subparacompact (metacompact) (cf. Alster [A, Theorem]). Let Y be a space. Then $Y \times \prod_{n\in\omega} X_n$ ($\prod_{n\in\omega} X_n$) is a closed subset of $Y \times X^\omega$ (X^ω) and hence, if $Y \times X^\omega$ (X^ω) is subparacompact (metacompact), then $Y \times \prod_{n\in\omega} X_n$ ($\prod_{n\in\omega} X_n$) is also subparacompact (metacompact).

Let X be a Čech-scattered space and Y be a space. A subset A of $Y \times X^n$ is said to be *rectangle* if $A = \tilde{A} \times \prod_{i\leq n} A_i$ such that $\tilde{A} \subset Y$ and for each $i \leq n, A_i \subset X$. A subset $A = \tilde{A} \times \prod_{i\in\omega} A_i$ of $Y \times X^\omega$ is said to be *basic open* (*basic closed*) if $\tilde{A}$ is an open (closed) subset of Y, and there is an $n \in \omega$ such that A_i is an open (closed) subset of X for each $i < n$ and $A_i = X$ for each $i \geq n$. Let

$$n(A) = \inf\{i : A_j = X \quad \text{for each} \quad j \geq i\}.$$

Let $n \in \omega$. If $A = \prod_{i\leq n} A_i$ ($\prod_{i\in\omega} A_i$) is a subset of X^n (X^ω) such that for each $i \leq n$ ($i \in \omega$), A_i is topped, then we denote

$$\mathrm{Top}(A) = \prod_{i\leq n} \mathrm{Top}(A_i) \ (\prod_{i\in\omega} \mathrm{Top}(A_i)).$$

3. Subparacompactness

An open cover $\mathcal{U}$ of a space X is said to be *well-monotone* if $\mathcal{U}$ is well-ordered by inclusion. In order to prove subparacompactness of spaces, the following is useful: A space X is subparacompact if and only if every well-monotone open cover has a σ-locally finite closed refinement (cf. Yajima [Y2, Lemma 2.4]).

Firstly, we shall consider subparacompactness of countable products. By the Reduction, it suffices to prove the following.

Theorem 1. *If Y is a perfect subparacompact space and X is a subparacompact Čech-scattered space with* $\mathrm{Top}(X) = \{a\}$*, then the product $Y \times X^\omega$ is subparacompact.*

Proof: Let $\mathcal{U}$ be a well-monotone open cover of $Y \times X^\omega$. Define $(R, (\mathcal{A}(R)_{i,m})) \in \mathcal{C}$ if $R = \tilde{R} \times \prod_{i\in\omega} R_i$ is a basic closed subset of $Y \times X^\omega$ such that for each $i \in \omega$, R_i is topped and $(\mathcal{A}(R)_{i,m})$ is a sequence of open (in $\mathrm{Top}(R_i)$) covers of $\mathrm{Top}(R_i)$, satisfying Lemma 1.

Take an $(R, (\mathcal{A}(R)_{i,m})) \in \mathcal{C}$ and $R = \tilde{R} \times \prod_{i\in\omega} R_i$. Let $i < n(R)$. For each $A \in \mathcal{A}(R)_{i,1}$, take an open subset A' of R_i such that $A' \cap \mathrm{Top}(R_i) = A$. Then $\{A' : A \in \mathcal{A}(R)_{i,1}\} \cup \{R_i - \mathrm{Top}(R_i)\}$ is an open (in R_i) cover of R_i. By Lemma 2(1), there is a σ-locally finite cover $\mathcal{F}(R)_i$ of R_i, consisting of topped, closed subsets such that $\mathcal{F}(R)_i$ refines $\{A' : A \in \mathcal{A}(R)_{i,1}\} \cup \{R_i - \mathrm{Top}(R_i)\}$. In order to lengthen $n(R)$, take a σ-locally finite topped, closed cover $\mathcal{F}(R)_{n(R)}$ of X such that there is a proper element $F \in \mathcal{F}(R)_{n(R)}$ with $a \in F$ and for each $F' \in \mathcal{F}(R)_{n(R)} - \{F\}$, $a \notin F'$.

Then $\mathcal{F}(R) = \prod_{i\le n(R)} \mathcal{F}(R)_i$ is a σ-locally finite cover of $\prod_{i\le n(R)} R_i$, consisting of closed rectangles such that for $F = \prod_{i\le n(R)} F_i \in \mathcal{F}(R)$ and $i \le n(R)$, F_i is topped. Take an $F = \prod_{i\le n(R)} F_i \in \mathcal{F}(R)$ with $\mathrm{Top}(F) \cap \mathrm{Top}(\prod_{i\le n(R)} R_i) \ne \emptyset$ and hence, for each $i \le n(R)$, $\mathrm{Top}(F_i) \cap \mathrm{Top}(R_i) \ne \emptyset$. For each $i \le n(R)$, since $\mathrm{Top}(F_i) \cap \mathrm{Top}(R_i) = F_i \cap \mathrm{Top}(R_i) = \mathrm{Top}(F_i)$, there is a subset $A \in \mathcal{A}(R)_{i,1}$ such that $\mathrm{Top}(F_i) \subset A$. Let $\hat{F} = F \times X \times \cdots = \prod_{i\in\omega} \hat{F}_i$. Then $\hat{F}$ is a basic closed subset of X^ω with $\mathrm{Top}(\hat{F}) = \mathrm{Top}(F) \times \{a\} \times \cdots$. For each $y \in \tilde{R}$, let $F_y = \{y\} \times \mathrm{Top}(\hat{F})$. Define the condition (*) as follows: F_y satisfies (*) if there are basic open set B in $Y \times X^\omega$ and $U \in \mathcal{U}$ such that $F_y \subset B \subset \overline{B} \subset U$. Let

$$n(F_y) = \min\{n(B) : B \text{ is a basic open subset of } Y \times X^\omega \text{ such that } F_y \subset B \subset \overline{B} \subset U \text{ for some } U \in \mathcal{U}\}.$$

We say that F *satisfies* (*) if there is a $y \in \tilde{R}$ such that F_y satisfies (*). Let $y \in \tilde{R}$ and assume that F_y satisfies (*). Take a basic open set $B(F_y) = \widetilde{B(F_y)} \times \prod_{i\in\omega} B(F_y)_i$ in $Y \times X^\omega$ and $U(F_y) \in \mathcal{U}$ such that

(1) (a) $F_y \subset B(F_y) \subset \overline{B(F_y)} \subset U(F_y)$,
(b) $n(F_y) = n(B(F_y))$.

Define

$$r(F_y) = \max\left\{n(R)+1, n(F_y)\right\}.$$

Let $m \in \omega$ and $W(F)_m = \{y \in \tilde{R} : n(F_y) = m\}$. Since $\bigcup_{i\le m} W(F)_i = \bigcup\{\widetilde{B(F_y)} \cap \tilde{R} : n(F_y) \le m\}$, every $W(F)_m$ is an F_σ-set in Y. Since Y is a perfect subparacompact space, there is a collection $\mathcal{G}(F)_m$ of closed subsets of Y such that: for each $m \in \omega$,

(2) (a) $W(F)_m = \bigcup \mathcal{G}(F)_m$,
(b) $\mathcal{G}(F)_m$ refines $\{\widetilde{B(F_y)} \cap \tilde{R} : n(F_y) = m\}$,
(c) $\mathcal{G}(F)_m$ is σ-locally finite in Y.

For each $G \in \mathcal{G}(F)_m$, take a $y(G) \in W(F)_m$ such that $G \subset \widetilde{B(F_{y(G)})} \cap \tilde{R}$. Then $n(F_{y(G)}) = m$. Define $E(G)$ as follows:

$$E(G) = G \times \prod_{i<r(F_{y(G)})} \left(\hat{F}_i \cap \overline{B(F_{y(G)})_i}\right) \times X \times \cdots = G \times \prod_{i\in\omega} E(G)_i.$$

Then $E(G)$ is a basic closed subset of $Y \times X^\omega$ such that for each $i \in \omega$, $E(G)_i$ is topped and $G \times \mathrm{Top}(\hat{F}) \subset E(G)$. By a similar manner as in the proof of Aoki, Mori and Tanaka [AMT, Theorem 3.1] or Tanaka [T2, Theorem 4.1], we can obtain the following collection $\mathcal{R}(G)$ of basic closed subsets such that:

(3) (a) $\mathcal{R}(G)$ is σ-locally finite in $Y \times X^\omega$,
(b) $G \times \hat{F} - E(G) \subset \bigcup \mathcal{R}(G) \subset G \times \hat{F}$,

for each $R' = G \times \prod_{i\in\omega} R'_i \in \mathcal{R}(G)$,

(c) $n(R') = r(F_{y(G)}) > n(R)$,
(d) for each $i \in \omega, \alpha(R'_i) \le \alpha(R_i)$,
(e) $(R', (\mathcal{A}(R')_{i,m})) \in \mathcal{C}$ such that for each $i \le n(R)$, if $\alpha(R'_i) = \alpha(R_i)$, then $\mathrm{Top}(R'_i) \subset \mathrm{Top}(F_i)$ and for each $m \in \omega$, $\mathcal{A}(R')_{i,m} = \{A \cap R'_i : A \in \mathcal{A}(R)_{i,m+1}\}$,
(f) if $n(F_{y(G)}) < n(R)$, then there is an $i < n(F_{y(G)})$ such that $\alpha(R'_i) < \alpha(R_i)$.

Let $\mathcal{E}(F) = \{E(G) : G \in \bigcup_{m\in\omega} \mathcal{G}(F)_m\}$, $\mathcal{R}(F) = \bigcup\{\mathcal{R}(G) : G \in \bigcup_{m\in\omega} \mathcal{G}(F)_m\}$. If F does not satisfy (*) or $\mathrm{Top}(F) \cap \mathrm{Top}(\prod_{i\le n(R)} R_i) = \emptyset$, let $\mathcal{E}(F) = \{\emptyset\}$, $\mathcal{R}(F) = \{R'\}$, where $R' = \tilde{R} \times F \times X \times \cdots$. Take a sequence $(\mathcal{A}(R')_{i,m})$ such that $(R', (\mathcal{A}(R')_{i,m})) \in \mathcal{C}$ as (3)(e). Let

$$\mathcal{E}(R) = \bigcup\{\mathcal{E}(F) : F \in \mathcal{F}(R)\} \text{ and } \mathcal{R}(R) = \bigcup\{\mathcal{R}(F) : F \in \mathcal{F}(R)\}.$$

(4) (a) $\mathcal{E}(R)$ is a σ-locally finite collection of basic closed subsets of $Y \times X^\omega$ such that every element of $\mathcal{E}(R)$ is contained in some member of $\mathcal{U}$,
(b) $\mathcal{R}(R)$ is a σ-locally finite collection of basic closed subsets of $Y \times X^\omega$,
(c) $R - \bigcup \mathcal{E}(R) \subset \bigcup \mathcal{R}(R) \subset R$,

for $R' = \tilde{R}' \times \prod_{i\in\omega} R'_i \in \mathcal{R}(F), F = \prod_{i\le n(R)} F_i \in \mathcal{F}(R)$,

(d) $n(R') > n(R)$,
(e) for each $i \in \omega, \alpha(R'_i) \le \alpha(R_i)$,
(f) $(R', (\mathcal{A}(R')_{i,m})) \in \mathcal{C}$ such that for each $i \le n(R)$, if $\alpha(R'_i) = \alpha(R_i)$, then $\mathrm{Top}(R'_i) \subset \mathrm{Top}(F_i)$ and for each $m \in \omega$, $\mathcal{A}(R')_{i,m} = \{A \cap R'_i : A \in \mathcal{A}(R)_{i,m+1}\}$,
(g) if $R' = G \times \prod_{i\in\omega} R'_i$ for some $G \in \mathcal{G}(F)_m$, $m \in \omega$ and $n(F_{y(G)}) < n(R)$, then there is an $i < n(F_{y(G)})$ such that $\alpha(R'_i) < \alpha(R_i)$.

Let $\mathcal{E}_0 = \{\emptyset\}$, $R_0 = Y \times X^\omega$ and $\mathcal{R}_0 = \{R_0\}$. Put $\mathcal{A}_{i,m} = \{\{a\}\}$ for $i, m \in \omega$ and $Y(0) = \emptyset$. By the above construction, for each $n \ge 1$, we obtain collections $\mathcal{E}_n$ and $\mathcal{R}_n$ of basic closed subsets of $Y \times X^\omega$ and a subset $Y(n)$ of Y, satisfying the following:

(5) $\mathcal{E}_n = \bigcup\{\mathcal{E}(R) : R \in \mathcal{R}_{n-1}\}$ is σ-locally finite in $Y \times X^\omega$ such that every element of $\mathcal{E}_n$ is contained in some member of $\mathcal{U}$,
(6) $\mathcal{R}_n = \bigcup\{\mathcal{R}(R) : R \in \mathcal{R}_{n-1}\}$ is σ-locally finite in $Y \times \prod_{n\in\omega} X_n$,

for $R = \tilde{R} \times \prod_{i\in\omega} R_i \in \mathcal{R}_{n-1}$, $R' = \tilde{R}' \times \prod_{i\in\omega} R'_i \in \mathcal{R}(F)$, $F = \prod_{i\le n(R)} F_i \in \mathcal{F}(R)$,

(7) $(R, (\mathcal{A}(R)_{i,m})) \in \mathcal{C}$,
(8) $R - \bigcup \mathcal{E}(R) \subset \bigcup \mathcal{R}(R) \subset R$,
(9) $n(R) < n(R')$,
(10) for $i \in \omega, \alpha(R'_i) \le \alpha(R_i)$,
(11) $(R', (\mathcal{A}(R')_{i,m})) \in \mathcal{C}$ such that for each $i \le n(R)$, if $\alpha(R'_i) = \alpha(R_i)$, then $\mathrm{Top}(R'_i) \subset \mathrm{Top}(F_i)$ and for each $m \in \omega$, $\mathcal{A}(R')_{i,m} = \{A \cap R'_i : A \in \mathcal{A}(R)_{i,m+1}\}$,
(12) $Y(R,F) = \{y \in \tilde{R} : F_y \text{ satisfies } (*)\}$ for $F \in \mathcal{F}(R)$ and $Y(n) = \bigcup\{Y(R,F) : R \in \mathcal{R}_{n-1} \text{ and } F \in \mathcal{F}(R)\}$,
(13) if $y \in Y(R,F), F \in \mathcal{F}(R)$ and $n(F_y) < n(R)$, then there is an $i < n(F_y)$ such that $\alpha(R'_i) < \alpha(R_i)$.

Let $\mathcal{E} = \bigcup_{n\in\omega} \mathcal{E}_n$. We shall show that $\mathcal{E}$ is a σ-locally finite basic closed refinement of $\mathcal{U}$. By (5), it suffices to show that $\mathcal{E}$ is a cover of $Y \times X^\omega$. Assume that $\mathcal{E}$ does not cover $Y \times X^\omega$. Take a $(y, (x_t)) \in Y \times X^\omega - \bigcup \mathcal{E}$. Then there are sequences $\{R_n : n \in \omega\}$, $\{F_n : n \ge 1\}$, $\{y_n : n \ge 1\}$ (if possible) such that: for each $n \ge 1$,

(14) (a) $(y, (x_t)) \in R_n = \widetilde{R_n} \times \prod_{i\in\omega} R_{n,i} \in \mathcal{R}(F_n)$ and $F_n = \prod_{i\le n(R_{n-1})} F_{n,i} \in \mathcal{F}(R_{n-1})$,

(b) $n(R_{n-1}) < n(R_n)$,
(c) for each $i \in \omega, \alpha(R_{n,i}) \leq \alpha(R_{n-1,i})$,
(d) for each $i \leq n(R_{n,i})$, if $\alpha(R_{n+1,i}) = \alpha(R_{n,i})$, then $\mathrm{Top}(R_{n+1,i}) \subset \mathrm{Top}(F_{n,i})$ and for each $m \in \omega$, $\mathcal{A}(R_{n+1})_{i,m} = \{A \cap R_{n+1,i} : A \in \mathcal{A}(R_n)_{i,m+1}\}$,
(e) if F_{ny} satisfies (*), then $n(F_{ny}) = n(F_{ny_n})$ and furthermore, if $n(F_{ny}) < n(R_{n-1})$, then there is an $i < n(F_{ny})$ such that $\alpha(R_{n,i}) < \alpha(R_{n-1,i})$.

Let $i \in \omega$. For each $n \geq 1$, by (14)(c), $\alpha(R_{n,i}) \leq \alpha(R_{n-1,i})$. So, by (14)(b), there is an $n_i \in \omega$ such that $i < n(R_{n_i})$ and $\alpha(R_{n,i}) = \alpha(R_{n_i,i})$ for $n \geq n_i$. Then by (14)(d), $\mathrm{Top}(R_{n+1,i}) \subset \mathrm{Top}(F_{n,i})$. Then there is a sequence $\{A_n : n \geq n_i\}$ of closed subsets of X such that for each $n \geq n_i$, $A_n \in \mathcal{A}(R_{n_i})_{i,n-n_i+1}$ and $\mathrm{Top}(F_{n,i}) \subset A_n$. It follows from Lemma 1 that $C_i = \bigcap_{n \geq n_i} \mathrm{Top}(R_{n,i}) = \bigcap_{n \geq n_i} \mathrm{Top}(F_{n,i})$ is nonempty and compact. Let $C = \{y\} \times \prod_{i \in \omega} C_i$. Then C is compact. Since $\mathcal{U}$ is a well-monotone open cover of $Y \times X^\omega$, there is a $U \in \mathcal{U}$ such that $C \subset U$. Then there is a basic open subset $B = \tilde{B} \times \prod_{i \in \omega} B_i$ such that $C \subset B \subset \overline{B} \subset U$ and $n(B)$ is minimal for this property. Take an $m \in \omega$ such that

(15) (a) $n(B) < n(R_m)$,
(b) for each $i < n(B), n_i \leq m$ and $\mathrm{Top}(R_{m,i}) \subset B_i$.

Then $F_{m+1_y} \subset B$ and hence, F_{m+1_y} satisfies (*). Then by (14) and (15), $n(F_{m+1_y}) = n(F_{m+1_{y_{m+1}}}) \leq n(B) < n(R_m)$. It follows from (14)(e) that there is an $i < n(F_{m+1_y})$ such that $\alpha(R_{m+1,i}) < \alpha(R_{m,i})$, which is a contradiction.

□

4. Metacompactness

Theorem 2. *If X is a metacompact Čech-scattered space with* $\mathrm{Top}(X) = \{a\}$, *then the product X^ω is metacompact.*

Proof: Let $\mathcal{U}$ be an open cover of X^ω, which is closed under finite unions. Define $(B, (\mathcal{A}(B)_{i,m})) \in \mathcal{C}$ if $B = \prod_{i \in \omega} B_i$ is a basic open subset of X^ω such that for each $i \in \omega$, $\overline{B_i}$ is topped and $(\mathcal{A}(B)_{i,m})$ is a sequence of open (in $\mathrm{Top}(\overline{B_i})$) covers of $\mathrm{Top}(\overline{B_i})$, satisfying Lemma 1.

Take a $(B, (\mathcal{A}(B)_{i,m})) \in \mathcal{C}$ and let $B = \prod_{i \in \omega} B_i$. Let $i < n(B)$. For each $A \in \mathcal{A}(B)_{i,1}$, take an open subset A' of $\overline{B_i}$ such that $A' \cap \mathrm{Top}(\overline{B_i}) = A$. By Lemma 2 (2), there is a point finite collection $\mathcal{H}(B)_i$ of open subsets of B_i such that:

(1) (a) $\mathcal{H}(B)_i$ covers B_i,
(b) for each element H of $\mathcal{H}(B)_i$, $\overline{H}$ is topped,
(c) $\overline{\mathcal{H}(B)_i} = \{\overline{H} : H \in \mathcal{H}(B)_i\}$ refines $\{A' : A \in \mathcal{A}(B)_{i,1}\} \cup \{\overline{B_i} - \mathrm{Top}(\overline{B_i})\}$.

Take a point finite open cover $\mathcal{H}(B)_{n(B)}$ of X such that:

(2) (a) for each $H \in \mathcal{H}(B)_{n(B)}, \overline{H}$ is topped,
(b) there is a proper element $H \in \mathcal{H}(B)_{n(B)}$ with $a \in H$ and for each $H' \in \mathcal{H}(B)_{n(B)} - \{H\}$, $a \notin \overline{H'}$.

Then $\mathcal{H}(B) = \prod_{i \leq n(B)} \mathcal{H}(B)_i$ is a point finite cover of $\prod_{i \leq n(B)} B_i$, consisting of open rectangles, such that for $H = \prod_{i \leq n(B)} H_i \in \mathcal{H}(B)$ and $i \in \omega$, $\overline{H_i}$ is topped. Take an $H = \prod_{i \leq n(B)} H_i \in \mathcal{H}(B)$ with $\mathrm{Top}(\overline{H}) \cap \mathrm{Top}(\overline{\prod_{i \leq n(B)} B_i}) = \mathrm{Top}(\overline{H}) \cap \mathrm{Top}(\prod_{i \leq n(B)} \overline{B_i}) \neq \emptyset$ and let $\hat{H} = H \times X \times \cdots = \prod_{i \in \omega} \hat{H}_i$. Then $\hat{H}$ is a basic open subset of X^ω with $n(\hat{H}) = n(B) + 1$ such that $\mathrm{Top}(\overline{\hat{H}}) = \mathrm{Top}(\overline{H}) \times \{a\} \times \cdots$. As before, define the condition (**) as follows: H satisfies (**) if there are basic open sets B_1, B_2 in X^ω with $n(B_1) = n(B_2)$ and $U \in \mathcal{U}$ such that $\mathrm{Top}(\overline{\hat{H}}) \subset B_1 \subset \overline{B_1} \subset B_2 \subset \overline{B_2} \subset U$. Let

$$\begin{aligned} k(H) = \ &\min\{n(B_1) : B_1, B_2 \text{ are basic open subsets of } X^\omega \text{ with } n(B_1) = n(B_2) \\ &\text{such that } \mathrm{Top}(\overline{\hat{H}}) \subset B_1 \subset \overline{B_1} \subset B_2 \subset \overline{B_2} \subset U \text{ for some } U \in \mathcal{U}\}. \end{aligned}$$

Assume that H satisfies (**). Take basic open sets $B_1(H) = \prod_{i \in \omega} B_1(H)_i$, $B_2(H) = \prod_{i \in \omega} B_2(H)_i$ in X^ω with $n(B_1(H)) = n(B_2(H))$ and $U(H) \in \mathcal{U}$ such that

(3) (a) $\mathrm{Top}(\overline{\hat{H}}) \subset B_1(H) \subset \overline{B_1(H)} \subset B_2(H) \subset \overline{B_2(H)} \subset U(H)$,
(b) $k(H) = n(B_1(H))$.

Let
$$r(H) = \max\{n(B) + 1,\ k(H)\}.$$

Define a basic open subset $G(H)$ as follows:

$$G(H) = \prod_{i < r(H)} (\hat{H}_i \cap B_2(H)_i) \times X \times \cdots = \prod_{i \in \omega} G(H)_i.$$

For each $i \in \omega$, $\overline{G(H)_i}$ is topped and $\overline{G(H)}$ is contained in $U(H)$. By (3)(a), using $B_1(H)$, we can also obtain the following collection $\mathcal{B}(H)$ of basic open subsets of X^ω such that:

(4) (a) $\mathcal{B}(H)$ is point finite in X^ω,
(b) $H - G(H) \subset \bigcup \mathcal{B}(H) \subset H$,

for each $B' = \prod_{i \in \omega} B'_i \in \mathcal{B}(H)$,

(c) $n(B') = r(H) > n(B)$,
(d) for each $i \in \omega, \alpha(\overline{B'_i}) \leq \alpha(\overline{B_i})$,
(e) $(B', (\mathcal{A}(B')_{i,m})) \in \mathcal{C}$ such that for each $i \leq n(B)$, if $\alpha(\overline{B'_i}) = \alpha(\overline{B_i})$,

then $\mathrm{Top}(\overline{B_i'}) \subset \mathrm{Top}(\overline{H_i})$ and for each $m \in \omega$, $\mathcal{A}(B')_{i,m} = \{A \cap \overline{B_i'} : A \in \mathcal{A}(B)_{i,m+1}\}$,

(f) if $k(H) < n(B)$, then there is an $i < k(H)$ such that $\alpha(\overline{B_i'}) < \alpha(\overline{B_i})$.

If H does not satisfy (**) or $\mathrm{Top}(\overline{H}) \cap \mathrm{Top}(\prod_{i \leq n(B)} \overline{B_i}) = \emptyset$, let $\mathcal{B}(H) = \{\hat{H}\}$. By Lemma 1 and (4)(a), take a sequence $(\mathcal{A}(B')_{i,m})$ such that $(B', (\mathcal{A}(B')_{i,m})) \in \mathcal{C}$. Let

$$\mathcal{G}(B) = \{G(H) : H \in \mathcal{H}(B) \text{ and } H \text{ satisfies } (^{**})\} \text{ and}$$
$$\mathcal{B}(B) = \bigcup \{\mathcal{B}(H) : H \in \mathcal{H}(B)\}.$$

(5) (a) $\mathcal{G}(B)$ is a point finite collection of basic open subsets of X^ω such that for each $G \in \mathcal{G}(B)$, $\overline{G}$ is contained in some member of $\mathcal{U}$,

(b) $\mathcal{B}(B)$ is a point finite collection of basic open subsets in X^ω,

(c) $B - \bigcup \mathcal{G}(B) \subset \bigcup \mathcal{B}(B) \subset B$,

for $B' = \prod_{i \in \omega} B_i' \in \mathcal{B}(H)$, $H = \prod_{i \leq n(B)} H_i \in \mathcal{H}(B)$,

(d) $n(B') > n(B)$,

(e) for each $i \in \omega, \alpha(\overline{B_i'}) \leq \alpha(\overline{B_i})$,

(f) $(B', (\mathcal{A}(B')_{i,m})) \in \mathcal{C}$ such that for each $i \leq n(B)$, if $\alpha(\overline{B_i'}) = \alpha(\overline{B_i})$, then $\mathrm{Top}(\overline{B_i'}) \subset \mathrm{Top}(\overline{H_i})$ and for each $m \in \omega$, $\mathcal{A}(B')_{i,m} = \{A \cap \overline{B_i'} : A \in \mathcal{A}(B)_{i,m+1}\}$,

(g) if H satisfies (**) and $k(H) < n(B)$, then there is an $i < k(H)$ such that $\alpha(\overline{B_i'}) < \alpha(\overline{B_i})$.

Let $\mathcal{G}_0 = \{\emptyset\}$, $B_0 = X^\omega$, $\mathcal{B}_0 = \{B_0\}$ and $\mathcal{A}_{i,m} = \{\{a\}\}$ for $i, m \in \omega$. By the above construction, for each $n \geq 1$, we obtain collections $\mathcal{G}_n$ and $\mathcal{B}_n$ of basic open subsets of X^ω, satisfying the following:

(6) $\mathcal{G}_n = \bigcup \{\mathcal{G}(B) : B \in \mathcal{B}_{n-1}\}$ is point finite in X^ω and for $G \in \mathcal{G}_n$, $\overline{G}$ is contained in some member of $\mathcal{U}$,

(7) $\mathcal{B}_n = \bigcup \{\mathcal{B}(B) : B \in \mathcal{B}_{n-1}\}$ is point finite in $\prod_{n \in \omega} X_n$,

for $B = \prod_{i \in \omega} B_i \in \mathcal{B}_{n-1}$, $B' = \prod_{i \in \omega} B_i' \in \mathcal{B}(H)$, $H = \prod_{i \leq n(B)} H_i \in \mathcal{H}(B)$,

(8) $(B, (\mathcal{A}(B)_{i,m})) \in \mathcal{C}$,

(9) $B - \bigcup \mathcal{G}(B) \subset \bigcup \mathcal{B}(B) \subset B$,

(10) $n(B) < n(B')$,

(11) for $i \in \omega, \alpha(\overline{B_i'}) \leq \alpha(\overline{B_i})$,

(12) $(B', (\mathcal{A}(B')_{i,m})) \in \mathcal{C}$ such that for each $i \leq n(B)$, if $\alpha(\overline{B_i'}) = \alpha(\overline{B_i})$, then $\mathrm{Top}(\overline{B_i'}) \subset \mathrm{Top}(\overline{H_i})$ and for each $m \in \omega$, $\mathcal{A}(B')_{i,m} = \{A \cap \overline{B_i'} : A \in \mathcal{A}(B)_{i,m+1}\}$,

(13) if H satisfies (**) and $k(H) < n(B)$, then there is an $i < k(H)$ such that $\alpha(\overline{B_i'}) < \alpha(\overline{B_i})$.

Let $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$. We shall show that $\mathcal{G}$ is a point finite basic open refinement of $\mathcal{U}$. By (6), every $\mathcal{G}_n$ is point finite and for each $G \in \mathcal{G}$, $\overline{G}$ is contained in some member of $\mathcal{U}$. Let $(x_t) \in X^\omega$. Since $(x_t) \in B_0 = X^\omega$, by (6), (7) and (9), there are finite subcollections $\mathcal{G}'_1 \subset \mathcal{G}_1$ and $\mathcal{B}'_1 \subset \mathcal{B}_1$ such that $(x_t) \in (\bigcup \mathcal{G}'_1) \cup (\bigcup \mathcal{B}'_1)$ and $(x_t) \notin (\bigcup(\mathcal{G}_1 - \mathcal{G}'_1)) \cup (\bigcup(\mathcal{B}_1 - \mathcal{B}'_1))$. If $\mathrm{ord}((x_t), \mathcal{B}'_1) = 0$, then $(x_t) \in \bigcup \mathcal{G}'_1$ and $(x_t) \notin \bigcup_{2\le n}(\bigcup \mathcal{G}_n)$. Assume that $1 \le \mathrm{ord}((x_t), \mathcal{B}'_1)$. By (6), (7) and (9) again, there are finite subcollections $\mathcal{G}'_2 \subset \mathcal{G}_2$ and $\mathcal{B}'_2 \subset \mathcal{B}_2$ such that $(x_t) \in (\bigcup \mathcal{G}'_2) \cup (\bigcup \mathcal{B}'_2)$ and $(x_t) \notin (\bigcup(\mathcal{G}_2 - \mathcal{G}'_2)) \cup (\bigcup(\mathcal{B}_2 - \mathcal{B}'_2))$. If $\mathrm{ord}((x_t), \mathcal{B}'_2) = 0$, then $(x_t) \in \bigcup \mathcal{G}'_2$ and $(x_t) \notin \bigcup_{3\le n}(\bigcup \mathcal{G}_n)$. Assume that this method can be continued infinitely. That is, for each $n \ge 1$, there is a finite subcollection $\mathcal{B}'_n \subset \mathcal{B}_n$ such that $(x_t) \in \bigcup \mathcal{B}'_n$. Then, by (7) and König's lemma [K], there are sequences $\{B_n : n \in \omega\}$, $\{H_n : n \ge 1\}$ such that: for $n \ge 1$,

(14) (a) $(x_t) \in B_n = \prod_{i\in\omega} B_{n,i} \in \mathcal{B}(H_n)$ and $H_n = \prod_{i\le n(B_{n-1})} H_{n,i} \in \mathcal{H}(B_{n-1})$,
(b) $n(B_{n-1}) < n(B_n)$,
(c) for each $i \in \omega, \alpha(\overline{B_{n,i}}) \le \alpha(\overline{B_{n-1,i}})$,
(d) $(B_n, (\mathcal{A}(B_n)_{i,m})) \in \mathcal{C}$ such that for each $i \le n(B_{n-1})$, if $\alpha(\overline{B_{n,i}}) = \alpha(\overline{B_{n-1,i}})$, then $\mathrm{Top}(\overline{B_{n,i}}) \subset \mathrm{Top}(\overline{H_{n,i}})$ and for each $m \in \omega$, $\mathcal{A}(B_n)_{i,m} = \{A \cap \overline{B_{n,i}} : A \in \mathcal{A}(B_{n-1})_{i,m+1}\}$,
(e) if H_n satisfies (**) and $k(H_n) < n(B_{n-1})$, then there is an $i < k(H_n)$ such that $\alpha(\overline{B_{n,i}}) < \alpha(\overline{B_{n-1,i}})$.

Let $i \in \omega$. For each $n \ge 1$, by (14)(c), $\alpha(\overline{B_{n,i}}) \le \alpha(\overline{B_{n-1,i}})$. So, by (14)(b), there is an $n_i \in \omega$ such that $i < n(B_{n_i})$ and $\alpha(\overline{B_{n,i}}) = \alpha(\overline{B_{n_i,i}})$ for $n \ge n_i$. Then by (14)(d), $\mathrm{Top}(\overline{B_{n+1,i}}) \subset \mathrm{Top}(\overline{H_{n,i}})$ for each $n \ge n_i$. As before, $\{\mathrm{Top}(\overline{B_{n,i}}) : n \ge n_i\}$ is a decreasing sequence of closed subsets of $\mathrm{Top}(\overline{B_{n_i}})$, satisfying the completeness. By Lemma 1, $C_i = \bigcap_{n\ge n_i+1} \mathrm{Top}(\overline{B_{n,i}}) = \bigcap_{n\ge n_i} \mathrm{Top}(\overline{H_{n,i}})$ is nonempty and compact. Let $C = \prod_{i\in\omega} C_i$. Then C is compact. Since $\mathcal{U}$ is an open cover of X^ω, which is closed under finite unions, there is a $U \in \mathcal{U}$ such that $C \subset U$. Since C is compact, there are basic open subsets $B = \prod_{i\in\omega} B_i$ and $B' = \prod_{i\in\omega} B'_i$ in X^ω with $n(B) = n(B')$ such that $C \subset B \subset \overline{B} \subset B' \subset \overline{B'} \subset U$ and $n(B)$ is minimal for this property. Take an $m \in \omega$ such that:

(15) (a) $n(B) < n(B_m)$,
(b) for each $i < n(B), n_i \le m$ and $\mathrm{Top}(\overline{B_{m,i}}) \subset B_i$.

Then H_{m+1} satisfies (**). Since $k(H_{m+1}) < n(B_m)$, by (14)(e), there is an $i < k(H_{m+1})$ such that $\alpha(\overline{B_{m+1,i}}) < \alpha(\overline{B_{m,i}})$, which is a contradiction.

So, our method is finished after finitely many times, that is , n times for some $n \ge 1$. Then $(x_t) \in \bigcup \mathcal{G}'_n$ and $(x_t) \notin \bigcup \mathcal{B}_n$ and hence, $(x_t) \notin \bigcup_{s\ge n+1}(\bigcup \mathcal{G}_s)$. Thus $\mathcal{G}$ is point finite. $\square$

A space X is said to be *submetacompact* (*weakly submetacompact*) if for every open cover $\mathcal{U}$ of X, there is a sequence $(\mathcal{V}_n)$ of open refinements (an open refinement $\bigcup_{n\in\omega}\mathcal{V}_n$) of $\mathcal{U}$ such that for each $x \in X$, there is an $n \in \omega$ with $\mathrm{ord}(x, \mathcal{V}_n) < \omega$ ($1 \leq \mathrm{ord}(x, \mathcal{V}_n) < \omega$). For a collection $\mathcal{A}$ of subsets of X and $x \in X$, let $\mathrm{ord}(x, A) = |\{A \in \mathcal{A} : x \in A\}|$. It is well known that every subparacompact (metacompact) space is submetacompact and every countably compact, weakly submetacompact space is compact (cf. [S]). The second author ([T3], [T4]) proved that if $\{X_n : n \in \omega\}$ is a countable collection of submetacompact C-scattered spaces, then the product $\prod_{n\in\omega} X_n$ is submetacompact and if Y is a hereditarily weakly submetacompact space and $\{X_n : n \in \omega\}$ is a countable collection of weakly submetacompact C-scattered spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is weakly submetacompact. So we raise the following problem.

Problem. (1) If $\{X_n : n \in \omega\}$ is a countable collection of submetacompact Čech-scattered spaces, then is the product $\prod_{n\in\omega} X_n$ submetacompact?

(2) If Y is a hereditarily weakly submetacompact space and $\{X_n : n \in \omega\}$ is a countable collection of weakly submetacompact Čech-scattered spaces, then is the product $Y \times \prod_{n\in\omega} X_n$ weakly submetacompact?

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