An infinitary version of Sperner's Lemma

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Abstract. We prove an extension of the well-known combinatorial-topological lemma of E. Sperner to the case of infinite-dimensional cubes. It is obtained as a corollary to an infinitary extension of the Lebesgue Covering Dimension Theorem.

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1. Introduction

The well-known lemma of E. Sperner on colourings of vertices of n-simplices is one of the cornerstones of simplicial algebraic topology, directly related to the invariance of homology groups under simplicial subdivisions. It was used by Sperner [21] to give a short proof that the topological dimension of an n-cell equals n. In fact, it can be considered as a combinatorial (discrete) version of Brouwer's Fixed Point Theorem, obtained from Sperner's Lemma by a simple argument using the compactness of the cubes I^n, as shown in [10]. There are several versions and extensions of this lemma ([2], [6], [12], [15], [22], [25]) and there are versions for matroids ([11], [13], [14]). The well-known extension of Brouwer's Fixed Point Theorem to infinite-dimensional (Banach) spaces, Schauder's Fixed Point Theorem, is stated for *compact* convex subsets and is based on the finite-dimensional theorem. Indeed, there is no "truly" infinitary Sperner's Lemma in the literature, and this is connected with the facts that Brouwer's Fixed Point Theorem fails in general (see e.g. [1]) and the ordinary homology groups $H_n(U, \mathbb{Z})$ vanish for unit spheres U of infinite-dimensional Banach spaces. However, we give here a natural extension of Sperner's Lemma to colourings of cubical triangulations of infinite-dimensional cubes. The problem can be reduced to a combinatorial problem about colourings $\varphi : [k]^\omega \to \{0,1\}^\omega$ (where k is a positive integer and $[k]$ denotes the set $\{0, \dots, k\}$) satisfying the *Sperner condition* that $\varphi(\sigma) \neq \varphi(\sigma')$ whenever the distance of σ and σ' in $[k]^\omega$ is maximal. For such colourings φ we prove that there is $\sigma \in [k]^\omega$ such that $\varphi(K_\sigma)$ is infinite, where K_σ is the "cube" corresponding to σ. We also indicate why this result is the best possible. The results are stated in this paper for the unit cube U_∞ of the Banach space ℓ_∞; we consider U_∞ as the ω-dimensional (combinatorial!) analogue of the finite-dimensional cubes I^n.

The Lebesgue Covering Dimension Theorem (cf., e.g., [4]) says that open covers of I^n by sufficiently small sets have order at least $n+1$. This theorem cannot be extended to open covers of the cube U_∞, since by paracompactness every open cover has a point-finite refinement. However, this theorem can be extended if only *uniform* covers are considered, replacing the lack of compactness of U_∞ by the condition of uniformity. In this paper, this extension is proved first and then used to obtain Sperner's Lemma as a direct corollary. The sets $[k]^\omega$ correspond to regular or uniform cubical subdivisions of U_∞, and, by the same token, Sperner's Lemma is not valid for non-uniform subdivisions. The general question (Stone [23], Isbell [9]) whether every uniform cover of a Banach space has a point-finite uniform refinement was answered in the negative independently by Pelant [16] and Ščepin [24]. It was handled in a different way later by Rödl [19], who by using results of [5] produced a counter-example of minimal cardinality.

2. Preliminaries

In this section we develop the necessary notation and background for our treatment of the infinitary covering dimension and Sperner's Lemma. Let us first state the classical version of this result. In this paper the symbol $[n]$ denotes the subset $\{0,\dots,n\}$ of integers. Let Δ be an n-simplex, and let $\mathcal{K}=\{\Delta_1,\dots,\Delta_m\}$ be a simplicial subdivision of Δ. Let $\varphi:\mathcal{K}^{(0)}\to[n]$ be a mapping ("colouring") of the vertices of the simplices in $\mathcal{K}$. If φ satisfies the condition

(Sperner Condition): $\varphi(\Delta^{(0)})=[n]$ (bijectivity) and if $v\in\mathcal{K}^{(0)}$ lies in an r-face F of Δ (where $0\le r<n$), then $\varphi(v)\in\varphi(F^{(0)})$,

then φ is called a Sperner colouring. (In some articles the notion "proper labelling" is used.) The classical Sperner's Lemma asserts that there is a simplex $\Delta_i\in\mathcal{K}$ such that $\varphi(\Delta_i^{(0)})=[n]$:

Theorem 2.1 (Sperner's Lemma [21])**.** *If $\mathcal{K}$ is a simplicial subdivision of an n-simplex Δ and $\varphi:\mathcal{K}^{(0)}\to[n]$ is a Sperner colouring, then there is a simplex Δ_i of $\mathcal{K}$ such that the vertices of Δ_i are coloured by φ with $n+1$ colours.*

In the sequel we loosely call colourings satisfying a suitable version of Sperner Condition *Sperner colourings*. We notice here that in this finite-dimensional case, Sperner Condition can be replaced by the condition that the cover $\mathcal{C}=\{\varphi^{-1}(k): k\in[n]\}$ determined by the colours of the vertices $v\in\Delta^{(0)}$ is a "bounded" cover of $\mathcal{K}^{(0)}$ in the following sense. The complex $\mathcal{K}$ becomes a *uniform complex* in the sense of Isbell [9] if for $p,q\in|\mathcal{K}|$ (here $|\mathcal{K}|$ denotes the underlying set of $\mathcal{K}$) we let the distance $d(p,q)$ be the maximum difference of their barycentric coordinates. Then $\mathcal{C}$ is called bounded if $\operatorname{diam}_d(\varphi^{-1}(k))<1$ for all $k\in[n]$. Notice that boundedness implies $\varphi\restriction\Delta^{(0)}$ is injective: if $v,w\in\Delta^{(0)}$ and $v\neq w$, then $d(v,w)=1$ and hence $\varphi(v)\neq\varphi(w)$. Thus, $\varphi\restriction\Delta^{(0)}$ is bijective.

It is natural to consider cubes $I^n = [0,1]^n$ instead of the simplices Δ. For a "cubical" version of Sperner's Lemma, see [12]. In the infinite-dimensional situation, the cubical form becomes a natural one. Let $\mathcal{K}$ be the cubical complex consisting of the single cube I^n. (We consider cubical complexes given by a set of *maximal* cubes such that any two intersecting cubes meet in a common cubical face.) Then $\mathcal{K}^{(0)}$ corresponds to the set $\{0,1\}^n$. A regular (or uniform) subdivision of I^n of sidelength $1/m$ ($m \geq 1$) is the set consisting of all products (called cubes of the subdivision)

$$K_\sigma \;=\; [i_1/m,(i_1+1)/m] \times \cdots \times [i_n/m,(i_n+1)/m],$$

where $i_1,\dots,i_n \in [m-1]$ and where the n-tuple $\sigma = (i_1,\dots,i_n)$ is called the *index* of the associated cube K_σ. For a regular subdivision $\mathcal{L}$ of I^n, the vertex set $\mathcal{L}^{(0)}$ corresponds to $[m]^n$ for some m; the correspondence is simply given by the map $f : [m]^n \to \mathcal{L}^{(0)}$ for which

$$f(i_1,\dots,i_n) = (i_1/m,\dots,i_n/m).$$

Thus, the colourings of $\mathcal{L}^{(0)}$ satisfying Sperner Condition lead to colourings $\varphi : [m]^n \to \{0,1\}^n$ with the following property: if $\sigma,\sigma' \in [m]^n$ and $\sigma(i) = 0$, $\sigma'(i) = m$ for some $i \in [n]$, then $\varphi(\sigma) \neq \varphi(\sigma')$. The natural way to extend these colourings to the infinitary situation is to consider colourings $\varphi : [m]^\omega \to \{0,1\}^\omega$, where $m \in \omega$. As above, the sets $[m]^\omega$ correspond to regular cubical subdivisions of the cube $[0,1]^\omega$, denoted here by U_∞. The topology of U_∞ is given by the ℓ_∞-norm defined by $\|\sigma-\tau\|_{\ell_\infty} = \sup\{|\sigma(i)-\tau(i)| : i \in \omega\}$. We denote the distance of two elements $\sigma,\tau \in [m]^\omega$ simply by $\|\sigma-\tau\|$.

3. Regular subdivisions

The classical Sperner's Lemma was extended to colourings of cubical triangulations by Kuhn in [12]. It is easy to show by using either the classical result that $\dim(I^n) = n$ or the results of Kuhn that for any colouring $\varphi : [m]^n \to \{0,1\}^n$ satisfying Sperner Condition there is $\sigma \in [m]^n$ such that $\varphi(K_\sigma)$ contains at least $n+1$ colours. (The result proved by Kuhn is stronger, see [12, p. 521].) However, this result is true for *any* (finite) cubical triangulation of I^n which is related to the fact that every open cover of I^n is a uniform cover. (Likewise every cubical triangulation of I^n with rational vertices has a subdivision which is regular in the above sense.)

On the other hand, regular subdivisions are *necessary* when we deal with infinite-dimensional cubes. Indeed, we will show that the straightforward extension of Sperner's Lemma to arbitrary cubical subdivisions of U_∞ is false. Let $\mathcal{V}$ be an open cover of U_∞ such that $\operatorname{diam}(V) < 1$ for each $V \in \mathcal{V}$. Since U_∞ is paracompact, we can assume that $\mathcal{V}$ is locally finite; let $\mathcal{W}$ be an open refinement of $\mathcal{V}$ such that each $W \in \mathcal{W}$ meets only finitely many members of $\mathcal{V}$. We can find a cubical subdivision $\mathcal{K}$ of U_∞ such that $\mathcal{K} \prec \mathcal{W}$; i.e., for each cube $K \in \mathcal{K}$ there is $W \in \mathcal{W}$ with $K \subset W$. Indeed, let $\mathcal{K}_1$ be the regular subdivision of U_∞ into cubes of sidelength 1/2. Let $\mathcal{K}_1'$ be the subset of all $K \in \mathcal{K}_1$ such that $K \subset W$ for some $W \in \mathcal{W}$, and let $\mathcal{K}_2'$ be the cubical complex obtained by subdividing each $K \in \mathcal{K}_1 \setminus \mathcal{K}_1'$ into cubes of sidelength 1/4. Let $\mathcal{K}_2$ denote the subset of all $K \in \mathcal{K}_2$ such that $K \subset W$ for some $W \in \mathcal{W}$, and define $\mathcal{K}_3$ as the subdivision of $\mathcal{K}_2 \setminus \mathcal{K}_2'$ into cubes of sidelength 1/8. Continue in this fashion ad infinitum. Then let $\mathcal{K}^* = \bigcup\{\mathcal{K}_n' : n \in \omega\}$. The set $\mathcal{K}^*$ is naturally partially ordered with respect to the relation of inclusion. With this partial order, $\mathcal{K}^*$ is a tree. Furthermore, this tree is well-founded, i.e. each maximal linearly ordered subset (that is, a decreasing sequence of cubes) is finite. To see this, suppose that $K_1 \supset K_2 \supset \ldots$ is a decreasing sequence of cubes. Since $\operatorname{diam}(K_n) \leq 2^{-n}$ and since U_∞ is complete as a metric space, there is $p \in \bigcap\{K_n : n \in \omega\}$. Let $p \in W_p$, $W_p \in \mathcal{W}$. Then $K_n \subset W_p$ already for some n, contradicting the assumption that $K_{n+1} \in \mathcal{K}^*$. Thus, $\mathcal{K}^*$ is well-founded. Consequently the minimal elements of $\mathcal{K}^*$ form the desired complex $\mathcal{K}$. We define a colouring $\varphi : \mathcal{K}^{(0)} \to \{0,1\}^\omega$ as follows. It is easy to see that the cardinality of the set $\mathcal{K}$ is 2^ω. Choose for each $p \in \mathcal{K}^{(0)}$ some $V_p \in \mathcal{V}$ such that $p \in V_p$, and let $\mathcal{V}' = \{V_p : p \in \mathcal{K}^{(0)}\}$. Then the cardinality of $\mathcal{V}'$ is 2^ω and there is a bijection $\phi : \mathcal{V}' \to \{0,1\}^\omega$. Define $\varphi(p) = \phi(V_p)$. Then φ satisfies Sperner Condition since $\operatorname{diam}(V) < 1$ for each $V \in \mathcal{V}'$. However, for each cube $K \in \mathcal{K}$ the set $\varphi(K)$ of colours is finite, since K meets only finitely many members of $\mathcal{V}$.

The above example also shows that the straightforward extension of Lebesgue's Covering Dimension Theorem to the infinite-dimensional setting is false: there is no $x \in U_\infty$ such that $\mathcal{V}_x = \{V \in \mathcal{V} : x \in V\}$ is infinite. Anyhow, we prove that this extension is true for *uniform* covers of U_∞. We note here that if unit cubes of other Banach spaces, e.g. $c_0(\omega)$ are considered, then the infinitary Sperner's Lemma does not hold even for regular cubical subdivisions. Indeed, $c_0(\omega)$ is separable and hence every uniform cover of $c_0(\omega)$ has a uniformly locally finite uniform refinement ([20], [26])*. Thus, by repeating the construction in the above example, we can find arbitrarily fine regular cubical subdivisions $\mathcal{K}$ of the unit cube of $c_0(\omega)$ with colourings $\varphi : \mathcal{K}^{(0)} \to \omega$ satisfying Sperner Condition such that $\varphi(K^{(0)})$ is finite for each $K \in \mathcal{K}$. Combinatorially these subdivisions correspond to the sets $[m]^{<\omega}$ of all sequences $\sigma \in [m]^\omega$ satisfying $\sigma(i) = 0$ for almost all $i \in \omega$. It is, however, possible to give modified versions of Sperner's Lemma even in these cases; we will return to this topic in Section 7.

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*It is remarkable that the basic result that every countable uniform cover has a point-finite uniform refinement was not yet contained in [9]. As pointed out in [20], it was essentially contained in the proof of 2.2 in R.E. Hodel: *Note on metric-dependent dimension functions*, Fund. Math. **61** (1967), p. 84.

4. Definitions

In this section we give definitions, in addition to those given in the previous section, necessary for the proof of the main result. Let $n > 1$ be fixed. The sum $\sigma + \tau$ of two elements $\sigma, \tau \in [n]^\omega$ is always understood relative to the interval $[n]$, i.e., $(\sigma + \tau)(i) = \min(n, \sigma(i) + \tau(i))$ for all $i \in \mathbb{N}$. We define for each $\sigma \in [n]^\omega$ the "positive cube" K_σ with index σ as the set of all $\sigma + \tau$, where $\tau \in \{0,1\}^\omega$. We also define a combinatorial generalization of metric balls. Let $\sigma \in [n]^\omega$, let $A \subset \omega$ and let $k \in [n]$. Then $B(\sigma, A, k)$ denotes the set of all $\tau \in [n]^\omega$ such that $|\sigma(i) - \tau(i)| \leq k$ for $i \in A$ and $\sigma(i) = \tau(i)$ for $i \in \omega \setminus A$. In the proof of 5.1 we primarily consider those subsets A of ω for which both A and $\omega \setminus A$ are infinite. For any subset $S \subset \omega$ let $\mathcal{A}(S)$ denote the collection of all $A \subset \omega$ such that $S \subset A$ and $|\omega \setminus A| = |A \setminus S| = \omega$. We define

$$\hat{B}(\sigma, A, L) \ = \ \bigcup \{B(\sigma, A' \setminus A, L) : A' \in \mathcal{A}(A)\}.$$

Suppose that $\mathcal{G}$ is a covering of $[n]^\omega$. We define here a number that can be called a *local relative Lebesgue number* of the cover $\mathcal{G}$. Given $\sigma \in [n]^\omega$ and $A \subset \omega$, we define

$$\ell(\sigma, A, \mathcal{G}) = \max\{k \in [n] : \exists G \in \mathcal{G}(\hat{B}(\sigma, A, k) \subset G)\}.$$

We observe that $A_1 \subset A_2$ implies $\ell(\sigma, A_1, \mathcal{G}) \leq \ell(\sigma, A_2, \mathcal{G})$. For each $\sigma \in [n]^\omega$ there is $A \in \mathcal{A}(\emptyset)$ such that $\ell(\sigma, A, \varphi) = \ell(\sigma, A', \varphi)$ for all $A' \in \mathcal{A}(A)$. We also notice that if $\sigma' \in \hat{B}(\sigma, A, k)$, say $\sigma' \in B(\sigma, A' \setminus A, k)$, then $\hat{B}(\sigma', A', k) \subset \hat{B}(\sigma, A, k)$, which implies $\ell(\sigma', A', \mathcal{G}) \geq \ell(\sigma, A, \mathcal{G})$.

To facilitate the proof of our first result, we define here a property M that depends on 5 parameters. (Unfortunately, simple arguments such as that of [3] do not seem applicable in this infinitary situation.) Let S be the set of all $\sigma \in [n]^\omega$ such that $\sigma(i) = 0$ for infinitely many $i \in \omega$. Let $\sigma \in S$, let $A \in \mathcal{A}(\emptyset)$, let $k \in \omega$, let $\mathcal{G}$ be a covering of $[n]^\omega$, and let $G \in \mathcal{G}$. Then $M(\sigma, A, k, G, \mathcal{G})$ iff $\hat{B}(\sigma, A, k) \subset G$ but $\hat{B}(\sigma', A', k+1) \not\subset G'$ for all $G' \in \mathcal{G}$ and for all extensions σ' of σ in $\hat{B}(\sigma, A, k)$, where $\sigma' \in S$ and $A' \in \mathcal{A}(A)$. (We call σ' an *extension* of σ if $\sigma(i) \neq 0$ implies $\sigma'(i) = \sigma(i)$; in other words, if the support E of σ is contained in that of σ' and these functions agree on E.)

Let $A \subset \omega$ and let $k \in \omega$. An (A, k)-function is a function $\chi : \omega \to [-k, k]$ such that $\chi(i) = 0$ for $i \in \omega \setminus A$. Thus, every element of $B(\sigma, A, k)$ can be represented in the form $\sigma + \chi$, where χ is an (A, k)-function.

5. Infinitary Covering Dimension

As will be seen in the remarks following the proof of 5.1 (Remark 5.4), the infinitary Sperner's Lemma does not yield a direct extension of Brouwer's Fixed Point Theorem. However, it is equivalent to an infinitary extension of Lebesgue's Covering Dimension Theorem. Here one has to consider uniform covers instead of open covers, because the cubes U_∞ are not compact. A uniform space μX (see [9] for terminology) is called point-finite if every uniform cover $\mathcal{U} \in \mu$ has a uniform refinement $\mathcal{V}$ such that $\mathcal{V}_x = \{V \in \mathcal{V} : x \in V\}$ is finite for each $x \in X$. Is every uniform (e.g., metric) space point-finite? This question of A.H. Stone [23] and J. Isbell [9] was answered in the negative by E.V. Ščepin [24] and J. Pelant [16]. Ščepin proved that $\ell_\infty(\beth_\omega)$ is not point-finite, where the beth number $\beth_\omega$ is defined inductively by $\beth_0 = \omega$, $\beth_{n+1} = 2^{\beth_n}$ and $\beth_\omega = \sup\{\beth_n : n \in \omega\}$. Pelant proved that even $\ell_\infty(\beth_1)$ is not point-finite, by using his combinatorial technique of cornets. By using graph-theoretic results of Erdös, Galvin and Hajnal [5], V. Rödl [19] has given a simple proof showing that there is a non-point-finite space of cardinality ω_1. By using 5.1, we can easily prove that $\ell_\infty(\omega)$ is not point-finite, by showing that the subspace U_∞ satisfies an infinitary version of Lebesgue's Covering Dimension Theorem. This result has also been announced by Pelant.

Theorem 5.1. *Let $\mathcal{U}$ be a uniform cover of U_∞ such that* $\operatorname{diam}(U) < 1$ *for each $U \in \mathcal{U}$. Then there is $x \in U_\infty$ such that $\mathcal{U}_x$ is infinite.*

Proof: To facilitate the argument, we move from the covering $\mathcal{U}$ of U_∞ to a covering $\mathcal{G}$ of $[n]^\omega$ for a suitable n. Let $n \geq 2$ be such that the metric balls $B(x, 2/n)$ of U_∞ refine $\mathcal{U}$. Then for each $U \in \mathcal{U}$ let $G_U \subset [n]^\omega$ consist of all σ such that $(\sigma(i)/n) \in U$, and define $\mathcal{G} = \{G_U : U \in \mathcal{U}\}$. We shall construct a sequence of 4-tuples $\langle \sigma_k, A_k, L_k, U_k \rangle$, where $\sigma_k \in [n]^\omega$, $A_k \in \mathcal{A}(A_{k-1})$, $L_k \in [n]$, $U_k \in \mathcal{U}$, such that $M(\sigma_k, A_k, L_k, G_{U_k}, \mathcal{G})$ (as defined above) holds for each k.

Let $A_0 = \emptyset$, let S be as above (i.e., the set of all $\sigma \in [n]^\omega$ such that $\sigma(i) = 0$ for infinitely many $i \in \omega$) and let

$$L_1 = \max\{\ell(\sigma, A, \mathcal{G}) : \sigma \in S, A \in \mathcal{A}(A_0)\};$$

say $L_1 = \ell(\sigma_1, A_1, \mathcal{G})$, and let $\sigma_1 \in S$ and $U_1 \in \mathcal{U}$ be such that $\hat{B}(\sigma_1, A_1, L_1) \subset G_{U_1}$. Notice in particular that $L_1 < n$; this follows from the assumption that $\operatorname{diam}(U) < 1$ for each $U \in \mathcal{U}$. We can assume that $\sigma_1(i) = 0$ for all $i \in \omega \setminus A_1$ and that there is an element $n_1 \in A_1$ such that $\sigma_1(n_1) = 0$.

It is clear that $\hat{B}(\sigma, A, L_1 + 1) \not\subset G_U$ for all $\sigma \in S$, $A \in \mathcal{A}(A_1)$ and $U \in \mathcal{U}$. It follows that $M(\sigma_1, A_1, L_1, G_{U_1}, \mathcal{G})$ holds. For the inductive hypothesis, assume that we have a sequence of 4-tuples $\langle \sigma_k, A_k, L_k, U_k \rangle$, $1 \leq k \leq m$, with the following properties:

1) $M(\sigma_k, A_k, L_k, G_{U_k}, \mathcal{G})$ holds for each $k \in \{1, \ldots, m\}$;
2) $L_1 \geq \ldots \geq L_m$;
3) $A_{k+1} \in \mathcal{A}(A_k)$ for each $k \in \{0, \ldots, m-1\}$;
4) there are fixed elements $n_i \in A_i$ such that $\sigma_i(n_i) = 0$, where we assume that $n_i \in A_i \setminus A_{i-1}$ for $i > 0$ and that $|A_i \setminus A_{i-1}| > 1$;

5) if $i \le k \le m$, then $\sigma_k \restriction A_i = \sigma_i \restriction A_i$;
6) if $i \in \omega \setminus A_m$, then $\sigma_k(i) = 0$ for all $k \in \{1, \dots, m\}$;
7) $\hat{B}(\sigma_{i+1}, A_{i+1}, 1) \not\subset G_{U_i}$ for all $i \in \{1, \dots, m-1\}$.

We shall construct a 4-tuple $\langle \sigma_{m+1}, A_{m+1}, L_{m+1}, U_{m+1} \rangle$ such that the above conditions 1) – 7) hold with m replaced by $m+1$. Notice that for each $A \in \mathcal{A}(A_m)$ and each $(A \setminus A_m, L_m)$-function $\chi : \omega \to [-L_m, L_m]$, one has $\sigma_m + \chi \in G_{U_m}$. We claim that there is $A'_m \in \mathcal{A}(A_m)$ and an $(A'_m \setminus A_m, L_m)$-function χ_m such that

$$B(\sigma_m + \chi_m, A'_m \setminus A_m, 1) \not\subset G_{U_m}.$$

Indeed, suppose that there is no such A'_m. Then $\hat{B}(\sigma_m, A_m, L_m + 1) \subset G_{U_m}$. To see this, let $\alpha \in \hat{B}(\sigma_m, A_m, L_m + 1)$. Thus, there is $A \in \mathcal{A}(A_m)$ such that $\alpha \in B(\sigma_m, A', L_m + 1)$, where $A' = A \setminus A_m$. We have $|\sigma_m(i) - \alpha(i)| \le L_m + 1$ for all $i \in A'$ and $\alpha(i) = \sigma_m(i)$ for $i \in \omega \setminus A'$. Define a function $\beta \in [n]^\omega$ by setting $\beta(i) = \sigma_m(i)$ for $i \in \omega \setminus A'$, set $\beta(i) = \alpha(i)$ for $i \in A'$ such that $|\alpha(i) - \sigma_m(i)| \le L_m$ and otherwise $\beta(i) = \sigma_m(i) - L_m$ or $\beta(i) = \sigma_m(i) + L_m$ depending on whether $\alpha(i) < \sigma_m(i)$ or $\alpha(i) > \sigma_m(i)$. Clearly $\beta \in B(\sigma_m, A', L_m)$ and $\|\alpha - \beta\| \le 1$, and thus $\alpha \in B(\sigma_m + \chi, A', 1)$ for the (A', L_m)-function $\chi = \beta - \sigma_m$. Therefore, by our assumption, we have $\alpha \in G_{U_m}$ and consequently this shows that $\hat{B}(\sigma_m, A_m, L_m + 1) \subset G_{U_m}$, which is a contradiction with the definition of L_m. (*Notice that for this contradiction we need the crucial property that* $L_m < n$.) Thus, the desired function χ_m and the desired set $A'_m \in \mathcal{A}(A_m)$ exist.

Let

$$L_{m+1} = \max\{\ell(\sigma, A, \mathcal{G}) : \sigma \in E, A \in \mathcal{A}(A'_m)\},$$

where E denotes the set of all extensions of $\sigma_m + \chi_m$ in $\hat{B}(\sigma_m, A_m, L_m)$. It is easy to see that $L_{m+1} \le L_m$. We can find σ_{m+1}, A_{m+1}, U_{m+1} with the following properties:

1) $A_{m+1} \in \mathcal{A}(A'_m)$;
2) $\hat{B}(\sigma_{m+1}, A_{m+1}, L_{m+1}) \subset G_{U_{m+1}}$;
3) $\hat{B}(\alpha, A, L_{m+1} + 1) \not\subset G_U$ for all $A \in \mathcal{A}(A_{m+1})$, all $U \in \mathcal{U}$ and all extensions $\alpha \in S$ of σ_{m+1} in $\hat{B}(\sigma_{m+1}, A_{m+1}, L_{m+1})$.

It follows that $M(\sigma_{m+1}, A_{m+1}, L_{m+1}, G_{U_{m+1}}, \mathcal{G})$ holds. Finally, we note that Condition 4) can easily be satisfied since A_{m+1} can be replaced by a larger infinite set. This finishes the inductive step.

As $L_{i+1} \le L_i$ for $i \in \mathbb{N}^*$, there is (the least) $i_0 \in \mathbb{N}$ such that $i \ge i_0$ implies $L_i = L_{i_0}$. (Moreover, notice that $L_i \ge 1$ for all i by the choice of n.) Define

$$\hat{\sigma} = \lim \sigma_i,$$

i.e., $\hat{\sigma}(i) = \sigma_k(i)$ for $i \in A_k$ and $\hat{\sigma}(i) = 0$ otherwise. Then $\hat{\sigma} \in \hat{B}(\sigma_i, A_i, L_{i_0})$ for all $i \geq i_0$. Indeed, the support of $\hat{\sigma}$ is contained in $A_\omega = \bigcup\{A_k : k \in \mathbb{N}\} \setminus \{n_k : k \in \mathbb{N}^*\}$, and this is — by the inductive construction — an element of $\mathcal{A}(\emptyset)$. For each $i \geq i_0$, we have $\hat{\sigma} = \sigma_i + \chi_i$, where χ_i is an (A, L_{i_0})-function with $A \in \mathcal{A}(A_i)$. We claim that $i, j \geq i_0$, $i \neq j$ implies $U_i \neq U_j$. To prove this, let us assume $i < j$ and $U_i = U_j$ to derive a contradiction. By our assumption and by the choice of the sets G_U, we have

$$\hat{B}(\sigma_j, A_j, L_i) \subset G_{U_j} = G_{U_i}, \tag{$*$}$$

and therefore $B(\sigma_i + \chi_i, A_i' \setminus A_i, 1) \subset G_{U_i}$. To prove this claim, suppose that $\xi \in B(\sigma_i + \chi_i, A_i' \setminus A_i, 1)$. For $k \in A_i' \setminus A_i$ we have by the definition of σ_{i+1} that $\sigma_j(k) = \sigma_i(k) + \chi_i(k)$. Notice that $\xi(k) = 0$ for all $k \in \omega \setminus A_i'$. If $k \in A_j \setminus A_i'$, then $|\sigma_i(k) - \sigma_j(k)| \leq L_{i_0}$. Finally, if $k \in A_i$, then $\sigma_j(k) = \sigma_i(k)$. It then follows from $L_{i_0} \geq 1$ that $|\xi(k) - \sigma_j(k)| \leq L_{i_0}$ for all $k \in A_j$, and hence by $(*)$ we have $\xi \in G_{U_i}$. This contradiction proves that $U_i \neq U_j$.

Finally, we have $|(\mathcal{G})_{\hat{\sigma}}| \geq \omega$. In fact, as $\|\sigma_i - \hat{\sigma}\| \leq L_{i_0}$ for all $i \geq i_0$ and by the inductive construction $\hat{B}(\sigma_i, A_i, L_{i_0}) \subset G_{U_i}$, we have $\hat{\sigma} \in G_{U_i}$ for all $i \geq i_0$. But then $x \in U_i$ for infinitely many i, where $x = (\hat{\sigma}(i)/n)$. This concludes the proof of 5.1. □

Remark 5.2. The statement of 5.1 is the best possible. One cannot prove consistently with ZFC that under the hypotheses of 5.1 there is $x \in U_\infty$ such that $|(\mathcal{U})_x| = \kappa > \omega$. Indeed, assume that **CH** (the continuum hypothesis) holds. Then the uniformity of U_∞ has a basis of covers of cardinality ω_1. But then a straightforward modification of the proof given for the uniformly locally finite refinements of countable uniform covers in [20] shows that the uniformity of U_∞ has a basis consisting of point-countable covers. Moreover, by using the method of Section 3 we can thus construct a regular cubical triangulation $\mathcal{K}$ of U_∞ and a Sperner colouring $\varphi : \mathcal{K}^{(0)} \to \{0,1\}^\omega$ such that $\varphi(K)$ is at most a countable set for each cube K of $\mathcal{K}$.

Remark 5.3. The simplest proof showing that a metric (uniform) space is not point-finite is given by Pelant and Rödl in [18]. In fact, they implicitly formulate and prove a "weak" infinitary Sperner theorem. Suppose that $m, n \in \omega$, $n > 0$, and let $\varphi : [\beth_{m+n-1}]^n \to \beth_m$ be a mapping ("colouring") such that $a \cap a' = \emptyset$ implies $\varphi(a) \neq \varphi(a')$ for all $a, a' \in [\beth_{m+n+1}]^n$ ("Sperner Condition"). Then there exists a subset ("simplex") $\Delta \subset [\beth_{m+n-1}]^n$ such that 1) $|\Delta| = \beth_m$; 2) $a \neq a'$ implies $\varphi(a) \neq \varphi(a')$ for all $a, a' \in \Delta$ and 3) $|\bigcap \Delta| = n-1$. The proof (by induction on n) easily follows from the strong assumptions. This result is used to show that $\ell_1(\beth_\omega)$ is not point-finite.

Remark 5.4. Theorem 5.1 as such does not imply a useful fixed point theorem for mappings $U_\infty \to U_\infty$. Indeed, the regular cubical triangulations correspond to uniformly continuous mappings $f : U_\infty \to U_\infty$, but the usual method of using Sperner's Lemma (see e.g. [10]) only yields that for each $\epsilon > 0$ there is an infinite set $A \subset \omega$ and $x \in U_\infty$ such that $|x_i - (f(x))_i| < \epsilon$ for $i \in A$. This can readily be proved without the use of 5.1. It has been shown ([1]) that there are even Lipschitz mappings $f : U_\infty \to U_\infty$ without approximate fixed point; i.e., there is $\epsilon > 0$ such that $\|x - f(x)\|_{\ell_\infty} \geq \epsilon$ for all $x \in U_\infty$. This leads us to the following question.

Question: Let $f : U_\infty \to U_\infty$ be a uniformly continuous mapping. Is there an infinite subset $A \subset \omega$ such that $(f(x))_i = x_i$ for $i \in A$?

Remark 5.5. By [17], a metric space is point-finite if and only if it can be uniformly embedded into $c_0(\Gamma)$ for some set Γ. Therefore, we obtain as a corollary to 5.1 that there is no uniform (save Lispchitz) embedding of ℓ_∞ into $c_0(\Gamma)$ for any Γ. As such, this seems to be a new result related to the uniform and Lipschitz classification of Banach spaces.

6. An infinitary version of Sperner's Lemma

In this section we state and prove our infinitary version of Sperner's Lemma. Let us recall that a mapping $\varphi : [n]^\omega \to \{0,1\}^\omega$ is a Sperner colouring if $\varphi(\sigma) \neq \varphi(\sigma')$ whenever $\|\sigma - \sigma'\| = n$.

Theorem 6.1. *Let $n \in \mathbb{N}$ and let $\varphi : [n]^\omega \to \{0,1\}^\omega$ be a Sperner colouring. Then there is $\sigma \in [n]^\omega$ such that $\varphi(K_\sigma)$ is infinite.*

Proof: We will define a uniform cover $\mathcal{U}$ of U_∞ and apply Theorem 5.1. For each $\sigma \in [n]^\omega$ define the set G_σ as the product

$$G_\sigma = \prod_{k \in \mathbb{N}} I_k(\sigma),$$

where for each $k \in \mathbb{N}$, $I_k(\sigma)$ is the open interval $]\frac{\sigma(k)-2/3}{n}, \frac{\sigma(k)+2/3}{n}[$ for $0 < \sigma(k) < n$, the interval $[0, 1/n[$ for $\sigma(k) = 0$ and $]1 - 1/n, 1]$ for $\sigma(k) = n$. For each $\tau \in \{0,1\}^\omega$, let

$$U_\tau = \bigcup \{G_\sigma : \varphi(\sigma) = \tau\}.$$

Then $\mathcal{U} = \{U_\tau : \tau \in \{0,1\}^\omega\}$ is a uniform (open) cover of U_∞, and $\operatorname{diam}(U_\tau) < 1$ for all τ, because φ is a Sperner colouring. By 5.1 there is $x \in U_\infty$ such that $(\mathcal{U})_x$ is infinite. Thus, x is contained in infinitely many sets G_σ, each mapped by φ to a different τ. Let Σ be an infinite set of elements σ such that $x \in G_\sigma$ and which (pairwise) map to distinct colours. Given two such elements σ_1, σ_2, we have $|\sigma_1(i) - \sigma_2(i)| \leq 1$ for all i, by the definition of the sets G_σ. It follows that there is a cube K_σ for which Σ forms a subset of vertices; indeed, we may define σ as the coordinatewise infimum of the elements of Σ. This proves 6.1. $\square$

Remark 6.2. In the same way as the classical Sperner' Lemma corresponds to a homology theory of simplicial complexes (see, e.g. [2]), our infinitary version of Sperner's Lemma (Theorem 6.1) corresponds to an infinitary homology theory of infinite-dimensional cubical complexes.

7. The case of c_0

As noted earlier, the regular cubical triangulations of the unit ball of the Banach space c_0 correspond to the sets $[n]^{<\omega}$ of all $\sigma \in [n]^\omega$ such that $s(\sigma) = \{k \in \omega : \sigma(k) \neq 0\}$ is finite. We also noted that the infinitary version of Sperner's Lemma does not hold for these sets. However, although there are Sperner colourings $\varphi : [n]^{<\omega} \to \omega$ such that for each cube K_σ the vertices re coloured with only finitely many colours, one can show that there is a sequence (K_{σ_k}) of cubes and a sequence $(\tau_k)_{k\in\mathbb{N}}$ of distinct colours $\tau_k \in \omega$ such that

1) $\{\tau_1, \dots, \tau_k\} \subset \varphi(K_{\sigma_k})$;
2) σ_{k+1} is an extension of σ_k for each k, i.e., $\sigma_{k+1}(i) = \sigma_k(i)$ for $i \in s(\sigma_k)$, where $s(\sigma)$ denotes the support of σ.

This result is obtained from the following version of Lebesgue's Covering Dimension Theorem for the unit cube $U(c_0)$ of c_0. It is established by virtually the same proof as that given for 5.1 except that the families $\mathcal{A}(S)$ are replaced by the families $\mathcal{F}(S)$ of finite subsets. (Let us note that even this result was announced by Pelant in 1986. The proof was based on his technique of cornets.)

Theorem 7.1. *Let $\mathcal{U}$ be a uniform covering of $U(c_0)$ such that* $\operatorname{diam}(U) < 1$ *for each $U \in \mathcal{U}$. Then there is a sequence $(U_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{U}$ such that for each $n \in \mathbb{N}$, we have $U_1 \cap \dots \cap U_n \neq \emptyset$.*

We will interpret 7.1 with respect to *Noetherian* covers of uniform spaces. Let X be a set, and let $\mathcal{V}$ be any point-finite cover of X. There is a natural partially ordered set $\mathcal{P}(\mathcal{V})$ associated with $\mathcal{V}$ which consists of all finite subsets $\{V_1, \dots, V_n\}$ such that $V_1 \cap \dots \cap V_n \neq \emptyset$ and which is ordered with respect to set inclusion. (The poset $\mathcal{P}(\mathcal{V})$ corresponds to a simplicial complex in which the finite intersecting subsets are regarded as simplices.) The cover $\mathcal{V}$ is called Noetherian if $\mathcal{P}(\mathcal{V})$ does not contain any infinite increasing chain. Theorem 7.1 implies that no bounded uniform cover of $U(c_0)$ is Noetherian. Since the Lebesgue covering dimension of an n-cube can be regarded as the minimum of the maximal length of chains in posets $\mathcal{P}(\mathcal{V})$, where $\mathcal{V}$ is a bounded open covering of the cube, Theorem 7.1 can again be considered an infinitary version of Lebesque's Covering Dimension Theorem. As in the case of $U(\ell_\infty)$, the extension fails for general open coverings. Indeed, any paracompact space has a base of open Noetherian coverings (this result has been established independently by J. Fried [7]). The idea of considering Noetherian covers of uniform spaces instead of simply point-finite covers is due to J. Vilímovský.

The infinite chains of intersecting sets are representatives of infinite-dimensional simplices of dimension ω, and 7.1 corresponds to an infinitary homology theory in the same way as the classical Sperner's Lemma is related to the classical simplicial homology theory. In this context, the cube $U(c_0)$ represents the *finitary boundary* of $U(\ell_\infty)$, to be compared with S^{n-1} as the boundary of I^n. These problems will be considered in another paper.

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Remark (added in proof): Unfortunately, Jan Pelant passed away when this paper was still being refereed. The problem of point-finiteness of uniform spaces was one of his main mathematical questions. He had already contributed to this problem by 1974 in the *Seminar Uniform Spaces* in Prague, and he later also announced the result proved in the present paper that ℓ_∞ is not point-finite as well as the corresponding result on c_0. To emphasize his contribution to this problem, we mention his definitive characterization [17] of point-finite metric spaces as the ones that can be uniformly embedded into $c_0(\Gamma)$ for some Γ. Regrettably, he will now not be able to publish his version of the results presented here. Fortunately, however, he was aware of our version through many conversations for which we express our deep gratitude, even though it comes too late.

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