A new relationship between decomposability and convexity

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Abstract. In the present work we prove that, in the space of Pettis integrable functions, any subset that is decomposable and closed with respect to the topology induced by the so-called Alexiewicz norm $|||\cdot|||$ (where $|||f||| = \sup_{[a,b]\subset[0,1]} \left\| \int_a^b f(s)ds \right\|$) is convex. As a consequence, any such family of Pettis integrable functions is also weakly closed.

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1. Introduction

The concept of decomposability was introduced, for the first time in the Bochner integrability setting, as a substitute (in several aspects) of the concept of convexity (see [Ole]). This notion was extensively studied (see e.g. [HiU], [Fry]), as well as the connection between decomposability and convexity. Thus, Theorem 2.3.17 in [HuP1] yields that any decomposable weakly-closed subset of the space $L^1(\mu, X)$ is convex.

The aim of this paper is to give, in the space of Banach-valued Pettis integrable functions defined on the unit interval (in particular, in the space of Bochner integrable functions too), a new relationship between decomposability and convexity. For this purpose, the functional space is considered provided with the topology of the Alexiewicz norm: $|||f||| = \sup_{[a,b]\subset[0,1]} \left\| \int_a^b f(s)\, ds \right\|$. Let us note that the latter norm allowed to obtain existence results for some kinds of abstract differential equations (see e.g. [Sch1], [Sch2]), as well as a relaxation theorem for non-convex-valued differential inclusions in Banach spaces (see [Chu]).

We prove that any decomposable and $|||\cdot|||$-closed family of Pettis integrable functions is convex. Remark that our result and the previously mentioned theorem in [HuP1] are independent, since the Alexiewicz norm topology and the weak topology of $L^1(\mu,\ X)$ are not, in general, comparable. However, the main result in [Gut] shows that the two topologies coincide on any subset $F \subset L^1(\mu,\ X)$ that is bounded, uniformly integrable and satisfies the condition (U): for every $\varepsilon > 0$, one can find a compact K_ε such that, for each $f \in F$, there is a measurable $\Omega_{f,\varepsilon}$ with $\mu\left([0,1] \setminus \Omega_{f,\varepsilon}\right) \leq \varepsilon$ and $f(t) \in K_\varepsilon$, for all $t \in \Omega_{f,\varepsilon}$.

We would like to note that, since an optimal control problem need not have a solution unless a convexity condition is satisfied, "automatic" convexity results play an important role in control theory, as well as in mathematical economics.

Moreover, as a consequence of our convexity result, we can prove the following topological property: any decomposable and $|||\cdot|||$-closed subset of $\mathrm{Pe}(\mu,\ X)$ (resp. $L^1(\mu,\ X)$) is closed with respect to the weak topology of $\mathrm{Pe}(\mu,\ X)$ (resp. $L^1(\mu,\ X)$).

Finally, let us point out that our results concern the Pettis integral which is, in the case of an infinitely dimensional space, a concept strictly more general than the Bochner integral. The Pettis integral has proved itself useful when the existence of solutions of integral or differential equations and inclusions is studied (see e.g. [Kni], [Sze], [ACT], [SiK]) and, recently, in problems in mathematical economics (see [Pod]).

2. Terminology and notations

Let X be a separable Banach space, X^* its topological dual, B^* the closed unit ball of X^* and $([0,1],\Sigma,\mu)$ the unit interval provided with the σ-algebra of Lebesgue measurable sets and with the Lebesgue measure.

We start by recalling the definitions of the notions of decomposability and decomposable hull, that we will use in the sequel.

Definition 1. A family M of measurable functions defined on the measure space $([0,1],\Sigma,\mu)$ is said to be *decomposable* if, for every $f_1, f_2 \in M$ and every $A \in \Sigma$, $f_1\chi_A + f_2\chi_{A^c} \in M$.

Definition 2. The *decomposable* (resp. *closed decomposable*, with respect to a given topology on the function space) *hull* of a set M of measurable functions is the smallest decomposable (resp. closed decomposable) set containing M. We denote it by $\operatorname{dec} M$ (resp. $\overline{\operatorname{dec}}\, M$).

Let us now remind the reader of the definition of Pettis integral:

Definition 3. A function $f : [0,1] \to X$ is said to be Pettis integrable if:

(1) f is scalarly integrable, i.e. for all $x^* \in X^*$, $\langle x^*, f(\cdot)\rangle \in L^1(\mu)$;

(2) for each $A \in \Sigma$, there exists $x_A \in X$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, f\rangle\, d\mu, \quad \forall\, x^* \in X^*.$$

We denote x_A by $\int_A f\, d\mu$ and we call it the Pettis integral of f on A.

Remark 4. Any Pettis integrable function f is scalarly measurable, i.e. for each $x^* \in X^*$, the real function $\langle x^*, f(\cdot)\rangle$ is measurable. The scalar measurability is, in the case of a separable Banach space, equivalent to the μ-measurability (f is μ-measurable if there exists a sequence of simple functions convergent in μ-measure to f) and, therefore, to the strong measurability (for every Borel set B of X, $f^{-1}(B) \in \Sigma$), see [CaV].

In the sequel, $L^1(\mu, X)$ (resp. $\mathrm{Pe}(\mu, X)$) denotes the family of X-valued Bochner integrable (resp. Pettis integrable) functions defined on $([0,1], \Sigma, \mu)$.

One can consider the space $\mathrm{Pe}(\mu, X)$ provided with the following topologies:
(1) the topology induced by the Pettis norm:

$$\|f\|_{\mathrm{Pe}} = \sup_{x^* \in B^*} \int_0^1 |\langle x^*, f\rangle| \, d\mu$$

which is equivalent to the norm $\sup\{\|\int_A f\,d\mu\|; A \in \Sigma\}$ since (see [Mus, p. 198])

$$\sup\left\{\left\|\int_A f\,d\mu\right\|; A \in \Sigma\right\} \le \|f\|_{\mathrm{Pe}} \le 2\sup\left\{\left\|\int_A f\,d\mu\right\|; A \in \Sigma\right\};$$

(2) the (weaker) topology of the Alexiewicz norm

$$|||f||| = \sup_{[a,b]\subset[0,1]} \left\|\int_a^b f(s)\,ds\right\|;$$

(3) the weak topology associated to the normed space $(\mathrm{Pe}(\mu, X), \|\cdot\|_{\mathrm{Pe}})$, i.e. the coarsest topology with respect to which all $\|\cdot\|_{\mathrm{Pe}}$-continuous linear functionals are continuous.

Let us recall the notion PUI of uniform integrability appropriate to the Pettis integral:

Definition 5. A subset $K \subset \mathrm{Pe}(\mu, X)$ is said to be *Pettis uniformly integrable* (shortly, PUI) if, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$\mu(A) \le \delta_\varepsilon \quad \Rightarrow \quad \left\|\int_A f\,d\mu\right\| \le \varepsilon, \ \forall\, f \in K.$$

We will also consider the trace of the above defined topologies on the subspace $L^1(\mu, X) \subset \mathrm{Pe}(\mu, X)$. Obviously, for each $f \in L^1(\mu, X)$, $|||f||| \le \|f\|_{\mathrm{Pe}} \le \|f\|_{L^1}$ and any uniformly integrable subset of $L^1(\mu, X)$ is PUI.

3. A new relationship between decomposability and convexity

The key lemma is a result similar to Lemma 1.1 in [Chu] (proved in the Bochner integrability setting) for the Pettis integral.

Lemma 6. *Let $E \subset \mathbb{R}$ be a bounded Lebesgue measurable set, $(f_i)_{i=1}^n \subset \mathrm{Pe}(\mu, X)$ defined on E and $(\lambda_i)_{i=1}^n \subset L^1_+(\mu)$ such that, for any $t \in E$, $\sum_{i=1}^n \lambda_i(t) = 1$. Then, for every $\varepsilon > 0$ there exists a measurable partition $(M_i)_{i=1}^n$ of E satisfying the inequality*

$$\sup_{[a,b]\subset\mathbb{R}} \left\|\int_{[a,b]\cap E} \Big(\sum_{i=1}^n \lambda_i(s) f_i(s) - \sum_{i=1}^n \chi_{M_i}(s) f_i(s)\Big)\, ds\right\| \le \varepsilon.$$

PROOF: Theorem 9.1 in [Mus] yields that the simple functions are $\|\cdot\|_{\mathrm{Pe}}$-dense in $\mathrm{Pe}(\mu, X)$. Moreover, thanks to the regularity of Lebesgue measure and to the fact that, on the real line, any open set can be written as a countable union of open intervals, one can find a collection $\left(\overline{f_i}\right)_{i=1}^{n}$ of simple functions that are constant on intervals such that $\sum_{i=1}^{n} \left\|f_i - \overline{f_i}\right\|_{\mathrm{Pe}} \leq \frac{\varepsilon}{4}$. In other words, $\sum_{i=1}^{n} \sup_{x^* \in B^*} \int_E \left|\left\langle x^*,\ f_i(s) - \overline{f_i}(s)\right\rangle\right| \, ds \leq \frac{\varepsilon}{4}$.

Suppose, at the first step, that E is an interval. Let $\left(I^j\right)_{j=1}^{p}$ be a partition of E whose elements are intervals ordered from left to right satisfying that, on each I^j, the functions $\overline{f_i}$ be constant and $\max_{j=1}^{p} \sum_{i=1}^{n} \int_{I^j} \left\|\overline{f_i}(s)\right\| \, ds \leq \frac{\varepsilon}{4}$.

By Lyapunov's theorem, there exists a partition $\left\{I_i^j, 1 \leq i \leq n, 1 \leq j \leq p\right\}$ of E such that, for every $i \in \{1, \ldots, n\}$ and every $j \in \{1, \ldots, p\}$, $\mu\left(I_i^j\right) = \int_{I^j} \lambda_i(s)\, ds$. Denote by $M_i = \bigcup_{j=1}^{p} I_i^j$, by $\phi(s) = \sum_{i=1}^{n} \lambda_i(s) f_i(s) - \sum_{i=1}^{n} \chi_{M_i}(s) f_i(s)$ and by $\overline{\phi}(s) = \sum_{i=1}^{n} \lambda_i(s) \overline{f_i}(s) - \sum_{i=1}^{n} \chi_{M_i}(s) \overline{f_i}(s)$. It is not difficult to see that $\phi \in \mathrm{Pe}(\mu, X)$ and that $\overline{\phi} \in L^1(\mu, X)$. For any j, $\int_{I^j} \overline{\phi}(s)\, ds = 0$ by the choice of the sets M_i. Moreover, for each measurable $A \subset I^j$,

$$\int_A \left\|\overline{\phi}(s)\right\| \, ds \leq \int_A \sum_{i=1}^{n} \left|\lambda_i(s) - \chi_{M_i}(s)\right| \left\|\overline{f_i}(s)\right\| \, ds \leq \int_{I^j} \sum_{i=1}^{n} \left\|\overline{f_i}(s)\right\| \, ds \leq \frac{\varepsilon}{4}\,.$$

Consequently, for all $a \in I^{j'}$ and $b \in I^{j''}$,

$$\begin{aligned}\left\| \int_a^b \overline{\phi}(s)\, ds \right\| &\leq \left\| \sum_{j=j'+1}^{j''-1} \int_{I^j} \overline{\phi}(s)\, ds \right\| + \int_{[a,b] \cap \left(I^{j'} \cup I^{j''}\right)} \left\|\overline{\phi}(s)\right\| \, ds \\ &\leq 2 \max_{j=1}^{p} \sum_{i=1}^{n} \int_{I^j} \left\|\overline{f_i}(s)\right\| \, ds \leq \frac{\varepsilon}{2}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\left\| \int_a^b \phi(s)\, ds \right\| \leq \left\| \int_a^b \overline{\phi}(s)\, ds \right\| + \left\| \int_a^b \left(\phi(s) - \overline{\phi}(s)\right) \, ds \right\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \left\| \int_a^b \lambda_i(s) \left(f_i(s) - \overline{f_i}(s)\right) \, ds \right\| + \sum_{i=1}^{n} \left\| \int_a^b \chi_{M_i}(s) \left(f_i(s) - \overline{f_i}(s)\right) \, ds \right\| \\ &\leq \frac{\varepsilon}{2} + \sum_{i=1}^{n} \sup_{x^* \in B^*} \int_a^b \left|\left\langle x^*, f_i(s) - \overline{f_i}(s)\right\rangle\right| \, ds \\ &+ \sum_{i=1}^{n} \sup_{x^* \in B^*} \int_{[a,b] \cap M_i} \left|\left\langle x^*, f_i(s) - \overline{f_i}(s)\right\rangle\right| \, ds\end{aligned}$$

and so, $\left\| \int_a^b \phi(s)\, ds \right\| \leq \frac{\varepsilon}{2} + 2\sum_{i=1}^n \left\| f_i - \overline{f}_i \right\|_{\mathrm{Pe}} \leq \varepsilon$.

Now, if E is not an interval, we extend the functions f_i by 0 up to an interval E' containing E and we extend also the functions λ_i to E', λ_1 by 1 and the others by 0. From the first part of the proof, it follows that the trace of the partition obtained for E' is the requested partition of E. □

We will also use the following technical result:

Lemma 7. *Let $f, (f_n)_{n\in\mathbb{N}}$ be measurable functions defined on $[0,1]$ such that, for every $t \in [0,1]$, $f(t) \in \operatorname{co}\{f_n(t), n \in \mathbb{N}\}$. Then, for any positive ε, there exist a measurable $T_\varepsilon \subset [0,1]$ with $\mu([0,1] \setminus T_\varepsilon) \leq \varepsilon$, a natural n_ε and a family of measurable positive functions $(\lambda_i)_{i=1}^{n_\varepsilon}$ such that $\sum_{i=1}^{n_\varepsilon} \lambda_i(t) = 1$ and $\left\| f(t) - \sum_{i=1}^{n_\varepsilon} \lambda_i(t) f_i(t) \right\| \leq \varepsilon$, for all $t \in T_\varepsilon$.*

Proof: Since $f(t) \in \operatorname{co}\{f_n(t), n \in \mathbb{N}\}$ for every $t \in [0,1]$, one can find a sequence of positive functions $(\lambda_i)_{i\in\mathbb{N}}$ such that, for any t, there exists $k(t) \in \mathbb{N}$ with

$$f(t) = \sum_{i=1}^{k(t)} \lambda_i(t) f_i(t) \quad \text{and} \quad \sum_{i=1}^{k(t)} \lambda_i(t) = 1.$$

Define, for every $n \in \mathbb{N}$, $T_n = \{t \in [0,1] : k(t) \leq n\}$; obviously, the sequence $(T_n)_{n\in\mathbb{N}}$ is increasing and $\bigcup_{n=1}^\infty T_n = [0,1]$, whence there are $n_\varepsilon \in \mathbb{N}$ and a measurable $T_\varepsilon \subset T_{n_\varepsilon}$ such that $\mu([0,1] \setminus T_\varepsilon) \leq \varepsilon$.

Note that, for every $t \in T_\varepsilon$, $f(t) = \sum_{i=1}^{n_\varepsilon} \lambda_i(t) f_i(t)$ (with the convention that if $k(t) < n_\varepsilon$, then $\lambda_i(t) = 0$ for all $k(t) < i \leq n_\varepsilon$).

Consider the nonempty closed-valued function $\Gamma : T_\varepsilon \to \mathbb{R}_+^{n_\varepsilon}$ given by

$$\Gamma(t) = \Big\{ (\alpha_1, \ldots, \alpha_{n_\varepsilon}) : \sum_{i=1}^{n_\varepsilon} \alpha_i = 1, \Big\| f(t) - \sum_{i=1}^{n_\varepsilon} \alpha_i f_i(t) \Big\| \leq \varepsilon \Big\}.$$

It is measurable. Indeed, for every $x = (x_1, \ldots, x_{n_\varepsilon}) \in \mathbb{R}_+^{n_\varepsilon}$, the real function $t \to d(x, \Gamma(t))$ is measurable, since

$$d(x,\Gamma(t)) = \inf \Big\{ d\left(x, (\alpha_1, .., \alpha_{n_\varepsilon})\right), (\alpha_1, .., \alpha_{n_\varepsilon}) \in \mathbb{Q}_+^{n_\varepsilon}, \sum_{i=1}^{n_\varepsilon} \alpha_i = 1, \Big\| f(t) - \sum_{i=1}^{n_\varepsilon} \alpha_i f_i(t) \Big\| \leq \varepsilon \Big\}.$$

Therefore, Γ has a measurable selection, that satisfies the conclusion of the lemma. □

Proposition 8. *Let $(f_n)_{n\in\mathbb{N}}$ and f be X-valued Pettis integrable functions defined on $[0,1]$.*
(1) If there exists a sequence $\left(\widetilde{f_n}\right)_n$ of measurable functions such that, for μ-almost every $t\in[0,1]$, $\widetilde{f_n}(t)\in\operatorname{co}\{f_n(t),n\in\mathbb{N}\}$ strongly converges to $f(t)$, then there exists $\widetilde{\widetilde{f_n}}\in\operatorname{dec}\{f_n,n\in\mathbb{N}\}$ convergent to f with respect to the $|||\cdot|||$ norm topology.
(2) If $(f_n)_n$ is PUI *and there exists a sequence $\left(\widetilde{f_n}\right)_n$ of measurable functions such that, for μ-almost every $t\in[0,1]$, $\widetilde{f_n}(t)\in\operatorname{co}\{f_m(t),m\geq n\}$ is strongly convergent to $f(t)$, then there exists $\widetilde{\widetilde{f_n}}\in\operatorname{dec}\{f_m,m\geq n\}$ $|||\cdot|||$-convergent to f.*

Proof: (1) By the Pettis integrability assumption we can find, for every $\varepsilon>0$, a $\delta_\varepsilon>0$ such that, for every measurable A with $\mu(A)\leq\delta_\varepsilon$,

$$\Big\|\int_A f(s)\,ds\Big\|\leq\varepsilon\quad\text{ and }\quad\Big\|\int_A f_1(s)\,ds\Big\|\leq\varepsilon.$$

By Egorov's theorem, the sequence $\left(\widetilde{f_n}\right)_n$ is μ-almost uniformly convergent to f, therefore, we are able to find T_ε with $\mu\left([0,1]\setminus T_\varepsilon\right)\leq\frac{\delta_\varepsilon}{2}$ and $n_\varepsilon\in\mathbb{N}$ such that $\left\|\widetilde{f_n}(t)-f(t)\right\|\leq\varepsilon$, for every $t\in T_\varepsilon$ and $n\geq n_\varepsilon$.

Since for every $n\geq n_\varepsilon$, $\widetilde{f_n}(t)\in\operatorname{co}\{f_n(t),n\in\mathbb{N}\}$, by Lemma 7, there exist a measurable $\overline{T}_{\varepsilon,n}\subset[0,1]$ with $\mu\left([0,1]\setminus\overline{T}_{\varepsilon,n}\right)\leq\frac{\delta_\varepsilon}{2}$, a natural $k_{\varepsilon,n}$ and a family $\left(\lambda_i^n\right)_{i=1}^{k_{\varepsilon,n}}$ of measurable positive functions such that, for every $t\in\overline{T}_{\varepsilon,n}$, $\sum_{i=1}^{k_{\varepsilon,n}}\lambda_i^n(t)=1$ and $\left\|\widetilde{f_n}(t)-\sum_{i=1}^{k_{\varepsilon,n}}\lambda_i^n(t)f_i(t)\right\|\leq\varepsilon$.

By Lemma 6 we can find, for every $n\geq n_\varepsilon$, a measurable partition $\left(M_i^n\right)_{i=1}^{k_{\varepsilon,n}}$ of $T_\varepsilon\cap\overline{T}_{\varepsilon,n}$ satisfying that

$$\sup_{[a,b]\subset\mathbb{R}}\Big\|\int_{[a,b]\cap(T_\varepsilon\cap\overline{T}_{\varepsilon,n})}\Big(\sum_{i=1}^{k_{\varepsilon,n}}\lambda_i^n(s)f_i(s)-\sum_{i=1}^{k_{\varepsilon,n}}\chi_{M_i^n}(s)f_i(s)\Big)ds\Big\|\leq\varepsilon.$$

Then, the function $\widetilde{\widetilde{f_n^\varepsilon}}=\sum_{i=1}^{k_{\varepsilon,n}}\chi_{M_i^n}f_i+\chi_{[0,1]\setminus(T_\varepsilon\cap\overline{T}_{\varepsilon,n})}f_1\in\operatorname{dec}\{f_n,n\in\mathbb{N}\}$ satisfies, for every $n\geq n_\varepsilon$,

$$\begin{aligned}\Big|\Big|\Big|\widetilde{\widetilde{f_n^\varepsilon}}-f\Big|\Big|\Big|&=\sup_{[a,b]\subset[0,1]}\Big\|\int_a^b\Big(\widetilde{\widetilde{f_n^\varepsilon}}(s)-f(s)\Big)\,ds\Big\|\\&\leq\sup_{[a,b]\subset[0,1]}\Big\|\int_{[a,b]\cap(T_\varepsilon\cap\overline{T}_{\varepsilon,n})}\Big(\sum_{i=1}^{k_{\varepsilon,n}}\chi_{M_i^n}(s)f_i(s)-\sum_{i=1}^{k_{\varepsilon,n}}\lambda_i^n(s)f_i(s)\Big)\,ds\Big\|\end{aligned}$$

$$+ \sup_{[a,b]\subset[0,1]} \Big\| \int_{[a,b]\cap(T_\varepsilon\cap\overline{T}_{\varepsilon,n})} \Big(\sum_{i=1}^{k_{\varepsilon,n}} \lambda_i^n(s) f_i(s) - \widetilde{f_n}(s) \Big)\, ds \Big\|$$

$$+ \int_{T_\varepsilon\cap\overline{T}_{\varepsilon,n}} \Big\| \widetilde{f_n}(s) - f(s) \Big\|\, ds + \sup_{A\subset[0,1]\setminus(T_\varepsilon\cap\overline{T}_{\varepsilon,n})} \Big\| \int_A f(s)\, ds \Big\|$$

$$+ \sup_{A\subset[0,1]\setminus(T_\varepsilon\cap\overline{T}_{\varepsilon,n})} \Big\| \int_A f_1(s)\, ds \Big\| \leq 5\varepsilon.$$

Hence, by taking $\varepsilon = \frac{1}{5m}$, we find $\widetilde{\widetilde{f_n^m}} \in \operatorname{dec}\{f_n, n\in\mathbb{N}\}$ such that, for each $n \geq n_m$, $|||\widetilde{\widetilde{f_n^m}} - f||| \leq \frac{1}{m}$.

(2) The proof is similar, the only difference is that, by the PUI assumption, for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that, for every measurable A with $\mu(A) \leq \delta_\varepsilon$,

$$\Big\| \int_A f(s)\, ds \Big\| \leq \varepsilon \quad \text{and} \quad \Big\| \int_A f_n(s) ds \Big\| \leq \varepsilon, \forall\, n \in \mathbb{N}$$

and we can set the function $\widetilde{\widetilde{f_n^\varepsilon}} = \sum_{i=1}^{k_{\varepsilon,n}} \chi_{M_i^n} f_{n+i} + \chi_{[0,1]\setminus(T_\varepsilon\cap\overline{T}_{\varepsilon,n})} f_n \in \operatorname{dec}\{f_m, m \geq n\}$ in order to find the requested sequence. □

Corollary 9. *Every decomposable $|||\cdot|||$-closed subset K of $L^1(\mu, X)$ is sequentially closed with respect to the weak topology of $L^1(\mu, X)$.*

Proof: Consider $(f_n)_n \subset K$ weakly convergent towards $f \in L^1(\mu, X)$. We can find a sequence $\big(\widetilde{f_n}\big)_n$ of convex combinations which converges with respect to the strong topology of $L^1(\mu, X)$, so, we can extract a subsequence convergent almost everywhere. There exists therefore $N \subset [0,1]$ of null measure such that, for every $t \in [0,1]\setminus N$, $\widetilde{f_n}(t) \to f(t)$ and, by the previous proposition, there exists $\widetilde{\widetilde{f_n}} \in \operatorname{dec}\{f_n, n\in\mathbb{N}\}$ convergent to f with respect to the $|||\cdot|||$ norm topology. As the subset is supposed to be decomposable and $|||\cdot|||$-closed, it follows that $f \in K$ and so, the assertion is proved. □

In order to obtain the main result of the paper, we need an easy lemma.

Lemma 10. *The decomposable hull of a convex set K of measurable functions is convex.*

Proof: Let us arbitrarily consider the collection $\{f_i, g_i\}_{i=1}^n \subset K$, $\lambda \in [0,1]$ and $\{A_i\}_{i=1}^n \subset \Sigma$ a measurable partition of the space. Then

$$\lambda\Big(\sum_{i=1}^n f_i\chi_{A_i}\Big) + (1-\lambda)\Big(\sum_{i=1}^n g_i\chi_{A_i}\Big) = \sum_{i=1}^n (\lambda f_i + (1-\lambda)\, g_i)\, \chi_{A_i} \in \operatorname{dec} K$$

since K is convex. Hence, $\operatorname{dec} K$ is convex. □

The main theorem of the paper yields the convexity of decomposable $|||\cdot|||$-closed subsets of $\mathrm{Pe}(\mu, X)$. This result will allow us to obtain a new relationship between the $|||\cdot|||$-topology and the weak topology on the spaces of Pettis, resp. Bochner integrable functions.

Theorem 11. *Every decomposable, $|||\cdot|||$-closed subset $K \subset \mathrm{Pe}(\mu, X)$ is convex.*

Proof: The Banach space being separable, $\mathrm{Pe}(\mu, X)$ is, so we are able to find a sequence $(f_n)_n \subset \mathrm{Pe}(\mu, X)$ such that $K = \overline{(\{f_n, n \in \mathbb{N}\})}^{\|\cdot\|_{\mathrm{Pe}}}$.

Since the Pettis norm topology is stronger than the topology given by the previous norm, $K \subset \overline{\operatorname{dec}}^{\|\cdot\|_{\mathrm{Pe}}} \{f_n, n \in \mathbb{N}\} \subset \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}$ and, by the hypothesis, $\overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\} \subset K$. Therefore, $K = \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}$.

Now, we prove that

$$\operatorname{co} \{f_n, n \in \mathbb{N}\} \subset \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}.$$

For this purpose, consider $f \in \operatorname{co} \{f_n, n \in \mathbb{N}\}$ and fix $\varepsilon > 0$. By applying Lemma 6, we obtain a measurable partition $(M_k)_{k=1}^N$ of $[0, 1]$ such that

$$\sup_{[a,b] \subset [0,1]} \left\| \int_a^b \left(f - \sum_{k=1}^N f_k \chi_{M_k} \right) d\mu \right\| \leq \varepsilon,$$

hence $f_\varepsilon = \sum_{k=1}^N f_k \chi_{M_k} \in \operatorname{dec} \{f_n, n \in \mathbb{N}\}$ and satisfies the inequality $|||f - f_\varepsilon||| \leq \varepsilon$.

Consequently, $f \in \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}$, and so the inclusion is proved.

By passing to the closed decomposable hull, we immediately obtain that $\overline{\operatorname{dec}}^{|||\cdot|||} (\operatorname{co} \{f_n, n \in \mathbb{N}\}) \subset \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}$ and, since the inclusion in the other sense is obvious, it follows that

$$\overline{\operatorname{dec}}^{|||\cdot|||} (\operatorname{co} \{f_n, n \in \mathbb{N}\}) = \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}.$$

As a final argument, by Lemma 10, the set $\operatorname{dec} (\operatorname{co} \{f_n, n \in \mathbb{N}\})$ is convex. We deduce that $K = \overline{\operatorname{dec}}^{|||\cdot|||} \{f_n, n \in \mathbb{N}\}$ is convex. □

Remark 12. Since $L^1(\mu, X) \subset \mathrm{Pe}^1(\mu, X)$, it follows that every decomposable $|||\cdot|||$-closed subset $K \subset L^1(\mu, X)$ is convex.

On the other side, by Corollary 9, any such K is sequentially closed with respect to the weak topology of $L^1(\mu, X)$, but not closed, therefore the convexity cannot be obtained as a consequence of Theorem 2.3.17 in [HuP].

This convexity result allows us to obtain a relationship between the Alexiewicz norm topology and the weak topology on the space of Pettis, resp. Bochner integrable functions:

Corollary 13. (1) *Any* $|||\cdot|||$*-closed decomposable* $K \subset \mathrm{Pe}(\mu, X)$ *is closed with respect to the weak topology of* $\mathrm{Pe}(\mu, X)$.
(2) *Any subset* $K \subset L^1(\mu, X)$ *decomposable and* $|||\cdot|||$*-closed is closed with respect to the weak topology of* $L^1(\mu, X)$.

Proof: We will prove only the first assertion. K is supposed to be closed with respect to a topology weaker than the topology of the Pettis norm, so it is $\|\cdot\|_{\mathrm{Pe}}$-closed and then, since Theorem 11 yields that K is convex, it is weakly closed. □

Finally, let us give the following

Remark 14. A result similar to the main theorem in [Gut] can be proved for the Pettis integrability setting. More precisely, the weak topology of the space $\mathrm{Pe}(\mu, X)$ and the $|||\cdot|||$-topology coincide on any subset $F \subset \mathrm{Pe}(\mu, X)$ that is PUI and relatively compact with respect to the weak topology of the space $\mathrm{Pe}(\mu, X)$.

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