Homomorphic images of $\mathbb{R}$-factorizable groups

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Abstract. It is well known that every $\mathbb{R}$-factorizable group is ω-narrow, but not vice versa. One of the main problems regarding $\mathbb{R}$-factorizable groups is whether this class of groups is closed under taking continuous homomorphic images or, alternatively, whether every ω-narrow group is a continuous homomorphic image of an $\mathbb{R}$-factorizable group. Here we show that the second hypothesis is definitely false. This result follows from the theorem stating that if a continuous homomorphic image of an $\mathbb{R}$-factorizable group is a P-group, then the image is also $\mathbb{R}$-factorizable.

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1. Introduction

A topological group G is said to be *ω-narrow* if it can be covered by countably many translates of an arbitrary neighbourhood of the identity in G. It is worth mentioning that ω-narrow groups were introduced by I. Guran in [4] who called them *ω-bounded*. Since the latter term has several different meanings in topology, we change the terminology. The class of ω-narrow groups is stable with respect to all basic operations — it is closed under taking direct products, subgroups, and continuous homomorphic images ([4]).

$\mathbb{R}$-factorizable groups constitute a proper subclass of ω-narrow groups. A topological group G is called *$\mathbb{R}$-factorizable* ([8], [9]) if for every continuous function $f: G \to \mathbb{R}$, one can find a continuous homomorphism $p: G \to H$ onto a second-countable topological group H and a continuous function $h: H \to \mathbb{R}$ such that $f = h \circ p$. The class of $\mathbb{R}$-factorizable groups includes all totally bounded groups, all Lindelöf groups, arbitrary subgroups of σ-compact groups ([9]), and many others.

Unlike the case of ω-narrow groups, it is not known whether the class of $\mathbb{R}$-factorizable groups is closed with respect to topological products or taking continuous homomorphic images (but quotients of $\mathbb{R}$-factorizable groups are $\mathbb{R}$-factorizable, by [9, Theorem 3.10]). Our aim here is to consider continuous homomorphic images of $\mathbb{R}$-factorizable groups that are clearly ω-narrow. If, hypothetically, the class of $\mathbb{R}$-factorizable groups were not closed with respect to taking continuous homomorphic images, it could happen that every ω-narrow group would be a continuous image of an $\mathbb{R}$-factorizable group. Here we show that it is not the case by describing ω-narrow groups of weight $\aleph_1$ that cannot be represented in such a form. This solves in the negative the second part of Problem 3.9 in [9].

Our approach to the problem of finding such counterexamples is based on the fact that every topological group G naturally acts on the space $C_p(G)$ of continuous real-valued functions on G endowed with the topology of pointwise convergence, and this action is separately continuous. We show in Proposition 2.1 that if the group G is $\mathbb{R}$-factorizable, then the orbit Gf of every function $f \in C_p(G)$ has a countable network. Then we prove in Theorem 2.2 that a non-complete P-group of weight $\aleph_1$ cannot be a continuous homomorphic image of an $\mathbb{R}$-factorizable group by finding a continuous real-valued function on the group with the orbit of uncountable network weight. A simple construction of an ω-narrow non-complete P-group of weight $\aleph_1$ is presented in Example 2.3.

Applying more subtle methods, we improve Theorem 2.2 in Section 3 as follows: If a P-group G is a continuous homomorphic image of an $\mathbb{R}$-factorizable group, then G is also $\mathbb{R}$-factorizable. A new ingredient there is generating the topology of a given P-group by two clopen complementary subsets of the group (see Lemma 3.4).

As a matter of fact, our results rise more questions than resolve them. We finish the article with a brief discussion of open problems.

1.1 Notation and terminology. Given a subset A of a group G, the minimal subgroup of G containing A is denoted by $\langle A\rangle$, while $\langle x\rangle$, for $x \in G$, is the cyclic subgroup of G generated by x. If the set A does not contain the identity of G and the intersection $\langle x\rangle \cap \langle A_x\rangle$ is trivial for each $x \in A$, where $A_x = A \setminus \{x\}$, the set A is called *independent*.

A topological group H is *complete* if it is complete in the two-sided group uniformity, i.e., if H is Raĭkov complete. We say that H is a *P-group* if every G_δ-set in H is open.

The network weight of a space X is the minimal infinite cardinal number κ such that X has a network of cardinality less than or equal to κ. Given a completely regular space X, we denote by $C_p(X)$ the family of continuous real-valued functions on X endowed with the pointwise convergence topology ([1]).

2. Actions of groups. An example

It turns out that the $\mathbb{R}$-factorizable groups as well as their continuous homomorphic images have a common property which can be formulated in terms of a natural action of a topological group G on the space $C_p(G)$. Here are some preliminary definitions and simple facts.

Let G be a topological group. Then the natural left action of G on the space

$C_p(G)$ is defined by the formula

$$(y * f)(x) = f(y^{-1}x),$$

where $x, y \in G$ and $f \in C_p(G)$. The function $y * f \in C_p(G)$ is usually denoted by f_y and is called the *y-shift* of f. This action is separately continuous in the sense that the mapping

$$\alpha: G \times C_p(G) \to C_p(G),\ \alpha(y, f) = f_y,$$

is separately continuous. Given $f \in C_p(G)$, the set

$$Gf = \{y * f : y \in G\}$$

is called the *orbit* of f and is considered as a subspace of $C_p(G)$.

Proposition 2.1. *Let $\pi: G \to H$ be a continuous onto homomorphism of topological groups, where the group G is $\mathbb{R}$-factorizable. Then the orbit Hf of any element $f \in C_p(H)$ under the natural left action of H on $C_p(H)$ has a countable network.*

PROOF: Take any $f \in C_p(H)$ and let $g = f \circ \pi$. Clearly, $g \in C_p(G)$. Since the group G is $\mathbb{R}$-factorizable, we can find a continuous homomorphism $p: G \to K$ onto a second-countable topological group K and a continuous real-valued function h on K such that $g = h \circ p$.

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "p"'] \arrow[dr, "g"] & H \arrow[d, "f"] \\ K \arrow[r, "h"] & \mathbb{R} \end{tikzcd}$$

Consider the continuous mapping $p^*: C_p(K) \to C_p(G)$ defined by the rule $p^*(\varphi) = \varphi \circ p$, for each $\varphi \in C_p(K)$. We claim that

$$(1) \qquad p^*(Kh) = Gg,$$

where Kh and Gg are the orbits of h and g in $C_p(K)$ and $C_p(G)$, respectively.

Indeed, it is clear that $p^*(h) = g$, since $g = h \circ p$. Take any element $x_0 \in G$ and let g_{x_0} be the x_0-shift of g. Put $y_0 = p(x_0)$ and let h_{y_0} be the y_0-shift of h. For every $x \in G$, we have:

$$\begin{aligned} p^*(h_{y_0})(x) &= (h_{y_0} \circ p)(x) = h_{y_0}(p(x)) = h(y_0^{-1}p(x)) \\ &= h(p(x_0)^{-1}p(x)) = h(p(x_0^{-1}x)) = g(x_0^{-1}x) = g_{x_0}(x). \end{aligned}$$

This proves that $p^*(h_{y_0}) = g_{x_0}$ and, since $h_{y_0} \in Kh$, equality (1) follows.

Since the group K is second-countable, it follows that the space $C_p(K)$ has a countable network [1, Theorem I.1.3] and, by the continuity of p^*,

$$nw(Gg) \le nw(Kh) \le nw(C_p(K)) \le \omega. \tag{2}$$

Similarly to (1), we also have the equality

$$\pi^*(Hf) = Gg, \tag{3}$$

where $\pi^*: C_p(H) \to C_p(G)$ is defined by $\pi^*(\psi) = \psi \circ \pi$, for each $\psi \in C_p(H)$. Since π is continuous, π^* is a homeomorphic embedding of $C_p(H)$ to $C_p(G)$ (this follows from [1, Proposition 0.4.6]). Therefore, (2) and (3) together imply that $nw(Hf) = nw(Gg) \le \omega$. This finishes the proof. $\square$

We now prove the following result that will be generalized in Section 3:

Theorem 2.2. *Let G be a Boolean P-group of character $\aleph_1$. If G is not complete, then it cannot be represented as a continuous homomorphic image of an $\mathbb{R}$-factorizable topological group.*

PROOF: According to [8, Proposition 5.3], every $\mathbb{R}$-factorizable group is ω-narrow, and the class of ω-narrow groups is evidently closed under taking continuous homomorphic images. Hence, we can assume without loss of generality that G is ω-narrow. Since the weight and character of ω-narrow topological groups coincide [11, Proposition 4.1], we have that $w(G) = \aleph_1$. We also note that all non-empty open sets in G are uncountable — otherwise the ω-narrow P-group G would be countable and discrete, contradicting the assumption that the character of G is equal to $\aleph_1$.

Denote by H the Raĭkov completion of the group G. Since G is not complete, the complement $H \setminus G$ is not empty. Let e be the neutral element of H. By virtue of [11, Lemma 2.1], H is also a P-group of character $\aleph_1$, and H has a decreasing local base of length ω_1 at e consisting of open invariant subgroups. Clearly, each open subgroup of H is closed. Hence, H has a base of clopen subsets.

Choose an element $h^* \in H \setminus G$. By recursion of length ω_1 one can easily define a sequence $\{x_\alpha : \alpha < \omega_1\}$ of points satisfying the following conditions for all $\alpha, \beta < \omega_1$:

(a) $x_\alpha \in G$;
(b) the set $\{x_\alpha : \alpha < \omega_1\}$ is independent;
(c) the elements $x_\alpha^{-1} x_{\alpha+1}$ converge to h^* when $\alpha \to \omega_1$;
(d) e and h^* are the only accumulation points of the set $\{x_\alpha^{-1} x_\beta : \alpha < \beta < \omega_1\}$ in H.

Since the group G is Boolean, each element $x \in G$ satisfies $x^{-1} = x$. However, we keep writing x^{-1} just to follow notation in the case of an arbitrary group (in fact, the argument below can easily be adapted to this more general situation).

It suffices to choose independent elements $x_\alpha \in G$, with $\alpha < \omega_1$ in such a way that the transfinite sequence $\{x_\alpha : \alpha \text{ is even}, \alpha < \omega_1\}$ converges to e, while $\{x_\alpha : \alpha \text{ is odd}, \alpha < \omega_1\}$ converges to h^*. To make the set $\{x_\alpha : \alpha < \omega_1\}$ independent, one chooses $x_\beta \in G \setminus \langle X_\beta \rangle$ at the stage $\beta < \omega_1$ of the construction, where $X_\beta = \{x_\alpha : \alpha < \beta\}$. This is possible, since the set $\langle X_\beta \rangle$ is countable, while each non-empty open set in G is uncountable.

Once we have defined the set $\{x_\alpha : \alpha < \omega_1\}$, it is clear that there exists a family $\{U_\alpha : \alpha < \omega_1\}$ of clopen sets in H satisfying the following conditions for all $\alpha, \beta < \omega_1$:

(i) $\langle h^* \rangle \cap U_\alpha = \emptyset$, for each $\alpha < \omega_1$;
(ii) $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$;
(iii) the sets U_α converge to h^* when $\alpha \to \omega_1$;
(iv) if $\alpha < \beta$ and $\nu < \omega_1$, then $x_\alpha^{-1} x_\beta \in U_\nu$ iff $\beta = \alpha + 1$ and $\nu = \alpha$;
(v) the family $\{G \cap U_\alpha : \alpha < \omega_1\}$ is discrete in G.

To choose the sets U_α, notice that by (b) and (c), the set $\{x_\alpha^{-1} x_{\alpha+1} : \alpha < \omega_1\}$ is closed and discrete in G. Since, by (d), all accumulation points of the set $\{x_\alpha^{-1} x_\beta : \alpha < \beta < \omega_1\}$ in the P-group H lie in $\langle h^* \rangle$, we can separate the elements $x_\alpha^{-1} x_{\alpha+1}$ by open neighbourhoods U_α in H satisfying (i)–(iv). Notice that the space G is paracompact, as every regular P-space of weight $\aleph_1$ ([12]). Hence, one can take smaller neighbourhoods of the elements $x_\alpha^{-1} x_{\alpha+1}$, if necessary, to fulfill (v).

Let f be a function on G such that $f(x) = 1$ if $x \in G \cap U_\alpha$ for some $\alpha < \omega_1$, and $f(x) = 0$ otherwise. By (v), the function f is continuous. It remains to show that the orbit Gf of f in $C_p(G)$ has uncountable network weight. To this end, it suffices to verify that the subspace $\{x_\nu f : \nu < \omega_1\}$ of Gf is not separable. In fact, we claim that for each $\alpha < \omega_1$, the element $x_\alpha f \in C_p(G)$ in not in the closure of the set $\{x_\nu f : \nu < \alpha\}$. In other words, we claim that the subspace $\{x_\alpha f : \alpha < \omega_1\}$ of $C_p(G)$ is naturally *left-separated*.

Indeed, given $\alpha < \omega_1$, we put

$$O_\alpha = \{\varphi \in C_p(G) : \varphi(x_\alpha) < \varphi(x_{\alpha+1})\}.$$

It follows from (i) that $e \notin U_\alpha$, so $(x_\alpha f)(x_\alpha) = f(x_\alpha^{-1} x_\alpha) = f(e) = 0$ and, by (iv), $(x_\alpha f)(x_{\alpha+1}) = f(x_\alpha^{-1} x_{\alpha+1}) = 1$. Hence, O_α is an open neighbourhood of $x_\alpha f$ in $C_p(G)$, for each $\alpha < \omega_1$. Similarly, if $\nu < \alpha$, then the definition of f and (iv) imply that

$$(x_\nu f)(x_\alpha) = f(x_\nu^{-1} x_\alpha) \geq 0 \quad \text{and} \quad (x_\nu f)(x_{\alpha+1}) = f(x_\nu^{-1} x_{\alpha+1}) = 0.$$

Hence, $(x_\nu f)(x_{\alpha+1}) \leq (x_\nu f)(x_\alpha)$, and we conclude that $x_\nu f \notin O_\alpha$, for each $\nu < \alpha$. This proves our claim and implies that G cannot be the image of an $\mathbb{R}$-factorizable topological group under a continuous homomorphism. $\square$

Here is a simple example of an ω-narrow P-group G of weight $\aleph_1$ which is not complete and, therefore, cannot be represented as a continuous homomorphic image of any $\mathbb{R}$-factorizable group. The weaker fact that the group G is not $\mathbb{R}$-factorizable can be deduced from [6, Theorem 3.1].

Example 2.3. *There exists an ω-narrow topological Abelian group G satisfying $w(G) = |G| = \aleph_1$ which is not a continuous homomorphic image of any $\mathbb{R}$-factorizable topological group.*

According to Theorem 2.2, it suffices to find a non-complete ω-narrow P-group G satisfying the required cardinal restrictions. Let $F = D^{\omega_1}$, where $D = \{0, 1\}$ is the discrete group with addition modulo 2. For every $x \in F$, put $\mathrm{supp}(x) = \{\alpha \in \omega_1 : x(\alpha) = 1\}$ and consider the subgroup

$$H = \{x \in F : |\,\mathrm{supp}(x)| < \omega\}$$

of the group F. The group H is known as the *σ-product* of ω_1 copies of the group D. For every $\alpha < \omega_1$, let

$$N_\alpha = \{x \in H : \mathrm{supp}(x) \cap \alpha = \emptyset\}.$$

Each N_α is a subgroup of H, and a direct verification shows that the decreasing transfinite sequence $\{N_\alpha : \alpha < \omega_1\}$ is a base for a Hausdorff topological group topology at the neutral element of H (known also as the *ω-box topology*). It follows from the definition of the topology on H that the intersection of every countable family open sets in H is open, i.e., H is a P-group. It is easy to see that $w(H) = |H| = \aleph_1$. In addition, the group H with this topology is Lindelöf ([3]).

We now define a subspace G of H by the formula

$$G = \{x \in H : |\,\mathrm{supp}(x)| \text{ is even}\}.$$

Since the groups D and H are Boolean, G is a subgroup of H. We claim that G is as required.

It is clear that G is Abelian and satisfies $w(G) = |G| = \aleph_1$. It is also clear that each N_α has countable index in H, so that the group H is ω-narrow. Therefore, the subgroup G of H is also ω-narrow. Clearly, G is a P-group. Notice that G is a proper dense subgroup of H, so that G is not complete. Finally, by Theorem 2.2, G cannot be the image of an $\mathbb{R}$-factorizable group under a continuous homomorphism. $\square$

3. General case

Let us now generalize Theorem 2.2 by eliminating the condition of non-completeness of the group G. All we need is to establish the next result that implies the preservation of $\mathbb{R}$-factorizability under taking continuous homomorphic images provided the images are P-groups:

Theorem 3.1. *If a P-group G is a continuous homomorphic image of an $\mathbb{R}$-factorizable topological group, then G itself is $\mathbb{R}$-factorizable.*

The proof of Theorem 3.1 requires several auxiliary results. The first of them is an easy combination of some well-known facts from the C_p-theory.

Given a space Y and a family F of continuous real-valued functions on Y, we say that F *generates the topology of* Y if

$$\{f^{-1}(U) : f \in F,\ U \text{ is open in } \mathbb{R}\}$$

is a subbase for the topology of Y.

Lemma 3.2. *Let X be a space, $F \subseteq C_p(X)$, and φ the diagonal product of the family F. Then the image $Y = \varphi(X) \subseteq \mathbb{R}^F$ satisfies $nw(Y) = nw(F)$.*

PROOF: For every $f \in F$, there exists a continuous real-valued function $\tilde{f}$ on Y such that $f = \tilde{f} \circ \varphi$ (in fact, $\tilde{f}$ is the restriction to Y of the projection of $\mathbb{R}^F$ onto the fth factor). Consider the dual mapping $\varphi^*: C_p(Y) \to C_p(X)$ defined by the rule $\varphi^*(h) = h \circ \varphi$, for each $h \in C_p(Y)$. It follows from [1, Proposition 0.4.6] that φ^* is a homeomorphic embedding of $C_p(Y)$ into $C_p(X)$. It is also clear that $F = \varphi^*(F_Y)$, where $F_Y = \{\tilde{f} : f \in F\}$.

It follows from the definition of F_Y that this family generates the topology of Y. Therefore, according to [1, Proposition 0.5.4], the evaluation mapping $\psi: Y \to C_p(F_Y)$, where $(\psi(y))(h) = h(y)$ for $h \in C_p(F_Y)$, is a homeomorphic embedding of Y into $C_p(F_Y)$. Then we apply [1, Theorem I.1.3] to conclude that $nw(C_p(F_Y)) = nw(F_Y)$, whence it follows that $nw(Y) \leq nw(C_p(F_Y)) = nw(F_Y)$. In addition, since $nw(Y) = nw(C_p(Y))$, we have that $nw(F_Y) \leq nw(C_p(Y)) = nw(Y)$. It follows that $nw(Y) = nw(F_Y) = nw(F)$. □

Lemma 3.3. *For a P-group G, the following are equivalent:*

(a) *G is $\mathbb{R}$-factorizable;*
(b) *G is ω-narrow and every continuous homomorphic image H of G satisfying $\psi(H) \leq \aleph_1$ is Lindelöf;*
(c) *G is ω-narrow and every continuous homomorphic image H of G satisfying $w(H) \leq \aleph_1$ is Lindelöf;*
(d) *G is ω-narrow and every continuous homomorphic image H of G satisfying $w(H) \leq \aleph_1$ is Lindelöf provided that H is a P-group.*

PROOF: The equivalence of (a) and (b) of the lemma follows from [11, Theorem 4.16]. It is also clear that (b) implies (c) and (c) implies (d). Hence, it suffices to show that (d) $\Rightarrow$ (b).

Suppose that G satisfies (d), and let $p: G \to H$ be a continuous homomorphism onto a group H with $\psi(H) \leq \aleph_1$. We can assume without loss of generality that H is a P-group. Indeed, let N be the kernel of the homomorphism p, and

$\pi: G \to G/N$ the canonical open homomorphism onto the quotient group G/N. Clearly, there exists a continuous isomorphism $i: G/N \to H$ satisfying $p = i \circ \pi$. Then G/N is a P-group according to [11, Lemma 2.1(c)]. Since i is one-to-one and continuous, we conclude that $\psi(G/N) \le \psi(H) \le \aleph_1$. Therefore, the Lindelöf property of H would follow if we knew that G/N is Lindelöf. In the sequel we assume that H is a P-group.

Let $\{U_\alpha : \alpha < \omega_1\}$ be a family of open sets in H such that $\{e\} = \bigcap_{\alpha<\omega_1} U_\alpha$, where e is the neutral element of H. The group H is ω-narrow as a continuous homomorphic image of the ω-narrow group G, so H has a local base at e consisting of open invariant subgroups [11, Lemma 2.1(b)]. Therefore, we can easily define by recursion a decreasing family $\xi = \{V_\alpha : \alpha < \omega_1\}$ of open invariant subgroups of H such that $V_\alpha \subseteq U_\alpha$, for each $\alpha < \omega_1$. Let τ be the topological group topology on H whose local base at the neutral element e is the family ξ. Then the topological group $K = (H, \tau)$ is a Hausdorff P-group and, by the definition of the topology τ, we have that $\chi(K) \le \aleph_1$. Hence, $w(K) \le \aleph_1$ ([10, Proposition 4.1]). It is also clear that the identity isomorphism $i: H \to K$ is continuous. Thus, the composition $i \circ p$ is a continuous homomorphism of G onto the P-group K of weight $\aleph_1$, and our assumption about G implies that K is Lindelöf. We claim that i is a homeomorphism (hence, a topological isomorphism).

Suppose by the way of contradiction that i is not a homeomorphism. Then there exists an open invariant subgroup U^* in H such that the image $i(U^*)$ is not open in K (otherwise i would be an open mapping, hence, a homeomorphism). Let τ^* be the topological group topology on H whose local base at e is the family $\xi^* = \{U^* \cap V_\alpha : \alpha < \omega_1\}$. Again, $K^* = (H, \tau^*)$ is a Hausdorff P-group of weight $\le \aleph_1$ and K^* is a continuous homomorphic image of G. Therefore, K^* is Lindelöf. It follows from our definition of the topologies τ and τ^* on K and K^* that the identity mapping $\varphi: K^* \to K$ is a continuous isomorphism. It also follows that the set U^* is open in K^*, while $\varphi(U^*) = i(U^*)$ is not open in K. However, it follows from [11, Lemma 2.4] that a continuous onto homomorphism of Lindelöf P-groups is open, so φ must be open as well. This contradiction proves that i is a topological isomorphism and, therefore, H is Lindelöf. □

Let H be a topological group and $\mathcal{F}$ a family of open subsets of H. We say that the topology of H is *generated by* $\mathcal{F}$ if the family

$$\{xU : U \in \mathcal{F},\ x \in H\}$$

is a subbase for the topology of H. A similar (but different) way of generating topological group topologies was considered in [2]. In a sense, the next lemma complements the results of [2].

Lemma 3.4. *Let H be an ω-narrow P-group of weight $\aleph_1$. If H is not Lindelöf, then the topology of H is generated by two complementary clopen sets W_0 and $W_1 = H \setminus W_0$.*

PROOF: It is clear that H, being a regular P-space, is zero-dimensional. Since H is a P-group of weight $\aleph_1$, it follows from [12] that H is paracompact and strongly zero-dimensional, i.e., $\dim H = 0$. Therefore, since H is not Lindelöf, we can find a disjoint open cover γ of H such that no countable subfamily of γ covers H.

Denote by $\mathcal{N}$ the family of open invariant subgroups of H. By [11, Lemma 2.1(b)], $\mathcal{N}$ is a local base at the neutral element e of H. Since H is a P-group and $\chi(H) = w(H) = \aleph_1$, we can find a family $\xi = \{U_\alpha : \alpha < \omega_1\}$ of elements of $\mathcal{N}$ such that ξ is a local base of H at e.

Take an element $g \in H$, $g \neq e$. We are going to construct by recursion sequences $\{x_\alpha : \alpha < \omega_1\}$ and $\{y_\alpha : \alpha < \omega_1\}$ of elements of H and a sequence $\{N_\alpha : \alpha < \omega_1\} \subseteq \mathcal{N}$ satisfying the following conditions for all $\alpha, \beta < \omega_1$:

(1) $N_\beta \subseteq N_\alpha$ if $\alpha < \beta$;
(2) $N_\alpha \subseteq U_\alpha$;
(3) $x_\alpha \in gN_\alpha$;
(4) $x_\alpha \notin gN_\beta$ if $\alpha < \beta$;
(5) $\langle T_\alpha \rangle \cap N_\alpha = \{e\}$, where $T_\alpha = \{g\} \cup \{x_\nu : \nu < \alpha\} \cup \{y_\nu : \nu \leq \alpha\}$;
(6) if $z \in \langle T_\alpha \rangle \cap V$ for some $V \in \gamma$, then $zN_\alpha \subseteq V$;
(7) the set $\langle Z_\alpha \rangle \cdot N_\alpha$ is covered by a countable subfamily of γ, where $Z_\alpha = T_\alpha \cup \{x_\alpha\}$;
(8) if $\alpha < \beta$ and $y_\beta \in V$ for some $V \in \gamma$, then $V \cap (\langle Z_\alpha \rangle \cdot N_\alpha) = \emptyset$ and, in particular, $y_\beta \notin y_\alpha N_\alpha$.

Notice that by (1)–(3), the sequence $\{x_\alpha : \alpha < \omega_1\}$ will converge to g.

Take an arbitrary element $y_0 \in H \setminus \langle g \rangle$. Clearly, $H_0 = \langle g, y_0 \rangle$ is a countable subgroup of the P-group H, so we can choose $N_0 \in \mathcal{N}$ such that $N_0 \subseteq U_0$, $N_0 \cap H_0 = \{e\}$, and if $z \in H_0 \cap V$ for some $V \in \gamma$, then $zN_0 \subseteq V$. In particular, the set $H_0 \cdot N_0$ is covered by a countable subfamily γ_0 of γ. Let $x_0 \in gN_0 \setminus H_0$ be arbitrary. Then conditions (1)–(6) and (8) are evidently satisfied at the initial stage. Condition (7) follows from our choice of N_0 and x_0. Indeed, since $x_0 \in gN_0$ and N_0 is an invariant subgroup of H, we have that $x_0N_0 = gN_0$ and

$$\langle Z_0 \rangle \cdot N_0 = \langle g, y_0, x_0 \rangle \cdot N_0 = \langle g, y_0 \rangle \cdot N_0 = H_0 \cdot N_0 \subseteq \bigcup \gamma_0.$$

Suppose that for some $\beta < \omega_1$ we have defined sequences $\{x_\alpha : \alpha < \beta\}$, $\{y_\alpha : \alpha < \beta\}$, and $\{N_\alpha : \alpha < \beta\}$ satisfying (1)–(8) at each stage less than β. It follows from (7) that the set $\bigcup_{\alpha<\beta} \langle Z_\alpha \rangle \cdot N_\alpha$ is covered by a countable subfamily λ_β of γ. Take an arbitrary element $y_\beta \in H \setminus \bigcup \lambda_\beta$ and consider the subgroup $H_\beta = \langle T_\beta \rangle$ of H, where $T_\beta = \{g\} \cup \{x_\alpha : \alpha < \beta\} \cup \{y_\alpha : \alpha \leq \beta\}$. Since H is a P-group, we can choose $N_\beta \in \mathcal{N}$ such that $N_\beta \subseteq U_\beta \cap \bigcap_{\alpha<\beta} N_\alpha$, $N_\beta \cap H_\beta = \{e\}$, and if $h \in H_\beta \cap V$ for some $V \in \gamma$, then $hN_\beta \subseteq V$. It follows that the set $H_\beta \cdot N_\beta$ is covered by a countable subfamily γ_β of γ. It remains to choose a point $x_\beta \in gN_\beta \setminus H_\beta$. As at the stage zero, we have that

$$\langle Z_\beta \rangle \cdot N_\beta = \langle T_\beta \cup \{x_\beta\} \rangle \cdot N_\beta = \langle T_\beta \rangle \cdot N_\beta = H_\beta \cdot N_\beta \subseteq \bigcup \gamma_\beta,$$

that is, (7) holds true at the stage β. Conditions (1)–(3) and (5)–(8) are satisfied because of our choice of y_β, x_β, and N_β. Let us verify (4). If $\alpha < \beta$, then $g, x_\alpha \in T_\beta$, whence $g^{-1}x_\alpha \in \langle T_\beta \rangle = H_\beta$. Since $x_\alpha \neq g$ and $N_\beta \cap H_\beta = \{e\}$, the element $g^{-1}x_\alpha$ is not in N_β and, hence, $x_\alpha \notin gN_\beta$. We have thus finished the recursive construction.

Let $W = \bigcup_{\nu<\omega_1} y_\nu N_\nu$. Conditions (6) and (8) imply that the set W is clopen in H. We claim that the following equality holds for each $\beta < \omega_1$:

$$(*) \qquad x_\beta W \cap gW = \bigcup_{\nu \le \beta} g y_\nu N_\nu.$$

Since $gW = \bigcup_{\alpha<\omega_1} g y_\alpha N_\alpha$, the above equality will follow if we show that

$$x_\beta y_\nu N_\nu \cap g y_\alpha N_\alpha = \begin{cases} g y_\nu N_\nu & \text{if } \nu = \alpha \le \beta; \\ \emptyset & \text{otherwise.} \end{cases}$$

To this end, we consider the following six cases.

Case 1. $\beta \le \nu < \alpha$. Then $x_\beta y_\nu N_\nu \cap g y_\alpha N_\alpha \neq \emptyset$ iff $y_\alpha \in g^{-1}x_\beta y_\nu N_\nu$ (note that by (1), $N_\alpha \subseteq N_\nu$). Since $g, x_\beta, y_\nu \in Z_\nu$, the latter implies that $y_\alpha \in \langle Z_\nu \rangle \cdot N_\nu$, thus contradicting (8).

Case 2. $\nu < \alpha$ and $\nu < \beta$. Then $x_\beta N_\nu = gN_\nu$ by (1) and (3), whence it follows that $x_\beta y_\nu N_\nu \cap g y_\alpha N_\alpha \neq \emptyset$ iff $y_\alpha \in g^{-1}x_\beta y_\nu N_\nu = g^{-1}(x_\beta N_\nu) y_\nu = g^{-1} g N_\nu y_\nu = y_\nu N_\nu$. This, however, contradicts (8).

Case 3. $\beta < \nu = \alpha$. Then $x_\beta y_\nu N_\nu \cap g y_\nu N_\nu \neq \emptyset$ iff $(x_\beta y_\nu)(g y_\nu)^{-1} \in N_\nu$ iff $x_\beta \in N_\nu g = gN_\nu$. The latter is impossible by (4).

Case 4. $\nu = \alpha \le \beta$. It follows from (1) and (3) that $x_\beta \in gN_\beta \subseteq gN_\nu$. Therefore, $x_\beta y_\nu N_\nu = x_\beta N_\nu y_\nu = gN_\nu y_\nu = g y_\nu N_\nu$. We conclude that $x_\beta y_\nu N_\nu \cap g y_\nu N_\nu = g y_\nu N_\nu$.

Case 5. $\beta \le \alpha < \nu$. Then $x_\beta y_\nu N_\nu \cap g y_\alpha N_\alpha \neq \emptyset$ iff $y_\nu \in x_\beta^{-1} g y_\alpha N_\alpha$. However, by (8), $y_\nu \notin \langle Z_\alpha \rangle \cdot N_\alpha \supseteq x_\beta^{-1} g y_\alpha N_\alpha$. It follows that the sets $x_\beta y_\nu N_\nu$ and $g y_\alpha N_\alpha$ are disjoint.

Case 6. $\alpha < \nu$ and $\alpha < \beta$. It follows from (1) and (3) that $N_\alpha x_\beta = x_\beta N_\alpha = gN_\alpha$. Hence, $x_\beta y_\nu N_\nu \cap g y_\alpha N_\alpha \neq \emptyset$ iff $y_\nu \in x_\beta^{-1} g y_\alpha N_\alpha = x_\beta^{-1} N_\alpha^{-1} g y_\alpha = (gN_\alpha)^{-1} g y_\alpha = N_\alpha y_\alpha = y_\alpha N_\alpha$ iff $y_\nu \in y_\alpha N_\alpha$. The latter is impossible by (4).

We have thus proved the equality $(*)$. To finish the proof of the lemma, it suffices to show that for each $\alpha < \omega_1$, the group $N_{\alpha+1}$ can be obtained from the clopen sets $W_0 = W$ and $W_1 = H \setminus W$ by means of finitely many translations and intersections. Indeed, it follows from $(*)$ that

$$(x_{\alpha+1} W \cap gW) \setminus x_\alpha W = (x_{\alpha+1} W \cap gW) \setminus (x_\alpha W \cap gW) = g y_{\alpha+1} N_{\alpha+1}$$

or, equivalently,

$$N_{\alpha+1} = y_{\alpha+1}^{-1}\left(g^{-1}x_{\alpha+1}W_0 \cap W_0 \cap g^{-1}x_\alpha W_1\right).$$

The lemma is proved. □

Corollary 3.5. *Suppose that H is a non-discrete P-group, and let W_0 and W_1 be clopen complementary subsets of H that generate the topology of H. If a function $f\colon H \to \{0,1\}$ is defined by $f(x) = i$ if $x \in W_i$, where $i = 0, 1$, then the orbit Hf in $C_p(H)$ has uncountable network weight.*

Proof: Evidently, f is continuous. For an element $a \in H$, let f_a be the a-shift of f, that is, the function on H defined by $f_a(x) = f(a^{-1}x)$, where $x \in H$. It is clear that $f_a^{-1}(0) = aW_0$ and $f_a^{-1}(1) = aW_1$. Therefore, the topology on H generated by the functions from the orbit Hf coincides with the topology on H generated by the sets W_0 and W_1 which is in its turn the original topology of H. In other words, the family $Hf \subseteq C_p(H)$ generates the topology of H and, by Lemma 3.2, $nw(Hf) = nw(H)$. Obviously, every non-discrete P-group has uncountable cellularity, so $\omega < c(H) \le nw(H) = nw(Hf)$, as claimed. □

Proof of Theorem 3.1: Suppose that the group G is a continuous homomorphic image of an $\mathbb{R}$-factorizable group. Then G is evidently ω-narrow and, according to Lemma 3.3, it suffices to verify that every continuous homomorphic image H of G satisfying $w(H) \le \aleph_1$ is Lindelöf provided H is a P-group. Again, H is ω-narrow. Suppose that H is not Lindelöf. Then, by Lemma 3.4, there exist complementary clopen sets W_0 and W_1 in H generating the topology of H. Let f be a function on H defined by $f(x) = i$ if $x \in W_i$, for $i = 0, 1$. Then f is continuous and Corollary 3.5 implies that the orbit $Hf \subseteq C_p(H)$ has uncountable network weight. According to Proposition 2.1 this means that H cannot be a continuous homomorphic image of an $\mathbb{R}$-factorizable group, which contradicts our choice of H. □

Curiously, Theorem 3.1 implies a corollary that seems to be more general than the theorem itself. Notice that the group G below need not be a P-group.

Corollary 3.6. *Suppose that the direct product $G = \prod_{i\in I} G_i$ of P-groups is a continuous homomorphic image of an $\mathbb{R}$-factorizable group. Then G is also $\mathbb{R}$-factorizable.*

Proof: Suppose that $p\colon K \to G$ is a continuous homomorphism of an $\mathbb{R}$-factorizable group K onto G. Taking compositions of p with projections of G onto the factors, we conclude that each G_i is a continuous homomorphic image of the group K. Since each G_i is a P-group, it follows from Theorem 3.1 that G_i is $\mathbb{R}$-factorizable. It remains to refer to Theorem 5.5 of [11] according to which direct products of $\mathbb{R}$-factorizable P-groups are $\mathbb{R}$-factorizable. □

It is unknown whether the class of $\mathbb{R}$-factorizable groups is productive [9, Problem 4.1]. If there exists a counterexample, then one of the factors must fail to be a P-group, by [11, Theorem 5.5].

4. Open problems

According to [9, Theorem 3.10], every quotient group of an $\mathbb{R}$-factorizable group is $\mathbb{R}$-factorizable as well. Continuous homomorphic images of $\mathbb{R}$-factorizable P-groups are again $\mathbb{R}$-factorizable [11, Corollary 5.9]. Further, by Theorem 3.1, a continuous homomorphic image of an $\mathbb{R}$-factorizable group remains $\mathbb{R}$-factorizable provided that the image is a P-group. The problem arises when the corresponding homomorphism is not open or the image is not a P-group (see [9, Problem 3.9]):

Problem 4.1. *Let $p: G \to H$ be a continuous onto homomorphism of topological groups, where the group G is $\mathbb{R}$-factorizable. Is the group H $\mathbb{R}$-factorizable?*

It is known that a P-group G is $\mathbb{R}$-factorizable iff it is *pseudo-ω_1-compact*, that is, every locally finite family of open sets in G is countable [11, Theorem 4.16]. We do not know whether a similar results holds in general, for all $\mathbb{R}$-factorizable groups:

Problem 4.2. *Is every $\mathbb{R}$-factorizable group pseudo-ω_1-compact?*

Continuous homomorphisms do not preserve completeness in topological groups. In fact, every topological Abelian group is a quotient of a complete topological Abelian group ([7]). Theorem 2.2 suggests therefore the following alternative approach to the proof of Theorem 3.1:

Problem 4.3. *Let G be an ω-narrow P-group (of weight $\aleph_1$), and suppose that G is not Lindelöf. Does there exist a continuous homomorphism $p: G \to H$ onto a P-group H that fails to be complete?*

The condition in the above problem that G is not Lindelöf is easy to explain. First, every Lindelöf P-group is complete (see the last part of Section 3 in [5]). Therefore, every continuous homomorphic image H of a Lindelöf P-group is complete provided H is a P-group. Second, every Lindelöf topological group is $\mathbb{R}$-factorizable ([11, Theorem 5.5]).

Although it is not important for the problems regarding $\mathbb{R}$-factorizability, one can try to improve Lemma 3.4 as follows:

Problem 4.4. *Let H be an ω-narrow P-group of weight $\aleph_1$. If H is not Lindelöf, does H contain a single clopen subset that generates the topology of H?*

It is easy to see that the topology of a non-discrete Lindelöf P-group cannot be generated by a countable family of clopen sets.

One possibility to generalize Theorem 3.1 is to consider continuous images of $\mathbb{R}$-factorizable groups:

Problem 4.5. *Let $f\colon G \to H$ be a continuous mapping (not necessarily a homomorphism) of an $\mathbb{R}$-factorizable group G onto a P-group H. Is H then $\mathbb{R}$-factorizable?*

Notice that by Lemma 3.3, the affirmative answer to Problem 4.2 would imply that the answer to Problem 4.5 is also affirmative.

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