Isomorphic and isometric copies of $\ell_\infty(\Gamma)$ in duals of Banach spaces and Banach lattices

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Abstract. Let X and E be a Banach space and a real Banach lattice, respectively, and let Γ denote an infinite set. We give concise proofs of the following results: (1) The dual space X^* contains an isometric copy of c_0 iff X^* contains an isometric copy of ℓ_∞, and (2) E^* contains a lattice-isometric copy of $c_0(\Gamma)$ iff E^* contains a lattice-isometric copy of $\ell_\infty(\Gamma)$.

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1. Introduction

Let X, E, and Γ have the same meanings as in the Abstract. If Γ' is an infinite subset of Γ then $c_0(\Gamma')$ denotes the subspace of $c_0(\Gamma)$ consisting of all the elements with supports included in Γ'; a similar meaning has the symbol $\ell_\infty(\Gamma')$. By $\mathbb{N}$ we denote the set of positive integers; then the spaces $c_0(\mathbb{N})$ and $\ell_\infty(\mathbb{N})$ are simply denoted by c_0 and ℓ_∞, respectively. All operators are assumed to be linear and continuous.

The main goal of this paper is to provide concise and short proofs of the statements (1) and (2) given in the Abstract. These equivalences are immediate consequences of more general facts, presented in the Theorem below, concerning the possibility of extensions of isomorphisms $T : c_0(\Gamma) \to X^*$ to isomorphisms $\widetilde{T} : \ell_\infty(\Gamma) \to X^*$ with the norms $\|\widetilde{T}\|$ and $\|\widetilde{T}^{-1}\|$ controlled by $\|T\|$ and $\|T^{-1}\|$, respectively.

Theorem. (a) *Let ℓ_∞ be a real or complex space, let c_0 denote its respective subspace, and let $T : c_0 \to X^*$ be an isomorphism. Then there is an infinite subset A of $\mathbb{N}$ such that the restricted operator $T_A := T_{|c_0(A)}$ extends to an isomorphism $S : \ell_\infty(A) \to X^*$ with $\|S\| = \|T_A\|$ and $\|T_A^{-1}\| \leq \|S^{-1}\| \leq \|T^{-1}\|$.*

(b) *Let $c_0(\Gamma)$ and $\ell_\infty(\Gamma)$ denote real spaces, and let $T : c_0(\Gamma) \to E^*$ be a lattice isomorphism. Then T extends to a lattice isomorphism $\widetilde{T} : \ell_\infty(\Gamma) \to E^*$ with $\|\widetilde{T}\| = \|T\|$ and $\|\widetilde{T}^{-1}\| = \|T^{-1}\|$.*

Thus, the extended operator S in item (a) acts on a *subspace* of ℓ_∞, while the isomorphism $\widetilde{T}$ in item (b) acts on the *whole* space $\ell_\infty(\Gamma)$. Part (a) of the Theorem is obtained by an appeal to a result by Rosenthal included in [8, Remark 2, p. 17]; cf. Lemma 2 below. Another Rosenthal's result, for Γ uncountable [8, Proposition 1.2], gives a somewhat weaker conclusion than in item (b) (here the space $\ell_\infty(\Gamma)$ is real or complex, and $c_0(\Gamma)$ is its respective subspace):

(a_ε) *Let Γ be an uncountable set, and let $T : c_0(\Gamma) \to X^*$ be an isomorphism. Then, for every $\varepsilon \in (0,1)$ there exists a subset Γ_ε of Γ with* $\mathrm{card}(\Gamma_\varepsilon) = \mathrm{card}(\Gamma)$ *such that the restricted operator $T_\varepsilon := T_{|c_0(\Gamma_\varepsilon)}$ extends to an isomorphism $S_\varepsilon : \ell_\infty(\Gamma_\varepsilon) \to X^*$ with $\|S_\varepsilon\| = \|T_\varepsilon\|$ and $\|T_\varepsilon\| \le \|S_\varepsilon^{-1}\| \le \|T^{-1}\|/(1-\varepsilon)$.*

The equivalence, which follows immediately from our item (a), that X^* *contains an isometric copy of c_0 iff X^* contains an isometric copy of ℓ_∞* was obtained in 2000 by Dowling [4, Theorem 1] as a result of six equivalent conditions in an isometric version of the classical Bessaga and Pełczyński theorem [3]; [5, Proposition 2.e.8] on copies of c_0 in X^*. The properties included in the above items (a_ε), (b), and (2) are new.

A comment concerning part (b) of the Theorem is necessary. For $\Gamma = \mathbb{N}$ there is a lattice version of the above-mentioned theorem of Bessaga and Pełczyński asserting (*via* eleven equivalent conditions; see [2, Theorem 14.21]) that E^* *contains a lattice copy of c_0 iff E contains a lattice copy of ℓ_1 iff E^* contains a lattice copy of ℓ_∞*, but there are no connections between the norms of operators in question (here "lattice copy" means "both lattice and homeomorphic copy"). The proof of this equivalence uses essentially the so-called property (u) of Pełczyński which is, however, of the *countable* nature and therefore cannot be extended to the case when the lattices c_0, ℓ_1, and ℓ_∞ are replaced by $c_0(\Gamma)$, $\ell_1(\Gamma)$, and $\ell_\infty(\Gamma)$, respectively, with Γ uncountable. On the other hand, in 1970 Rosenthal proved that, for X a Banach space and Γ uncountable, *if X^* contains a copy of $c_0(\Gamma)$ then X^* contains a copy of $\ell_\infty(\Gamma)$* ([7, Corollary 1.2]; [8, Theorem 1.3]). In the context of our item (b) and Rosenthal's result, it seems to be an open question if the containment of an isomorphic copy of $\ell_\infty(\Gamma)$ by the dual E^* of a Banach lattice E (or, more generally, by E whenever E is Dedekind complete) implies the containment of the *lattice* copy of $\ell_\infty(\Gamma)$ (the case $\Gamma = \mathbb{N}$ has a positive answer: see [2, Theorem 14.9]; cf. [9, Theorem]).

It should be stressed that our proof of item (b) is completely independent on the above-cited results of Bessaga and Pełczyński, and Rosenthal, and it follows only from the Fatou property and monotone completeness of the dual norm of E^*.

2. Notations and terminology

We follow standard notations and terminology (for Banach spaces see [5]). For the basic results concerning Banach lattices we refer to the monographs [2], [6].

For the convenience of the reader we recall some definitions.

Let G, H be two linear lattices. An injective operator $T : G \to H$ is a lattice isomorphism provided that both T and T^{-1} are positive; equivalently, $|Tx| = T(|x|)$ for all $x \in G$. The lattice G is Dedekind complete if every nonempty subset V of G bounded from above has a supremum in G. If $E = (E, \| \ \|)$ is a Banach lattice then the dual space E^*, endowed with the dual norm $\| \ \|^*$, is a Dedekind complete Banach lattice with respect to the ordering $x^* \le y^*$ iff $x^*(x) \le y^*(x)$ for all $x \in E^+$. The norm $\| \ \|$ is said to be *monotone complete* ([6, p. 96]) if every norm-bounded and upward directed set $(x_i)_{i\in I}$ in E^+ has a supremum (this property appears in the literature under many different names, e.g. to be a *Levi norm*; see [1, p. 282]). The lattice E has the *Fatou property* if for every upward directed set $(x_i)_{i\in I}$ in E^+ with $\sup_{i\in I} x_i = x$ it follows that $\sup_{i\in I} \|x_i\| = \|x\|$. In the proof of part (b) of the Theorem we shall apply the following result (see [6, Theorems 2.4.19 and 2.4.21]):

Lemma 1. *For every Banach lattice E, the dual E^* has the Fatou property and the dual norm $\| \ \|^*$ is monotone complete.*

The symbol $\circ$ will denote composition of operators.

3. Proof of the Theorem

We start with the cited in Section 1 results by Rosenthal. Part (i) of the lemma below is included in [8, Remark 2 on p. 17], while part (ii) is a quantitative version of [8, Proposition 1.2] obtained from the following modification of its proof: on page 17 of [8], lines 14–19 from above, one should apply [8, Lemma 1.1] with $\varepsilon \in (0, 1)$ *arbitrary* instead of (as in the original proof) fixed $\varepsilon = 1/2$. Here the space $\ell_\infty(\Gamma)$ is real or complex.

Lemma 2. *Let B be a Banach space, and let $R : \ell_\infty(\Gamma) \to B$ be an operator such that $R_0 := R_{|c_0(\Gamma)}$ is an isomorphism.*

(i) *If $\Gamma = \mathbb{N}$, there is an infinite subset A of $\mathbb{N}$ such that the restricted operator $R_A := R_{|\ell_\infty(A)}$ is an isomorphism with $\|R_A^{-1}\| \le \|R_0^{-1}\|$.*

(ii) *If Γ is uncountable, for every $\varepsilon \in (0, 1)$ there is a subset Γ_ε of Γ with $\mathrm{card}(\Gamma_\varepsilon) = \mathrm{card}(\Gamma)$ such that the restricted operator $S_\varepsilon := R_{|\ell_\infty(\Gamma_\varepsilon)}$ is an isomorphism with $\|S_\varepsilon^{-1}\| \le \|R_0^{-1}\|/(1-\varepsilon)$.*

In proofs of items (a) and (a$_\varepsilon$) we follow an idea of the proof of [8, Theorem 1.3], and we identify $\ell_\infty(\Gamma)$ with $c_0(\Gamma)^{**}$. By π and π_1, respectively, we denote the canonical embeddings of X into X^{**} and X^* into X^{***}, respectively, and P denotes the well-known projection, with $\|P\| = 1$, from X^{***} onto $\pi_1(X^*)$ of the form $P(x^{***}) = \pi_1(x^{***} \circ \pi)$. Then the operator $R := \pi_1^{-1} \circ P \circ T^{**}$ maps $\ell_\infty(\Gamma)$ into X^*, and its the restriction $R_0 := R_{|c_0(\Gamma)}$ is an isomorphism because

$$R_0 = T. \tag{1}$$

Let $\Gamma = \mathbb{N}$ (i.e., we consider now item (a)), and let R_A be the isomorphism obtained from Lemma 2(i). From the identification of ℓ_∞ with c_0^{**} we obtain that $R_A = \pi_1^{-1} \circ P \circ T_A^{**}$, where $T_A := T_{|c_0(A)}$. Hence $R_{A|c_0(A)} = T_A$, i.e., R_A is an extension of T_A; thus $\|R_A\| \geq \|T_A\|$, but the form of R_A implies that $\|R_A\| \leq \|T_A\|$. Finally

$$\|R_A\| = \|T_A\|. \tag{2}$$

From (1) and Lemma 2 (i) we also have

$$\|R_A^{-1}\| \leq \|R_0^{-1}\| = \|T^{-1}\|. \tag{3}$$

On the other hand, since the isomorphism R_A is an extension of T_A, the inversed operator R_A^{-1} is an extension of T_A^{-1}; hence

$$\|T_A^{-1}\| \leq \|R_A^{-1}\|. \tag{4}$$

If we put now $S = R_A$, then from (2), (3) and (4) we obtain the conclusion in part (a).

The result in item (a$_\varepsilon$) can be proven in a similar way (applying part (ii) of Lemma 2).

For the proof of part (b) of the Theorem, let T be a lattice isomorphism from $c_0(\Gamma)$ into E^*, and let $f_\gamma = Te_\gamma$, where e_γ denotes the standard γth unit vector of $c_0(\Gamma)$. Let $\mathcal{G}$ be the class of all finite subsets of Γ. For every positive element $x = (t_\gamma)_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ and every $G \in \mathcal{G}$, we define the element $x_G = \sup_{\gamma\in G} t_\gamma e_\gamma$. Then for the element $f_G := \sup_{\gamma\in G} t_\gamma f_\gamma$ we have $f_G = T(x_G)$, and hence

$$\|f_G\| = \|T(x_G)\| \leq \|T\| \cdot \|x_G\|_\infty \leq \|T\| \cdot \|x\|_\infty \,. \tag{5}$$

Moreover, since T is positive and $x_{G_1} \leq x_{G_2}$ for $G_1 \subset G_2$, we have $f_{G_1} \leq f_{G_2}$ for $G_1 \subset G_2$. It follows that the set $(f_G)_{G\in\mathcal{G}}$ is both upward directed and norm-bounded (see (5)). By Lemma 1, the supremum $\sup_{G\in\mathcal{G}} f_G = \sup_{\gamma\in\Gamma} t_\gamma f_\gamma$ exists in E^*, and hence the formula

$$R_1(x) = \sup_{G\in\mathcal{G}} T(x_G) \tag{6}$$

defines an additive (positive) injective operator R_1 from the positive cone $\ell_\infty(\Gamma)^+$ into E^*. By [2, Theorem 1.7], the operator $\widetilde{T}(x) := R_1(x^+) - R_1(x^-)$ is a linear positive mapping from $\ell_\infty(\Gamma)$ into E^*. Since the elements $R_1(x^+)$ and $R_1(x^-)$ are disjoint, we have $|\widetilde{T}(x)| = \widetilde{T}(|x|) = R_1(|x|)$; it follows that $\widetilde{T}$ is a lattice isomorphism. In order to calculate the norm of $\widetilde{T}$ we apply the fact that, for every $x \in \ell_\infty(\Gamma)$, the net $(T(|x_G|))_{G\in\mathcal{G}}$ is directed upward with, by (6), $\sup_{G\in\mathcal{G}} T(|x_G|) = \widetilde{T}(|x|)$. Now we apply (5) and the Fatou property of E^* which imply that $\|\widetilde{T}(x)\| = \|\, |\widetilde{T}(x)|\, \| = \|\widetilde{T}(|x|)\| \leq \|T\| \cdot \|x\|_\infty$, and hence $\|\widetilde{T}\| \leq \|T\|$. The reversed inequality is obvious (because $\widetilde{T}$ extends T), and so $\|\widetilde{T}\| = \|T\|$, as claimed.

The second equality, $\|\widetilde{T}^{-1}\| = \|T^{-1}\|$, is obtained in a similar way: we notice that $\widetilde{T}^{-1}$ extends T^{-1}, and we apply Lemma 1 to $\ell_\infty(\Gamma)$ instead of E^* to show that, for every $f \geq 0$ in the norm-closed sublattice $\widetilde{T}(\ell_\infty(\Gamma))$, we have $\|\widetilde{T}^{-1}(f)\| = \sup_{G\in\mathcal{G}} \|T^{-1}(f_G)\|$, where f_G is defined as above.

References

[1] Abramovich Y.A., Wickstead A.W., *When each continuous operator is regular. II*, Indag. Math., N.S. **8** (1997), 281–294.

[2] Aliprantis C.D., Burkinshaw O., *Positive Operators*, Academic Press, New York, 1985.

[3] Bessaga C., Pełczyński A., *On basis and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

[4] Dowling P.N., *Isometric copies of c_0 and ℓ_∞ in duals of Banach spaces*, J. Math. Anal. Appl. **244** (2000), 223–227.

[5] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces. I*, Springer, Berlin, 1977.

[6] Meyer-Nieberg P., *Banach Lattices*, Springer, Berlin, 1991.

[7] Rosenthal H.P., *On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ*, Acta Math. **124** (1974), 205–247.

[8] Rosenthal H.P., *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1970), 13–36.

[9] Wójtowicz M., *The Sobczyk property and copies of ℓ_∞ in locally convex-solid Riesz spaces*, Arch. Math. **75** (2000), 376–379.

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