A semifilter approach to selection principles II: τ^*-covers

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Abstract. Developing the idea of assigning to a large cover of a topological space a corresponding semifilter, we show that every Menger topological space has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ provided ($\mathfrak{u}<\mathfrak{g}$), and every space with the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ is Hurewicz provided ($\mathrm{Depth}^+([\omega]^{\aleph_0})\le\mathfrak{b}$). Combining this with the results proven in cited literature, we settle all questions whether (it is consistent that) the properties P and Q [do not] coincide, where P and Q run over $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$, and $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.

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Introduction

Following [15] we say that a topological space X has the property $\bigcup_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, where $\mathcal{A}$ and $\mathcal{B}$ are collections of covers of X, if for every sequence $(u_n)_{n\in\omega}\in\mathcal{A}^\omega$ there exists a sequence $(v_n)_{n\in\omega}$, where each v_n is a finite subset of u_n, such that $\{\bigcup v_n : n\in\omega\}\in\mathcal{B}$. Throughout this paper "cover" means "open cover" and $\mathcal{A}$ is equal to the family $\mathcal{O}$ of all open covers of X. Concerning $\mathcal{B}$, we shall also consider the collections Γ, T, $\mathrm{T}^\star$, T^*, and Ω of all open γ-, τ-, $\tau^\star$, τ^*-, and ω-covers of X. For technical reasons we shall use the collection Λ of countable large covers. The most natural way to define these types of covers uses the Marczewski "dictionary" map introduced in [13]. Given an indexed family $u=\{U_n : n\in\omega\}$ of subsets of a set X and element $x\in X$, we define the Marczewski map $\mu_u : X\to\mathcal{P}(\omega)$ letting $\mu_u(x)=\{n\in\omega : x\in U_n\}$ ($\mu_u(x)$ is nothing else but $I_s(x,u)$ in notations of [23]). Recall that $A\subset^* B$ means that $|A\setminus B|<\aleph_0$. A family $\mathcal{A}\subset\mathcal{P}(X)$ of subsets of a set X is a *refinement* of a family $\mathcal{B}\subset\mathcal{P}(X)$, if for every $B\in\mathcal{B}$ there exists $A\in\mathcal{A}$ such that $A\subset B$. Depending on the properties of $\mu_u(X)=\{\mu_u(x) : x\in X\}$ a family $u=\{U_n : n\in\omega\}$ is defined to be

- a *large cover* of X ([15]), if for every $x\in X$ the set $\mu_u(x)$ is infinite;
- a *γ-cover* of X ([9]), if for every $x\in X$ the set $\mu_u(x)$ is cofinite in ω, i.e. $\omega\setminus\mu_u(x)$ is finite;
- a *τ-cover* of X ([19]), if it is a large cover and the family $\mu_u(X)$ is linearly preordered by the almost inclusion relation $\subset^*$ in sense that for all $x_1,x_2\in X$ either $\mu_u(x_1)\subset^*\mu_u(x_2)$ or $\mu_u(x_2)\subset^*\mu_u(x_1)$;

- a τ^*-*cover* of X ([19]), if there exists a linearly preordered by $\subset^*$ refinement $\mathcal{J}$ of $\mu_u(X)$ consisting of infinite subsets of ω;
- an ω-*cover* ([9]), if the family $\mu_u(X)$ is centered, i.e. for every finite subset K of X the intersection $\bigcap_{x\in K}\mu_u(x)$ is infinite.

We also introduce a new type of covers situated between τ- and τ^*-covers. A family $u = \{U_n : n \in \omega\}$ is

- a $\tau^\star$-*cover* of X, if there exists a linearly preordered by $\subset^*$ refinement $\mathcal{J} \subset \mu_u(X)$ of $\mu_u(X)$ consisting of infinite subsets of ω.

Recall that $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ and $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ are nothing else but the well-known Hurewicz and Menger covering properties introduced in [10] and [14], respectively, at the beginning of 20-th century.

Since every γ-cover is a τ-cover, every τ-cover is a $\tau^\star$-cover, every $\tau^\star$-cover is a τ^*-cover, and every τ^*-cover is an ω-cover, the above properties are related as follows:

$$\begin{tikzcd}
\underset{(2)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})} \arrow[r, Rightarrow] & \underset{(3)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^\star)} \arrow[r, Rightarrow] & \underset{(4)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)} \arrow[r, Rightarrow] & \underset{(5)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)} \arrow[d, Rightarrow] \\
\underset{(1)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)} \arrow[u, Rightarrow] & & & \underset{(6)}{\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})}
\end{tikzcd}$$

By a *tower* we understand a $\subset^*$-decreasing transfinite sequence of infinite subsets of ω, i.e. a sequence $(T_\alpha)_{\alpha<\lambda}$ such that $T_\alpha \subset^* T_\beta$ for all $\alpha \geq \beta$. The cardinality λ is called the *length* of this tower. The subsequent theorem, which is the main result of this paper, describes when some of the above properties coincide.

Theorem 1. (1) *Under* ($\mathfrak{u} < \mathfrak{g}$) *the selection principles* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^\star)$ *and* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ *coincide.*

(2) *Under Filter Dichotomy the selection principles* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^\star)$ *and* $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$ *coincide.*

(3) *The selection principles* $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ *and* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ *coincide iff each semifilter generated by a tower is meager.*

The following statement describes some partial cases of Theorem 1(3).

Corollary 1. (1) *The selection principles* $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ *and* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ *coincide if the inequality* $\mathrm{Depth}^+([\omega]^{\aleph_0}) \leq \mathfrak{b}$ *holds.*

(2) *Under* ($\mathfrak{b} < \mathfrak{d}$) *(resp.* ($\mathfrak{t} = \mathfrak{d}$)*) there exists a set of reals with the property* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ *which fails to satisfy* $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ *(resp.* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})$*).*

Theorem 1 gives a partial answer to Problem 5.2 from [3]. Namely, it implies the subsequent

Corollary 2. *It is consistent that the property* $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})$ *is closed under unions of families of subspaces of the Baire space of size* $< \mathfrak{b}$.

Proof: Follows immediately from Theorem 1(3) and the fact that the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$ is preserved by unions of less than $\mathfrak{b}$ subspaces of the Baire space, see [11]. □

We refer the reader to [22] for definitions of all small cardinals and related notions we use. All notions concerning semifilters may be found in [1] and will be defined in the next section. The condition $(\mathfrak{u} < \mathfrak{g})$ is known to be consistent: $\mathfrak{u} = \mathfrak{b} = \mathfrak{s} < \mathfrak{g} = \mathfrak{d}$ in Miller's model and the inequality $(\mathfrak{u} < \mathfrak{g})$ implies $\mathfrak{u} = \mathfrak{b} < \mathfrak{g} = \mathfrak{d}$, see [4] and [22]. Moreover, $(\mathfrak{u} < \mathfrak{g})$ is equivalent to the assertion that all upward-closed neither meager nor comeager families of infinite subsets of ω are "similar", see [12], [4, 9.22], [1, 7.6.4, 12.2.4], or Theorem 3. This assertion together with the Talagrand's [18] characterization of meager and comeager upward-closed families is the so-called *trichotomy* for upward-closed families or *Semifilter Trichotomy* in terms of [1]. The *Filter Dichotomy* follows from the Semifilter Trichotomy and is formally stronger than the NCF principle introduced by A. Blass, see [4, § 9] and the references there in.

$\mathrm{Depth}^+([\omega]^{\aleph_0})$ denotes the smallest cardinality κ such that there is no tower of length κ. Thus $\mathfrak{t} < \mathrm{Depth}^+([\omega]^{\aleph_0})$. A model with $\mathfrak{b} \geq \mathrm{Depth}^+([\omega]^{\aleph_0})$ was constructed in [6]. Some other applications of $\mathrm{Depth}^+([\omega]^{\aleph_0})$ in Selection Principles may be found in [16].

Theorem 1 with results proven in [11], [19], [21], and [23], enable us to settle almost all questions whether (it is consistent that) the properties P and Q [do not] coincide, where P and Q run over $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\Omega)$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^\star)$, $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})$, and $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$. (In fact, we settle all of the questions omitting $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^\star)$.) Some sufficient conditions for $\mathsf{P} = \mathsf{Q}$ and $\mathsf{P} \neq \mathsf{Q}$ are summarized in Table 1. Each entry $((i),(j))$, $i \neq j$, contains:

- A condition which implies $(i) = (j)$ (resp. $(i) \neq (j)$) provided $i < j$ (resp. $i > j$) or "?" if no such condition is known;
- ZFC, if $(i) \neq (j)$ in ZFC and $i > j$;
- $-$, if $(i) \neq (j)$ in ZFC and $i < j$;

and a reference to where this is proven. For example, "$[x]+[y]$, $[z]$" means that the sufficiency of the corresponding condition was proven in $[z]$, and it can be simply derived by combining results of $[x]$ and $[y]$. Throughout the table, λ stands for $\mathrm{Depth}^+([\omega]^{\aleph_0})$.

Table 1

	(1)	(2)	(3)	(4)	(5)	(6)
(1)		$(\lambda \le \mathfrak{b})$ Cor. 1	$(\lambda \le \mathfrak{b})$ Cor. 1	$(\lambda \le \mathfrak{b})$ Cor. 1	– [2], [5], [21]	– [2],[5],[21]
(2)	$(\mathfrak{b} < \mathfrak{s})$ [19]+[16]		$(\lambda \le \mathfrak{b})$ Cor. 1	$(\lambda \le \mathfrak{b})$ Cor. 1	– [21]	– [21]
(3)	$(\mathfrak{b} < \mathfrak{s}) \vee (\mathfrak{u} < \mathfrak{g})$ [19]+[16],[21]+Th. 1	$(\mathfrak{u} < \mathfrak{g})$ [21]+Th. 1		$(\lambda \le \mathfrak{b})$ Cor. 1	Filter Dich. Th. 1	$(\mathfrak{u} < \mathfrak{g})$ Th. 1
(4)	$(\mathfrak{t} = \mathfrak{d}) \vee (\mathfrak{b} < \mathfrak{d})$ Cor. 1	$(\mathfrak{t} = \mathfrak{d}) \vee (\mathfrak{u} < \mathfrak{g})$ Cor. 1, [21]+Th. 1	?		Filter Dich. Th. 1	$(\mathfrak{u} < \mathfrak{g})$ Th. 1
(5)	ZFC [21],[5],[2]	ZFC [21]	$(\lambda \le \mathfrak{b})$ [21]+Cor. 1	$(\lambda \le \mathfrak{b})$ [21]+Cor. 1		$(\mathfrak{u} < \mathfrak{g})$ Th. 1,[23]
(6)	ZFC [21],[5],[2]	ZFC [21]	$(\lambda \le \mathfrak{b}) \vee$ CH [21]+Cor. 1, [11]	$(\lambda \le \mathfrak{b}) \vee$ CH [21]+Cor. 1, [11]	CH [11]	

Semifilters

Our main tool is the notion of a semifilter. Following [1], a family $\mathcal{F}$ of nonempty subsets of ω is called a *semifilter*, if for every $F \in \mathcal{F}$ and $A \stackrel{*}{\supset} F$ the set A belongs to $\mathcal{F}$. For example, each family $\mathcal{A}$ of infinite subsets of ω generates the minimal semifilter $\uparrow \mathcal{A} = \{B \subset \omega : \exists A \in \mathcal{A}(A \subset^* B)\}$ containing $\mathcal{A}$. The family SF of all semifilters contains the smallest element $\mathfrak{Fr}$ consisting of all cofinite subsets of ω, and the largest one, $[\omega]^{\aleph_0}$, i.e. the family of all infinite subsets of ω. Throughout this paper by a *filter* we understand a semifilter which is closed under finite intersections of its elements.

Since every semifilter $\mathcal{F}$ on ω is a subset of the powerset $\mathcal{P}(\omega)$, which can be identified with the Cantor space $\{0,1\}^\omega$, we can speak about topological properties of semifilters. Recall that a subset of a topological space is *meager* if it is a union of countably many nowhere dense subsets. The complements of meager subsets are called *comeager*. We shall often use the subsequent characterization of meagerness of semifilters due to Talagrand, see [18] and [1, 5.3.1].

Theorem 2. *A semifilter $\mathcal{F}$ on ω is meager if and only if there exists an increasing number sequence $(k_n)_{n\in\omega}$ such that every $F \in \mathcal{F}$ meets all but finitely many half-intervals $[k_n, k_{n+1})$.*

A crucial role in the proof of Theorem 1 belongs to the following fundamental result of C. Laflamme [12]. Following [1], a semifilter $\mathcal{F}$ on ω is said to be *bi-Baire*, if it is neither meager nor comeager. Note that there is no comeager filter on ω, see [1, 5.3.2].

Theorem 3. *The following conditions are equivalent:*

(1) $(\mathfrak{u} < \mathfrak{g})$;
(2) *for any bi-Baire semifilters $\mathcal{F}$ and $\mathcal{U}$ there exists an increasing number sequence $(k_n)_{n\in\omega}$ such that the sets $\{\{n \in \omega : F \cap [k_n, k_{n+1}) \neq \emptyset\} : F \in \mathcal{F}\}$ and $\{\{n \in \omega : U \cap [k_n, k_{n+1}) \neq \emptyset\} : U \in \mathcal{U}\}$ coincide.*

Thus the inequality $(\mathfrak{u} < \mathfrak{g})$ implies the *Filter Dichotomy* [4, 9.16], which is the abbreviation of the assertion of Theorem 3(2) for bi-Baire filters:

For arbitrary bi-Baire filters $\mathcal{F}$ and $\mathcal{U}$ there exists an increasing number sequence $(k_n)_{n\in\omega}$ such that the sets $\{\{n \in \omega : F \cap [k_n, k_{n+1}) \neq \emptyset\} : F \in \mathcal{F}\}$ and $\{\{n \in \omega : U \cap [k_n, k_{n+1}) \neq \emptyset\} : U \in \mathcal{U}\}$ coincide.

The main idea of the semifilter approach to selection principles is to assign to a topological space X the family $\{\uparrow \mu_u(X) : u \in \Lambda(X)\}$. As it was shown in [23], the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ of a space X may be characterized in terms of topological properties of elements of the above family.

Theorem 4 ([23, Theorem 3])**.** *Let X be a Lindelöf topological space. Then X has the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ if and only if for every $u \in \Lambda(X)$ so does the semifilter $\uparrow \mu_u(X)$.*

And finally, we define some properties of semifilters closely related to $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$ and $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^\star)$. We say that a family $\mathcal{B} \subset \mathcal{F}$ is a *base* of a semifilter $\mathcal{F}$ if $\mathcal{F} = \uparrow \mathcal{B}$. The *character* $\chi(\mathcal{F})$ of a semifilter $\mathcal{F}$ equals, by definition, the smallest size of a base of $\mathcal{F}$.

Definition 6. A filter $\mathcal{F}$ on ω is defined to be a *simple P-filter*, if there exists a linearly preordered with respect to $\subset^*$ base of $\mathcal{F}$.

The subsequent observation explains the importance of simple P-filters in studying the properties $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$ and $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^\star)$.

Observation 1. *A family $u = \{U_n : n \in \omega\}$ of subsets of X is a τ^*- (resp. $\tau^\star$-) cover of X if and only if $\mu_u(X)$ can be enlarged to (resp. generates) a simple P-filter.*

We shall also use the subsequent characterization of simple P-filters.

Theorem 5 ([1, 3.2.3])**.** *A filter $\mathcal{F}$ is a simple P-filter if and only if $\mathcal{F}$ has a base $\mathcal{B} = (B_\alpha)_{\alpha<\chi(\mathcal{F})}$ such that $B_\alpha \subset^* B_\beta$ for all $\beta \le \alpha < \chi(\mathcal{F})$.*

Next, we shall search for conditions when there are nonmeager simple P-filters, or conditions which imply that all of them are meager.

Proposition 1. *If* $\mathrm{Depth}^+([\omega]^{\aleph_0}) \le \mathfrak{b}$, *then each simple P-filter is meager.*

PROOF: Follows easily from Theorem 5, the definition of the cardinal $\mathrm{Depth}^+([\omega]^{\aleph_0})$, and the fact that each semifilter with character $< \mathfrak{b}$ is meager, see [1, 8.3.1] or [17]. □

Proposition 2. *There exists a nonmeager simple P-filter provided $\mathfrak{b} < \mathfrak{d}$ or $\mathfrak{t} = \mathfrak{b}$.*

PROOF: Follows immediately from [1, 8.3.2, 11.2.3]. □

The following simple characterization of the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \Gamma)$ is of crucial importance for the proof of Theorem 1(3). Let u be a cover of a set X. A subset B of X is *u-bounded*, if $B \subset \cup v$ for some finite $v \subset u$.

Proposition 3. *A topological space X has the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \Gamma)$ if and only if for every sequence $(u_n)_{n\in\omega}$ of open covers of X there exists a sequence $(v_n)_{n\in\omega}$ such that each v_n is a finite subset of u_n and the semifilter $\uparrow \mu_{\{\cup v_n : n\in\omega\}}(X)$ is meager.*

PROOF: Only the "if" part needs a proof. Let $(u_n)_{n\in\omega}$ be a sequence of open covers of X. Without loss of generality, u_{n+1} is a refinement of u_n for all $n \in \omega$. Let $w = \{B_n : n \in \omega\}$ be such that each B_n is u_n-bounded and $\uparrow \mu_w(X)$ is meager. Then there is an increasing number sequence $(k_n)_{n\in\omega}$ such that each element of $\uparrow \mu_w(X)$ meets all but finitely many half-intervals $[k_n, k_{n+1})$. Since u_{n+1} is a refinement of u_n for all $n \in \omega$, the union $C_n = \bigcup_{k\in[k_n,k_{n+1})} B_k$ is u_n-bounded. We claim that $\{C_n : n \in \omega\}$ is a γ-cover of X. Indeed, given any $x \in X$ find $n_0 \in \omega$ such that $\mu_w(x) \cap [k_n, k_{n+1}) \ne \emptyset$ for all $n \ge n_0$. The above means that for every $n \ge n_0$ we can find $k_x(n) \in [k_n, k_{n+1})$ with the property $x \in B_{k_x(n)}$, and hence $x \in B_{k_x(n)} \subset \bigcup_{k\in[k_n,k_{n+1})} B_k = C_n$ for all $n \ge n_0$. □

In the proof of Theorem 1 we shall use some properties of the *eventual dominance relation* $\le^*$ on ω^ω defined as follows: $x \le^* y$ whenever the set $\{n \in \omega : x_n > y_n\}$ is finite. A subset A of ω^ω is said to be

- *bounded*, if there exists $x \in \omega^\omega$ such that $a \le^* x$ for every $a \in A$;
- *dominating*, if for every $x \in \omega^\omega$ there exists $a \in A$ such that $x \le^* a$;
- a *scale*, if there exists an ordinal α and a bijection $\varphi : \alpha \to A$ such that $\varphi(\beta) \le^* \varphi(\eta)$ for all $\beta < \eta$. In case $\alpha = \mathfrak{b}$ the set A is said to be a $\mathfrak{b}$-*scale*.

PROOF OF THEOREM 1: Let X be a topological space and $(u_n)_{n\in\omega}$ be a sequence of open covers of X such that u_{n+1} is a refinement of u_n for all $n \in \omega$.

1. As it was mentioned in the introduction, $(\mathfrak{u} < \mathfrak{g})$ implies $(\mathfrak{b} < \mathfrak{d})$, and therefore there exists a nonmeager simple P-filter $\mathcal{F}$ by Proposition 2. By the definition of the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ there exists a large cover $w_1 = \{B_n : n \in \omega\}$ of X such that each B_n is u_n-bounded, see [15]. Applying Theorem 4 we conclude

that the semifilter $\mathcal{U} = \uparrow \mu_{w_1}(X)$ has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$, and consequently it is not comeager by [23, Proposition 2]. Two cases are possible.

(a) $\mathcal{U}$ Is bi-Baire. Then Theorem 3 supplies us with an increasing sequence $(k_n)_{n\in\omega}$ such that $\mathcal{G} := \phi(\mathcal{U}) = \phi(\mathcal{F})$, where $\phi : \omega \to \omega$ is such that $\phi^{-1}(n) = [k_n, k_{n+1})$ for all $n \in \omega$, and $\phi(\mathcal{A}) = \{\phi(A) : A \in \mathcal{A}\}$ for any family $\mathcal{A}$ of subsets of ω. Note that $\mathcal{G}$ is a simple P-filter being an image of $\mathcal{F}$ under ϕ.

Let $C_n = \bigcup_{k\in[k_n,k_{n+1})} B_k$. By our choice of $(u_n)_{n\in\omega}$, each C_n is u_n-bounded. We claim that $w_2 = \{C_n : n \in \omega\}$ is a $\tau^\star$-cover of X. Indeed, since $\mathcal{G} = \phi(\mathcal{U})$, $\mathcal{U}$ is generated by $\mu_{w_1}(X)$, and $\mu_{w_2}(x) = \phi(\mu_{w_1}(x))$ for all $x \in X$, we conclude that $\mathcal{G}$ is generated by $\mu_{w_2}(X)$. Now it suffices to apply Observation 1.

(b) $\uparrow \mu_{w_1}(X)$ is meager. Then in the same way as in the proof of Proposition 3 we can construct a γ-cover $\{C_n : n \in \omega\}$ of X such that each C_n is u_n-bounded.

2. In this case it suffices to find an ω-cover $w_1 = \{B_n : n \in \omega\}$ of X such that each B_n is u_n-bounded and apply to the filter $\uparrow \mu_{w_1}(X)$ the same arguments as in the proof of the first item.

3. Let us assume that each simple P-filter is meager and X has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$. Then there exists a τ^*-cover $w = \{B_n : n \in \omega\}$ of X such that each B_n is u_n-bounded. By Observation 1 this implies that the semifilter $\mathcal{U} = \uparrow \mu_w(X)$ can be enlarged to a simple P-filter $\mathcal{F}$, which is meager by our assumption, and hence so is $\mathcal{U}$. Applying Proposition 3 we conclude that X has the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$.

Next, suppose that there exists a nonmeager simple P-filter $\mathcal{F}$. The rest of the proof falls naturally into two parts.

(a) $(\mathfrak{b} = \mathfrak{d})$. In this case the assertion follows from [21, 8.10], which supplies us with a subspace Y of the Baire space with the following properties:

(i) Y does not have the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T})$;
(ii) for any sequence $(w_n)_{n\in\omega}$ of open covers of Y there exists a family $w = \{B_n : n \in \omega\}$ such that each B_n is w_n-bounded and $\uparrow \mu_w(X) \subset \mathcal{F}$.

(b) $(\mathfrak{b} < \mathfrak{d})$. In this case the assertion follows from the subsequent two statements.

(i) There exists a subspace of the Baire space of size $\mathfrak{b}$ which does not have the property $\bigcup_{\mathrm{fin}}(\mathcal{O},\Gamma)$.
(ii) $(\mathfrak{b} < \mathfrak{d})$ implies that every subspace Y of the Baire space satisfies $\bigcup_{\mathrm{fin}}(\mathcal{O},\mathrm{T}^*)$ provided $|Y| \le \mathfrak{b}$.

The first of them may be found in [15]. To prove the second one, find a (probably not bijective) enumeration $\{y_\alpha : \alpha < \mathfrak{b}\}$ of Y. Recall from [19] that a subset $Z \subset \omega^\omega$ has a *weak excluded middle property* if there exists $x \in \omega^\omega$ such that the family $\{[z \le x] : z \in Z\}$ can be enlarged to a simple P-filter, where for a relation R on ω $[z\colon R\colon x] = \{n \in \omega : z(n)\colon R\colon x(n)\}$.

Let $f : Y \to \omega^\omega$ be continuous. By transfinite induction over $\mathfrak{b}$ construct a $\mathfrak{b}$-scale $B = \{b_\alpha : \alpha < \mathfrak{b}\}$ such that $f(y_\alpha), b_\beta \le^* b_\alpha$ for all $\beta \le \alpha < \mathfrak{b}$. Since $\mathfrak{b} < \mathfrak{d}$,

B is not dominating, which means that there exists $c \in \omega^\omega$ such that $c \leq^* b_\alpha$ for no $\alpha < \mathfrak{b}$, and hence $[b_\alpha < c]$ is infinite for all α. Observe that for arbitrary $\beta \leq \alpha < \mathfrak{b}$ the equation $b_\beta \leq^* b_\alpha$ implies $[b_\alpha < c] \subset^* [b_\beta < c]$, and therefore $\mathcal{T} = ([b_\alpha < c])_{\alpha<\mathfrak{b}}$ is a tower. Moreover, $[b_\alpha < c] \subset^* [f(y_\alpha) \leq c]$, consequently the family $\{[f(y_\alpha) \leq c] : \alpha < \mathfrak{b}\} = \{[f(y) \leq c] : y \in Y\}$ is a subset of the simple P-filter generated by $\mathcal{T}$, and hence $f(Y)$ has a weak excluded middle property. Applying [19, Theorem 7.8] asserting that a subset Z of the Baire space satisfies $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$ provided for every continuous $\phi : Z \to \omega^\omega$ the image $\phi(Z)$ has the weak excluded middle property, we conclude that Y has the property $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathrm{T}^*)$. □

Proof of Corollary 1:

1. Follows immediately from Proposition 1 and Theorem 1(3).
2. Under $(\mathfrak{b} < \mathfrak{d})$ the assertion follows from Proposition 2 and Theorem 1(3).

Under $(\mathfrak{t} = \mathfrak{d})$ it suffices to use the $(\mathfrak{t} = \mathfrak{b})$–part of Proposition 2 to find a nonmeager simple P-filter and then apply the same arguments as in the proof of the $(\mathfrak{b} = \mathfrak{d})$–part of Theorem 1(3). □

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