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On the Borel-Cantelli Lemma and moments

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Abstract. We present some extensions of the Borel-Cantelli Lemma in terms of moments. Our result can be viewed as a new improvement to the Borel-Cantelli Lemma. Our proofs are based on the expansion of moments of some partial sums by using Stirling numbers. We also give a comment concerning the results of Petrov V.V., *A generalization of the Borel-Cantelli Lemma*, Statist. Probab. Lett. **67** (2004), no. 3, 233–239.

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1. Introduction

The Borel-Cantelli lemmas play the central role in the proofs of many probability laws including the law of large numbers and the law of the iterated logarithm. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, that is a triple consisting of a space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω, and a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. If X is a nonnegative random variable, the expectation of X, denoted $\mathbb{E}(X)$, is

$$\mathbb{E}(X) = \int X d\mathbb{P}.$$

Recall that

Theorem 1.1 (Borel-Cantelli Lemmas)**.** *Let $A_1, A_2, \dots$ be an infinite sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote the probability of A_k by p_k.*

(1) *If $\sum p_k$ converges, then with probability one only finitely many of the events A_k occur.*
(2) *If the events A_k are mutually independent, and if $\sum p_k$ diverges, then with probability one, infinitely many of the events A_k occur.*

Many attempts were made in order to weaken the independence condition in the second part of the Borel-Cantelli Lemma. This condition means mutual independence of events.

In 1959, Erdös and Rényi [2] found that the condition of pairwise independence of events $A_1, A_2, \dots$ can be replaced by the weaker condition $\mathbb{P}(A_k \cap A_j) \leq \mathbb{P}(A_k)\mathbb{P}(A_j)$ for every k and j such that $k \neq j$.

In 1962, Rényi [8, Lemma C, p. 391] showed that, if $A_1, A_2, \dots$ are arbitrary events fulfilling the conditions

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty \tag{1.1}$$

and

$$\liminf_{n\to\infty} \frac{\sum_{1\le i,j\le n} \mathbb{P}(A_i \cap A_j)}{\left(\sum_{i=1}^{n} \mathbb{P}(A_i)\right)^2} = 1, \tag{1.2}$$

then $\mathbb{P}(\limsup A_n) = 1$.

In 1963, Lamperti [4] formulated the following proposition. If $A_1, A_2, \dots$ is a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $\mathbb{P}(A_k \cap A_j) \le C\mathbb{P}(A_k)\mathbb{P}(A_j)$ for all $k, j > N$ and some constants C and N, then $\mathbb{P}(\limsup A_n) > 0$.

In 1964, Kochen and Stone [3], see also Spitzer [9, P3, p. 317], proved the following result. If condition (1.1) is satisfied and if

$$\liminf_{n\to\infty} \frac{\sum_{1\le i,j\le n} \mathbb{P}(A_i \cap A_j)}{\left(\sum_{i=1}^{n} \mathbb{P}(A_i)\right)^2} \le C$$

then $\mathbb{P}(\limsup A_n) \ge \frac{1}{C}$.

In 1983, Ortega and Wschebor [5] proved that if conditions (1.1) and

$$\liminf_{n\to\infty} \frac{\sum_{1\le i<j\le n} \mathbb{P}(A_i \cap A_j) - \mathbb{P}(A_i)\mathbb{P}(A_j)}{\left(\sum_{i=1}^{n} \mathbb{P}(A_i)\right)^2} \le 0 \tag{1.3}$$

are satisfied, then $\mathbb{P}(\limsup A_n) = 1$. Note that this result can be obtained from Rényi's one.

In 2002, Petrov [6] formulated the following result. If $A_1, A_2, \dots$ is a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $\mathbb{P}(A_k \cap A_j) \le C\mathbb{P}(A_k)\mathbb{P}(A_j)$ for all $k, j > L$ such that $k \ne j$ and some constants $C \ge 1$ and L, then $\mathbb{P}(\limsup A_n) \ge \frac{1}{C}$.

In 2004, Petrov [7] “improved” these results as follows:

Theorem 1.2. *Let $A_1, A_2, \dots$ be a sequence of events satisfying condition* (1.1). *Let H be an arbitrary real constant. Put*

$$\alpha_H = \liminf \frac{\sum_{1\le i<j\le n} \mathbb{P}(A_i \cap A_j) - H\mathbb{P}(A_i)\mathbb{P}(A_j)}{\left(\sum_{k=1}^{n} \mathbb{P}(A_k)\right)^2}.$$

Then

$$\mathbb{P}(\limsup A_n) \ge \frac{1}{H + 2\alpha_H}.$$

We show below that

$$H + 2\alpha_H = \liminf \frac{\sum_{1\le i,j\le n} \mathbb{P}(A_i \cap A_j)}{(\sum_{k=1}^{n} \mathbb{P}(A_k))^2}.$$

In this paper, we present two extensions in terms of moment of order p as follows:

2. Main result and comments

Theorem 2.1. *If $A_1, A_2, \ldots$ is a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n)$ diverges and*

$$\mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j}\Big) \le C \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) \tag{2.1}$$

for all $i_p > i_{p-1} > \cdots > i_1 > L$ and some constants $C \ge 1$ and L, then

$$\mathbb{P}(\limsup A_n) \ge \frac{1}{C^{1/(p-1)}}.$$

Let $\mathbb{I}_{A_n}$ be the indicator of the event A_n. We put $S_n := \sum_{k=1}^{n} \mathbb{I}_{A_k}$.

Theorem 2.2. *Let $A_1, A_2, \ldots$ be a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Let $p \ge 2$ be an arbitrary integer. Put*

$$\alpha := \liminf_{n\to\infty} \frac{\sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathbb{P}(\bigcap_{j=1}^{p} A_{i_j})}{(\mathbb{E}(S_n))^p}.$$

Then we have

$$\mathbb{P}(\limsup A_n) \ge \frac{1}{(p!\alpha)^{1/(p-1)}}.$$

Theorem 2.3. *Let $A_1, A_2, \ldots$ be a sequence of events such that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. Let $p \ge 2$ be an arbitrary integer. Then we have*

$$\lim_{n\to\infty} \frac{1}{(\mathbb{E}(S_n))^p} \sum_{1\le i_1<i_2<\cdots<i_p\le n} \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) = \frac{1}{p!}$$

and

$$\lim_{n\to\infty} \frac{1}{\mathbb{E}(S_n^p)} \sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j}\Big) = \frac{1}{p!}.$$

Obviously we have $\mathbb{E}(\mathbb{I}_{A_n}) = \mathbb{P}(A_n)$, thus $\mathbb{E}S_n = \sum_{k=1}^{n} \mathbb{P}(A_k)$. By the Cauchy-Schwarz inequality we get

$$\mathbb{E}(S_n) = \mathbb{E}(S_n \mathbb{I}_{\bigcup_{k=1}^{n} A_k}) \leq \mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big)^{1/2} \Big(\mathbb{E} \sum_{i,j=1}^{n} \mathbb{I}_{A_i \cap A_j}\Big)^{1/2}$$

for arbitrary events $A_1, A_2, \dots A_n$, and hence

$$\mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big) \geq \frac{(\mathbb{E}(S_n))^2}{\sum_{i,j=1}^{n} \mathbb{P}(A_i \cap A_j)}$$

which is the Chung Erdös inequality [1]. Which gives

$$\limsup \frac{(\mathbb{E}(S_n))^2}{\sum_{i,j=1}^{n} \mathbb{P}(A_i \cap A_j)} \leq 1.$$

From this inequality, the fact that the condition (1.1) is satisfied,

$$\mathbb{E}(S_n^2) = \sum_{i,j=1}^{n} \mathbb{P}(A_i \cap A_j) = \mathbb{E}(S_n) + 2 \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i \cap A_j)$$

and

$$2 \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i)\mathbb{P}(A_j) = (\mathbb{E}S_n)^2 - \sum_{i=1}^{n} \mathbb{P}(A_i)^2$$

we get, the conditions (1.2) and (1.3) are equivalent, and

$$\lim_{n\to\infty} \frac{1}{(\mathbb{E}(S_n))^2} \sum_{1 \leq i < j \leq n} \mathbb{P}(A_i)\mathbb{P}(A_j) = \frac{1}{2}$$

which gives

$$H + 2\alpha_H = \liminf \frac{\sum_{1 \leq i,j \leq n} \mathbb{P}(A_i \cap A_j)}{(\sum_{k=1}^{n} \mathbb{P}(A_k))^2}$$

because of (if (a_n) converges and (b_n) arbitrary, then $\liminf(a_n + b_n) = \lim a_n + \liminf b_n$). Thus the result of [7] is the same as those of [9] and [3].

3. Stirling numbers and moments of S_n

In order to obtain an exact expression of $\mathbb{E}(S_n^p)$, we need the following notions on the Stirling numbers which can be found in [10].

For each positive integer n, let

$$(t)_n := t(t-1)\dots(t-n+1) \in \mathbb{Q}[t]$$

be the descending (falling) factorial. Also define $(t)_0 = 1$. *Stirling numbers of first kind*, denoted by $s(n,k)$, and *Stirling numbers of the second kind*, denoted $S(n,k)$ with $n,k \in \mathbb{N}$, are defined to be the coefficients in the expression

$$(t)_n = \sum_{k=0}^{n} s(n,k)t^k$$

and in the expression

$$t^n = \sum_{k=0}^{n} S(n,k)(t)_k.$$

We know also that if $c(n,k)$ denotes the number of permutations π of $\{1,2,\dots,n\}$ with exactly k cycles, then $s(n,k) = (-1)^{n-k}c(n,k)$. And if we denote by $P(n,k)$ the set of all partitions of an n-set into k nonempty subsets (blocs), then

$$S(n,k) = |P(n,k)|.$$

So we just mention that the two groups of numbers have similar properties and their generating functions are given by

$$\sum_{n=k}^{\infty} S(n,k)\frac{z^n}{n!} = \frac{1}{k!}(\exp(z)-1)^k$$

and

$$\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!} = \frac{1}{k!}[\log(1+z)]^k.$$

We will be mostly concerned with Stirling numbers of the first and second kind in the following theorem.

Theorem 3.1. *Let $A_1, A_2, \dots, A_n$ be a sequence of measurable sets, and p be a positive integer. Then we have*

$$\left(\sum_{k=1}^{n} \mathbb{I}_{A_k}\right)^p = \sum_{k=0}^{p} S(p,k)k! \sum_{1\le i_1<i_2<\cdots<i_k\le n} \mathbb{I}_{\bigcap_{j=1}^{k} A_{i_j}}$$

and

$$p! \sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathbb{I}_{\bigcap_{j=1}^{p} A_{i_j}} = \sum_{k=0}^{p} s(p,k)\left(\sum_{k=1}^{n} \mathbb{I}_{A_k}\right)^k.$$

Proof: Remark that for all $\omega \in \Omega$, we have $\sum_{k=1}^{n} \mathbb{I}_{A_k}(\omega) = t$ if and only if

$$k! \sum_{1\le i_1<i_2<\cdots<i_k\le n} \mathbb{I}_{\bigcap_{j=1}^{k} A_{i_j}}(\omega) = (t)_k$$

which gives the result. $\square$

By taking the expectation, the following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let $A_1, A_2, \ldots, A_n$ be a sequence of events, and p be a positive integer. Then we have*

$$\mathbb{E}\Big(\sum_{k=1}^{n} \mathbb{I}_{A_k}\Big)^p = \sum_{k=0}^{p} S(p,k) k! \sum_{1\le i_1<i_2<\cdots<i_k\le n} \mathbb{P}\Big(\bigcap_{j=1}^{k} A_{i_j}\Big)$$

and

$$p! \sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j}\Big) = \sum_{k=0}^{p} s(p,k)\mathbb{E}\Big(\sum_{k=1}^{n} \mathbb{I}_{A_k}\Big)^k.$$

4. Proofs of theorems

We shall often need Jensen's inequality which is as follows. If g is a convex function and X random variable such that $\mathbb{E}|g(X)| < \infty$ then

$$g(\mathbb{E}X) \le \mathbb{E}(g(X)).$$

Recall that $S_n = \sum_{k=1}^{n} \mathbb{I}_{A_k}$, and assume the sequence of events $A_1, A_2, \ldots$ satisfies (1.1).

To prove our Theorems, we need the following lemmas:

Lemma 4.1. *We have*

$$\mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big)^{(p-1)} \ge \frac{(\mathbb{E}S_n)^p}{\mathbb{E}(S_n^p)}.$$

Proof: By using Hölder's inequality we have

$$\mathbb{E}(S_n) = \mathbb{E}(S_n \mathbb{I}_{\bigcup_{k=1}^{n} A_k}) \le \mathbb{P}\Big(\bigcup_{k=1}^{n} A_k\Big)^{(p-1)/p} (\mathbb{E}(S_n^p))^{1/p}$$

which proves the lemma. $\square$

Lemma 4.2. *Let $p > 1$ be a real number, and I be an infinite subset of $\mathbb{N}$. If there exists $c \geq 0$ such that $\mathbb{E}(S_n^p) \leq c(\mathbb{E}S_n)^p$ for all $n \in I$, then*

$$\lim_{I \ni n \to \infty} \frac{\mathbb{E}(S_n^q)}{(\mathbb{E}S_n)^p} = 0$$

for all $0 < q < p$.

Proof: Let $n \in I$. From Jensen's inequality it follows that

$$\mathbb{E}(S_n^q) \leq (\mathbb{E}(S_n^p))^{\frac{q}{p}}.$$

Because of the assumption of the lemma it follows that

$$\mathbb{E}(S_n^q) \leq c^{\frac{q}{p}} (\mathbb{E}S_n)^q$$

and hence

$$\frac{\mathbb{E}(S_n^q)}{(\mathbb{E}S_n)^p} \leq c^{\frac{q}{p}} (\mathbb{E}S_n)^{q-p}$$

which proves our statement since $\lim \mathbb{E}S_n = \infty$. □

Lemma 4.3. *Let $p > 1$ be an integer, and I be an infinite subset of $\mathbb{N}$. If there exists $c \geq 0$ such that $\mathbb{E}(S_n^p) \leq c(\mathbb{E}S_n)^p$ for all $n \in I$, then*

$$\lim_{I \ni n \to \infty} \frac{\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbb{P}(\bigcap_{j=1}^k A_{i_j})}{(\mathbb{E}S_n)^p} = 0$$

for any integer $0 < k < p$.

Proof: By using Corollary 3.2, we get

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbb{P}\Bigg(\bigcap_{j=1}^k A_{i_j}\Bigg) = \frac{1}{k!} \sum_{j=0}^k s(k,j) \mathbb{E}\Bigg(\sum_{i=1}^n \mathbb{I}_{A_i}\Bigg)^j.$$

Hence

$$\frac{1}{(\mathbb{E}S_n)^p} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \mathbb{P}\Bigg(\bigcap_{j=1}^k A_{i_j}\Bigg) = \frac{1}{k!} \sum_{j=0}^k s(k,j) \frac{\mathbb{E}S_n^j}{(\mathbb{E}S_n)^p}$$

and by applying Lemma 4.2 we get the result. □

Lemma 4.4. *Let $m \geq 1$ be an integer, and I be an infinite subset of $\mathbb{N}$. If there exists $c \geq 0$ such that $\mathbb{E}(S_n^p) \leq c(\mathbb{E}S_n)^p$ for all $n \in I$, then*

$$\lim_{I \ni n \to \infty} \frac{1}{(\mathbb{E}S_n)^p} \Bigg(\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j} \Big) - \sum_{m \leq i_1 < i_2 < \cdots < i_p \leq n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j} \Big) \Bigg) = 0.$$

Proof: This follows from the fact that

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j} \Big) - \sum_{m \leq i_1 < i_2 < \cdots < i_p \leq n} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j} \Big) \leq m \sum_{1 \leq i_1 < i_2 < \cdots < i_{p-1} \leq n} \mathbb{P}\Big(\bigcap_{j=1}^{p-1} A_{i_j} \Big)$$

and Lemma 4.3. □

Lemma 4.5. *For every integer $m \geq 1$, we have*

$$\lim_{n \to \infty} \frac{1}{(\mathbb{E}S_n)^p} \Bigg(\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) - \sum_{m \leq i_1 < i_2 < \cdots < i_p \leq n} \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) \Bigg) = 0.$$

Proof: This follows from the fact that

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) - \sum_{m \leq i_1 < i_2 < \cdots < i_p \leq n} \prod_{j=1}^{p} \mathbb{P}(A_{i_j}) \leq m \sum_{1 \leq i_1 < i_2 < \cdots < i_{p-1} \leq n} \prod_{j=1}^{p-1} \mathbb{P}(A_{i_j}) \leq \frac{m}{(p-1)!} (\mathbb{E}S_n)^{p-1}.$$

□

The main part of the proof of Theorem 2.3 (second part) is the following lemma.

Lemma 4.6. *Let $a_1, a_2, \ldots, a_n$ be positive numbers and p a positive integer. Then the following inequality*

$$\left(\sum_{i=1}^{n} a_i \right)^p - p! \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \prod_{j=1}^{p} a_{i_j} \leq \sum_{j=2}^{p} \binom{p}{j} \left(\sum_{i=1}^{n} a_i \right)^{p-j} \sum_{i=1}^{n} a_i^j \tag{4.1}$$

holds. In particular if $a_i \in [0,1]$ *for* $i = 1, 2, \dots, n$, *we have*

$$\left(\sum_{i=1}^n a_i\right)^p - p! \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \prod_{j=1}^p a_{i_j} \le \sum_{j=2}^p \binom{p}{j} \left(\sum_{i=1}^n a_i\right)^{p+1-j}.$$

PROOF: Remark that the left side of inequality (4.1) is less than or equal to

$$\sum_{k=1}^n \sum_{j=2}^p \binom{p}{j} a_k^j \left(-a_k + \sum_{i=1}^n a_i\right)^{p-j}.$$

Now by the fact that $-a_k + \sum_{i=1}^n a_i \le \sum_{i=1}^n a_i$ we obtain the first inequality.

The second part follows from the first one by using $\sum_{i=1}^n a_i \ge \sum_{i=1}^n a_i^j$ for $j \ge 2$ and $a_i \in [0,1]$. □

Lemma 4.7. *We have*

$$\left|\mathbb{E}(S_n^p) - p! \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \mathbb{P}\left(\bigcap_{j=1}^p A_{i_j}\right)\right| \le (p! - 1)\left(\mathbb{E}(S_n^p)\right)^{(p-1)/p}$$

for all n *such that* $\mathbb{E}(S_n) \ge 1$.

PROOF: First we have $s(p,p) = 1$. Now, by applying Corollary 3.2, we get

$$\left|\mathbb{E}(S_n^p) - p! \sum_{1 \le i_1 < i_2 < \cdots < i_p \le n} \mathbb{P}\left(\bigcap_{j=1}^p A_{i_j}\right)\right| \le \sum_{k=0}^{p-1} |s(p,k)| \mathbb{E}(S_n^k)$$

and, by using Jensen's inequality, we obtain $\mathbb{E}(S_n^k) \le (\mathbb{E}S_n^p)^{k/p}$ for all $0 \le k \le p$. Remark that if $\mathbb{E}S_n \ge 1$ then $\mathbb{E}S_n^p \ge 1$ and thus $\left(\mathbb{E}S_n^p\right)^{(p-1)/p} \ge \mathbb{E}(S_n^k)$ for all $0 \le k \le p-1$. We have also $\sum_{k=0}^{p-1} |s(p,k)| = p! - 1$ which completes the proof. □

PROOF OF THEOREM 2.3: Remark that $\mathbb{E}(S_n) \le \mathbb{E}(S_n^p)$ for large n, and so the second part of the theorem follows from Lemma 4.7.

The first part of the theorem follows from Lemma 4.6 by $a_i = \mathbb{P}(A_i)$ and this completes the proof of Theorem 2.3. □

PROOF OF THEOREM 2.1: By applying Lemma 4.1, we get

$$\mathbb{P}\left(\bigcup_{k=m+1}^N A_k\right)^{(p-1)} \ge \frac{(\mathbb{E}(S_N - S_m))^p}{\mathbb{E}(S_N - S_m)^p}$$

and, by applying Theorem 2.3 we have

$$(\mathbb{E}(S_N - S_m))^p \sim_{N\to\infty} p! \sum_{m+1\le i_1<i_2<\cdots<i_p\le N} \prod_{j=1}^{p} \mathbb{P}(A_{i_j})$$

and

$$\mathbb{E}(S_N - S_m)^p \sim_{N\to\infty} p! \sum_{m+1\le i_1<i_2<\cdots<i_p\le N} \mathbb{P}\Big(\bigcap_{j=1}^{p} A_{i_j}\Big).$$

Combining these with equation (2.1) we obtain

$$\mathbb{P}\Big(\bigcup_{k=m+1}^{\infty} A_k\Big)^{(p-1)} \ge \frac{1}{C}$$

which terminates the proof. □

Lemma 4.8. *If $\alpha < \infty$, then there exist an infinite subset I of positive integers and a constant C such that*

$$\mathbb{E}S_n^p \le C(\mathbb{E}S_n)^p.$$

Proof: It follows from the assumption $\alpha < \infty$ that one can choose an infinite I of positives integers such that

$$\alpha = \lim_{I\ni n\to\infty} \frac{\sum_{1\le i_1<i_2<\cdots<i_p\le n} \mathbb{P}(\bigcap_{j=1}^{p} A_{i_j})}{(\mathbb{E}(S_n))^p}.$$

Now by applying Theorem 2.3 we prove the lemma. □

Proof of Theorem 2.2: By applying Lemma 4.1, we get

$$\mathbb{P}\Big(\bigcup_{k=m+1}^{N} A_k\Big)^{(p-1)} \ge \frac{(\mathbb{E}(S_N - S_m))^p}{\mathbb{E}(S_N - S_m)^p}.$$

By applying Lemma 4.8, then Lemmas 4.3, 4.4 and 4.5 we get

$$\alpha = \liminf \frac{\mathbb{E}(S_n^p)}{p!(\mathbb{E}S_n)^p},$$

thus

$$\mathbb{P}\Big(\bigcup_{k=m+1}^{\infty} A_k\Big)^{(p-1)} \ge \frac{1}{p!\alpha}$$

and the proof of the theorem is complete. □

We complete this article with the following result which can be obtained by the same method.

Proposition 4.9. *Let $A_1, A_2, \ldots$ be a sequence of events such that $\sum_n \mathbb{P}(A_n)$ diverges. Let $p > 1$ be a real number. Then we have*

$$\mathbb{P}(\limsup A_n)^{(p-1)} \geq \limsup \frac{(\mathbb{E}S_n)^p}{\mathbb{E}S_n^p}\,.$$

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