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Spaces of continuous characteristic functions

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Abstract. We show that if X is first-countable, of countable extent, and a subspace of some ordinal, then $C_p(X,2)$ is Lindelöf.

Keywords: $C_p(X,Y)$, subspace of ordinals, countable extent, Lindelöf space

Classification: 54C35, 54D20, 54F05

1. Introduction

One of the main problems in C_p-theory is to find the properties of X which force the space $C_p(X)$ to be Lindelöf. A natural way to approach this problem is to analyze it within classes of spaces with richer structures. For example, Nahmanson [NAH] proved that a compact LOTS has Lindelöf C_p iff it is metrizable. This theorem suggests that characterizing spaces with Lindelöf C_p might be an attainable goal in the class of all LOTS or GO-spaces. In the last section we will discuss some trivial necessary conditions and make conjectures about non-trivial conditions.

In [BUZ], the author proved that a countably compact first-countable subspace of an ordinal has Lindelöf C_p. This result suggests to replace "countable compact" with "countable extent". We do not know if this replacement leads to a theorem, however our main result speaks in favor of "yes". In the main result (Section 2), we prove that if X is first countable, of countable extent, and a subspace of some ordinal, then $C_p(X,2)$ is Lindelöf.

In notation and terminology we will follow [AR1] and [ENG]. By 2 we denote the discrete space $\{0,1\}$. As usual, $C_p(X,Y)$ is the space of all continuous functions from X to Y endowed with the topology of point-wise convergence. All spaces considered are Tychonoff. A space X has *countable extent* if any closed discrete subset of X is countable.

2. Main result

For spaces X and Y, U is a *standard open set* in $C_p(X,Y)$ if there exist $x_1,\dots,x_n \in X$ and open $B_1,\dots,B_n$ in Y such that $U = \{f \in C_p(X,Y) : f(x_i) \in B_i, i = 1,\dots,n\}$. In this case, we say that U *depends* on $\{x_1,\dots,x_n\}$. If $\mathcal{U}$ is a collection of standard open sets in $C_p(X,2)$ and $A \subset X$, by $\mathcal{U}(A)$ we denote the family of all elements of $\mathcal{U}$ that depend on a subset of A.

For brevity, let X be a fixed subspace of some ordinal, first countable, and of countable extent. Also, let $\mathcal{U}$ be a fixed open cover of $C_p(X,2)$ by standard open sets. In this section, we will work with these fixed structures.

We may assume that X is dense in some fixed ordinal χ, that is, X is obtained from χ by removing some of limit ordinals, while all isolated ordinals of χ are in X. Since for countable X our main result is trivial, we may assume that χ is uncountable and $\mathrm{cf}(\chi) > \omega$. The latter assumption does not put any additional restrictions on X. Indeed, uncountability and first-countability of X imply that X contains a clopen subset homeomorphic to an uncountable subspace of ω_1. We can simply move this set at the end of X (or "ahead", depending on one's view of order).

For any $A \subset X$, $\sup A$ is calculated in the class of ordinals. By $[\alpha,\beta]_X$, we denote $[\alpha,\beta] \cap X$. The same concerns open and half-open intervals. Closed intervals of ordinals will be also called segments. If $A \subset X$, by A^- we denote the set $\{\alpha : \alpha + 1 \in A, \mathrm{cf}(\alpha) > \omega\} \cup \{\chi\}$. In words, α belongs to A^- iff $\alpha = \chi$, or α has uncountable cofinality and is the immediate predecessor of an element of A.

Let us start with the following technical definitions.

Definition 2.1. Let $A \subset X$. The family $\mathcal{S}(A) \subset C_p(X,2)$ is defined as follows: $S \in \mathcal{S}(A)$ iff there exist $\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n \in A \cup A^-$ and $b_1, \ldots, b_n \in 2$ such that $S = \{f \in C_p(X,2) : f([\alpha_i,\beta_i]_X) = \{b_i\}, i = 1, \ldots, n\}$.

Notice that $\mathcal{S}(A)$ is countable if A is countable.

Definition 2.2. For $A \subset X$, a set $B \subset X$ is an ω-support of A if

1. B is countable and contains $A \cup \{0\}$;
2. if $\beta \in B$ is limit in X then β is limit in B;
3. if $\beta \in B$ then any countable $[\alpha,\beta]_X$ is a subset of B.

Lemma 2.3. *If $A \subset X$ is countable then there exists B an ω-support of A.*

Proof: For each $\alpha \in A$, let m_α be the smallest ordinal such that $[m_\alpha,\alpha]_X$ is countable. Let $B' = \bigcup_{\alpha\in A}[m_\alpha,\alpha]_X \cup \{0\}$. Clearly, $1, 3$ are met. If $\beta \in B'$ is limit in X but not limit in B' then $\beta = m_\alpha$ for some $\alpha \in A$. By the definition of m_α, any neighborhood of m_α, and therefore that of β, is uncountable. Since $\beta \in X$ and X is first-countable, there exists a strictly increasing sequence $\{\beta_n\}_n$ of ordinals of χ that converges to β. Since every neighborhood of β is uncountable and $\mathrm{cf}(\beta) = \omega$, we can choose β_n's in $\chi \setminus X$ with uncountable cofinality. Let $B_\beta = \{\beta_n + 1 : n \in \omega\}$. Since we agreed that all isolated ordinals of χ are in X, the set B_β is a subset of X. Let $B = \bigcup_{\beta\in B'} B_\beta \cup B'$. Since, we added to B' only isolated ordinals, B meets $1, 2$. And it meets 3 because any $\beta \in B \setminus B'$ is the immediate successor of an ordinal of uncountable cofinality. $\square$

Notice that if $\mathcal{A}$ is a countable family of countable subsets of X that are ω-supports of themselves then $\bigcup \mathcal{A}$ is an ω-support of itself, too. If, additionally, each element of $\mathcal{A}$ contains a fixed $A \subset X$, then $\bigcup \mathcal{A}$ is an ω-support of A.

Lemma 2.4. *Let I be a segment in $(\chi + 1)$ with uncountable* $\mathrm{cf}(\sup I)$ *and let $A \subset X$ be countable. Then there exists $A_I \subset X$ with the following properties:*

1. $\sup(A_I \cap I)$ *belongs to X but is not in A_I;*
2. *if $S \in \mathcal{S}(A_I)$ and $S \subset U \in \mathcal{U}$ then some $U_S \in \mathcal{U}(A_I)$ contains S;*
3. $\sup(A_I \cap I) > \sup(\overline{A} \cap I)$ *if $\overline{A} \cap I \neq \emptyset$;*
4. *A_I is an ω-support of $\overline{A}$.*

Proof: Let $A_0 = \overline{A}$. Assume a countable A_β is defined for each $\beta < \alpha < \omega_1$.
Step $\alpha < \omega_1$: Let $B = \bigcup_{\beta<\alpha} A_\beta \cup \{s_\alpha\}$, where s_α is any in $I \cap X$ such that $[s_\alpha, \sup I]_X$ does not meet the closure of $\bigcup_{\beta<\alpha} A_\beta$. Such an s_α exists because each A_β is countable, while the right end point of I has uncountable cofinality. For each $S \in \mathcal{S}(B)$, fix $U_S \in \mathcal{U}$ (if exists) that contains S. Let A_α be an ω-support of $\bigcup\{B_S :$ a fixed U_S depends on $B_S\} \cup B$.

Let $\alpha \leq \omega_1$ be the first limit ordinal such that $\sup\{s_\beta : \beta < \alpha\} \in X$. This α exists because X has countable extent. Since X is first countable, $\alpha < \omega_1$. The set $A_I = \bigcup_{\beta<\alpha} A_\beta$ is desired. Indeed, 1 holds by the choice of α. Property 3 holds due to the presence of s_β in $A_\beta \cap I$ for $\beta < \alpha$. To verify 2, fix $S \in \mathcal{S}(A_I)$ that is contained in some element of $\mathcal{U}$. Since S is determined by a finite subset of A_I, there exists $\beta < \alpha$ such that S is determined by a finite subset of A_β. By our construction, some fixed $U_S \in \mathcal{U}$ containing S depends on a subset of A_β. Therefore, U_S belongs to $\mathcal{U}(A_I)$. Condition 4 is satisfied too, since A_I is the union of an increasing family of sets that are ω-supports of themselves and contain $\overline{A}$.
□

For our further discussion we need to define two expressions: "hits" and "local type". Let $A \subset X$ be countable, $\mathcal{A}$ a chain of countable subsets of X, and I a segment as in Lemma 2.4. If A_I is the smallest element of $\mathcal{A}$, we say that A_I *hits* I if A_I satisfies the conclusion of Lemma 2.4 with input segment I and countable set A. If $A_I \in \mathcal{A}$ is not the smallest element of $\mathcal{A}$, we say that A_I *hits* I if A_I satisfies the conclusion of Lemma 2.4 with input segment I and countable set A' for every $A' \in \mathcal{A}$ a proper subset of A_I. It will be clear what $\mathcal{A}$ and A are under consideration. Also, we say that an ordinal α has *local type* β if β is the smallest ordinal greater than 0 such that α has an open neighborhood homeomorphic to an open neighborhood of β. For example, any isolated ordinal has local type 1; ordinal $\omega + \omega$ has local type ω. Clearly, any $\alpha \in \omega^n + 1$ has local type $\omega^0 = 1$, or ω, or ω^2, ... , or ω^n. We will use the following fact: if α is a limit ordinal then any neighborhood of α contains an ordinal of any given local type less than the local type of α.

Lemma 2.5. *Let $\{I_0, \ldots, I_n\}$ be a collection of segments in $(\chi+1)$ with uncountable* $\mathrm{cf}(\sup I_i)$ *for each i. Let $A \subset X$ be countable. Then there exists a chain*

$\mathcal{A} = \{A^n_\alpha : \alpha \in \omega^n + 1\}$ *of subsets of* X *with the following properties:*

1. *if* α *is limit, then* $A^n_\alpha = \bigcup_{\beta<\alpha} A^n_\beta$*;*
2. *if* α *is of local type* ω^i *then* A^n_α *hits* I_{n-i}*.*

Proof: For one interval, the conclusion follows from Lemma 2.4. Assume that the conclusion is true for any appropriate collection of n intervals. Let us construct a required chain for $(n+1)$ intervals. Let $B_0 = A$. Assume for $0 < \gamma < \beta$, a chain $\mathcal{A}_\gamma$ is defined and $B_\gamma = \bigcup \mathcal{A}_\gamma$ is countable.

Step $\beta < \omega_1$: Let s_β be any in $I_0 \cap X$ such that $[s_\beta, I_0]_X$ does not meet $\bigcup_{\gamma<\beta} B_\gamma$. Such an s_β exists because each B_γ is countable and the right end-point of I_0 has uncountable cofinality. By our assumption, the conclusion of our lemma is true for any appropriate collection of n intervals and any countable subset of X. Therefore, there exists a chain $\mathcal{A}_\beta$ that satisfies the conclusions of our lemma with input intervals $\{I_1, \dots, I_n\}$ and countable $[\bigcup_{\gamma<\beta} B_\gamma] \cup \{s_\beta\}$. Put $B_\beta = \bigcup \mathcal{A}_\beta$. Since each element of $\mathcal{A}_\beta$ is an ω-support of any preceding one, B_β is countable.

Due to countable extent and first countability of X, there exists a limit $\beta < \omega_1$ such that $\sup\{s_\gamma : \gamma < \beta\}$ is in X. Choose a strictly increasing sequence $\{\beta_k\}_k$ converging to β. Let the chain $\mathcal{A}$ consist of all elements of $\bigcup_k \mathcal{A}_{\beta_k}$ and the element $A^n_{\omega^n} = \bigcup\{A' : A' \in \mathcal{A}_{\beta_k}$ for some $k\}$. Observe that $A^n_{\omega^n}$ is constructed exactly in the same manner as A_I in Lemma 2.4. Therefore, $A^n_{\omega^n}$ satisfies the conclusion of Lemma 2.4 with input interval I_0 and any element of $\mathcal{A}$ distinct from $A^n_{\omega^n}$. Let us tag elements of $\mathcal{A}$ in accordance with the conclusion of our lemma. Represent $\mathcal{A}_{\beta_0}$ as $\{A^n_\alpha : \alpha \le \omega^n\}$; $\mathcal{A}_{\beta_1}$ as $\{A^n_\alpha : \omega^n < \alpha \le \omega^n + \omega^n\}$; and so on. Since each $\mathcal{A}_{\beta_k}$ was chosen to satisfy the conclusion of our lemma with input intervals $\{I_1, \dots, I_n\}$, these representations are possible and properties $1, 2$ are met for any $\alpha \le \omega^n$. □

In the next lemma, Cl_χ is the closure operator in χ.

Lemma 2.6. *Let* $f : X \to 2$ *be continuous. Let* $\{I_n : n \in \omega\}$ *be a collection of disjoint segments in* $(\chi + 1)$*. For each* n*, let* $i_n \in 2$ *satisfy the following property:*

$$\text{there exists } S_n \subset X \text{ such that } \inf I_n \in \mathrm{Cl}_\chi(S_n) \text{ and } f(S_n) = \{i_n\}.$$

Then the function $c_f : X \to 2$ *is continuous, where*

$$c_f(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_n I_n \\ i_n & \text{if } x \in I_n \cap X. \end{cases}$$

Proof: Since we work with first-countable spaces, it is enough to show that $c_f(x_k) \to c_f(x)$ whenever $x_k \to x$ in X. Observe that c_f coincides with f on $\overline{X \setminus \bigcup_n I_n}$. Therefore, we may assume that all x_k's are in $\bigcup_n I_n$. If infinitely many x_k's are in the same I_n then we are done since c_f is constant within each I_n. Thus we may assume that all x_k's are in different I_n's. Then either $x \in X \setminus \bigcup_n I_n$ or $x = \inf I_m$ for some m. In any case, $f(x) = c_f(x)$. By the property of i_n's, for each x_k we can fix $y_k \in \bigcup_n S_n$ such that $c_f(x_k) = c_f(y_k) = f(y_k)$ and $y_k \to x$. By continuity of f, $c_f(y_k) \to f(x) = c_f(x)$. Hence, $c_f(x_k) \to c_f(x)$. $\square$

Before we give our last definition let us remind the definition of A^- from the beginning of the section. If $A \subset X$, by A^- we denote the set $\{\alpha : \alpha + 1 \in A, \operatorname{cf}(\alpha) > \omega\} \cup \{\chi\}$. In words, α belongs to A^- iff $\alpha = \chi$, or α has uncountable cofinality and is the immediate predecessor of an element of A.

Definition 2.7. For $A \subset X$, define $G(A) = \{[0, \alpha] : \alpha \in A^-\}$.

Lemma 2.8. *Let $A \subset X$ be an ω-support of itself. Then*

1. *for any $x \in X \setminus A$, $x \in [\sup(A \cap I), \sup I]$ for some $I \in G(A)$;*
2. *the sets $[\sup(A \cap I_1), \sup I_1]$ and $[\sup(A \cap I_2), \sup I_2]$ do not meet, for any distinct $I_1, I_2 \in G(A)$.*

Proof: Let us prove 1. Let $r' = \min\{\alpha \in A \cup \{\chi\} : \alpha > x\}$. If $r' = \chi$, put $r = \chi$. Otherwise, by condition 3 in the definition of ω-support, there exists $r < \chi$ such that $r' = r + 1$ and $\operatorname{cf}(r) > \omega$. In either case, $r \in A^-$. Therefore, $I = [0, r] \in G(A)$ is as desired.

Let us prove 2. Since I_1, I_2 are distinct, we may assume that $\sup I_1 < \sup I_2$. By the definition of $G(A)$, $\sup I_1 \in A^-$. Since $\sup I_1 < \sup I_2$, $\sup I_1 \neq \chi$. By the definition of A^-, $\sup I_1 + 1 \in A$. Thus, $\sup I_1 < \sup(A \cap I_2)$. $\square$

We are finally ready to prove our main result.

Theorem 2.9. *Let X be a subspace of some ordinal. If X is first-countable and of countable extent, then $C_p(X, 2)$ is Lindelöf.*

Proof: We continue working with our fixed X and $\mathcal{U}$. Inductively, we will define $A_n \subset X$ such that $\mathcal{U}(\bigcup_n A_n)$ will be a countable subcover. Recall that $\mathcal{U}(\bigcup_n A_n)$ is the family of all elements of $\mathcal{U}$ that depend on a subset of $\bigcup_n A_n$.

Step 0: Let A_0 be an ω-support of itself. Enumerate $G(A_0)$ by prime numbers. If we do not have enough elements in $G(A_0)$, place $[0, \chi]$ at the rest of allocated places.

Step n: Let $I_0, \dots, I_n$ be the first $n+1$ elements of $G(A_{n-1})$. Let $\{A^n_\alpha\}_{\alpha \in \omega^n + 1}$ satisfy the conclusion of Lemma 2.5 with input set A_{n-1} and the above intervals in the given order.

Let $A_n = A^n_{\omega^n}$. Enumerate $G(A_n) \setminus G(A_{n-1})$ by $(n+1)$-st powers of prime numbers. With this enumeration, elements of $G(A_{n-1})$ keep their old tags.

Let $A = \bigcup_n A_n$. Since each A_n is an ω-support of itself, A is an ω-support of itself too. To simplify further argument, let us agree on notation.

Notation: If $\lambda \in A$, put $o(\lambda) = \langle n, \alpha \rangle$, where $\overline{A}^n_\alpha$ is the first containing λ. The order on $\langle n, \alpha \rangle$'s is lexicographical.

To avoid constant referring to the previous lemmas, let us make two remarks to be used later in the proof.

Remark 1. Let $I_5 \in \bigcup_n G(A_n)$ be the fifth element. For each $n > 5$, let $\alpha_n \in \omega^n + 1$ be an ordinal of local type ω^{n-5}. Then $r^n_5 = \sup(A^n_{\alpha_n} \cap I_5)$ exists and is limit in A. Moreover, $\{r^n_5\}_{n>5}$ is a strictly increasing sequence converging to $\sup(A \cap I_5)$. Indeed, r^n_5 exists because, by our construction, $A^n_{\alpha_n}$ hits I_5. By 1 of Lemma 2.4, r^n_5 is limit in A. By 3 of Lemma 2.4, $r^{n+1}_5 > r^n_5$, whence the sequence in question is strictly increasing. Finally, since $A = \bigcup_n A^n_{\alpha_n}$, the sequence converges to $\sup(A \cap I_5)$.

Remark 2. Let $r = \sup(A^n_\alpha \cap I_5)$, where α is of local type ω^{n-5} and $n > 5$. Then there exists $l < r$ that belongs to A^n_β for some $\beta < \alpha$. Moreover, l can be chosen as close to r as we wish. Indeed, since ω^{n-5} is limit for $n > 5$, α is a limit ordinal. By Lemma 2.5, $A^n_\alpha = \bigcup_{\beta<\alpha} A^n_\beta$ and A^n_α hits I_5. By 1 of Lemma 2.4, r belongs to the closure of A^n_α but does not belong to A^n_α. Therefore, $l = \sup(A^n_\beta \cap I_5)$ is as desired for large $\beta < \alpha$.

Recall that we constructed $A = \bigcup_n A_n$. Clearly, $\mathcal{U}(A)$ is countable, so let us prove that it is a subcover. Fix any $f \in C_p(X, 2)$. Inductively we will define a continuous function c_f that coincides with f on A. Then to prove that $f \in \bigcup \mathcal{U}(A)$ it will suffice to show that $c_f \in \bigcup \mathcal{U}(A)$.

Definition of c_f: Put $c_f(x) = f(x)$ for all $x \in \overline{A}$. Since we used primes to enumerate $\bigcup_n G(A_n)$, some numbers are left unassigned. So re-enumerate the elements of $\bigcup_n G(A_n)$ by non-negative integers without changing the current order.

Step 0: Select an infinite $C_0 \subset \omega \setminus \{0\}$ and $i_0 \in 2$ such that for any $k \in C_0$ there exist distinct $l^k_0, r^k_0 \in A$ with the following properties:

1. $r^k_0 \to \sup(A \cap I_0)$ and $l^k_0 \to \sup(A \cap I_0)$;
2. $o(r^k_0) = \langle k, \alpha \rangle$, where α is of local type ω^k, and $r^k_0 = \sup(A^k_\alpha \cap I_0)$;
3. $f([l^k_0, r^k_0]_X) = \{i_0\}$;
4. $o(l^k_0) < o(r^k_0)$.

Such an infinite collection of intervals exists. Indeed, by Remark 1, there exists $\{r^k_0 : k > 0\}$ that satisfies 2 and the first half of 1. Since r^k_0 is limit and f is continuous, there exists $l^k_0 < r^k_0$ such that f is constant on $[l^k_0, r^k_0]_X$. By Remark 2, l^k_0 can be chosen in A^k_α with $\alpha < \omega^k$ so that $\{l^k_0\}_k$ is a strictly increasing sequence converging to the same point as $\{r^k_0\}_k$, that is, to $\sup(A \cap I_0)$. Thus, 4 and the other half of 1 can be achieved. Condition 3 can be achieved due to finiteness of the range space $\{0,1\}$ (this is the only place our argument breaks for the reals).

Let $J_0 = [\sup(A \cap I_0), \sup I_0]$. For all $x \in J_0 \cap X$ put $c_f(x) = i_0$.

Step n: For simplicity, let $n = 2$. Select an infinite $C_2 \subset C_1$ and $i_2 \in 2$ such that for any $k \in C_2$ there exist distinct $l_2^k, r_2^k \in A$ with the following properties:

1. $r_2^k \to \sup(A \cap I_2)$ and $l_2^k \to \sup(A \cap I_2)$;
2. $o(r_2^k) = \langle k, \alpha \rangle$, where α is of local type ω^{k-2}, and $r_2^k = \sup(A_\alpha^k \cap I_2)$;
3. $f([l_2^k, r_2^k]_X) = \{i_2\}$;
4. $o(l_1^k) < o(l_2^k) < o(r_2^k) < o(r_1^k)$.

To construct such a collection, fix a previously constructed segment $[l_1^k, r_1^k]$, where $k \in C_1$. As remarked after the definition of local type, there are infinitely many $\langle k, \gamma \rangle$ such that γ is of local type ω^{k-2} and $o(l_1^k) < \langle k, \gamma \rangle < o(r_1^k)$. Fix one such $\langle k, \gamma \rangle$ and put $r_2^k = \sup(A_\gamma^k \cap I_2)$. By our construction, A_γ^k hits I_2. By Remark 1, $r_2^k \to \sup(A \cap I_2)$. As at Step 0, we can choose l_2^k to satisfy 1, 3, and the first two inequalities of 4. The rest of the argument is the same as in Step 0.

By Lemma 2.8, the segment $J_2 = [\sup(A \cap I_2), \sup I_2]$ either coincides with some J_m for $m < n = 2$ or disjoint with all of them. In the latter case put $c_f(x) = i_2$ for all $x \in J_2 \cap X$.

The segments J_n's that participated in the definition of c_f are disjoint. This collection of segments together with i_n's and $S_n = \{r_n^k : k \in C_n\}$ satisfy the conditions of Lemma 2.7. Therefore, c_f is continuous. Let us show that $\mathcal{U}(A)$ covers c_f. There exists $U_{c_f} \in \mathcal{U}$ that contains c_f. Suppose $U_{c_f} = \{g : g(x_k) = i_k \in 2, k = 1,2,3\}$. Assume $x_3 \in A$ and $x_1, x_2 \in X \setminus A$. By Lemma 2.8, x_1, x_2 are in at most two of J_n's, say in $J_1 \cup J_2$. Put $r_1 = \sup J_1$ and $r_2 = \sup J_2$. By Definition 2.7 of $G(A_n)$, r_1 and r_2 are in A^-.

Case I ($x_1 \in J_1, x_2 \in J_2$): Assume J_2 is to the right of J_1. By 1 in the definition of c_f, there exists $m \in C_2$ such that $[l_2^m, r_2^m]$ is to the right of J_1. Since sequences of l's and r's in the definition of c_f are in fact increasing, l_1^m and l_2^m are to the left of $\sup(A \cap I_1)$ and $\sup(A \cap I_2)$, respectively. Therefore, $x_1 \in [l_1^m, r_1]$ and $x_2 \in [l_2^m, r_2]$. Since, l_2^m is to the right of r_1, the intervals $[l_1^m, r_1]$ and $[l_2^m, r_2]$ are disjoint. Finally, since $x_3 \in A$, we can select such an $m \in C_2$ that neither $[l_1^m, r_1]$ nor $[l_2^m, r_2]$ contains x_3. Therefore, the set

$$S = \{g \in C_p(X, 2) : g([l_1^m, r_1]_X) = \{i_1\}, g([l_2^m, r_2]_X) = \{i_2\}, g(x_3) = i_3\}$$

is contained in U_{c_f}. Let $\langle m, \alpha \rangle = o(l_2^m)$, that is, $l_2^m \in A_{\alpha+1}^m$. By 4, $l_1^m \in A_\alpha^m$. Since $x_3, r_1, r_2 \in A \cup A^-$, we may assume they are in $A_\alpha^m \cup (A_\alpha^m)^-$ (simply choose $m \in C_2$ large enough, which is possible due to infiniteness of C_2). By Definition 2.1, $S \in \mathcal{S}(A^m_{\alpha+1})$. By 2 of Lemma 2.4, some $U_S \in \mathcal{U}(A^m_{\alpha+1})$ contains S. But this is possible only if U_S is a finite intersection of open sets in the following forms:

$$\{g : g(x_3) = i_3\}; \{g : g(y) = i_1\}; \{g : g(z) = i_2\},$$

where $y \in [l^m_1, r_1] \cap A^m_{\alpha+1}$ and $z \in [l^m_2, r_2] \cap A^m_{\alpha+1}$. These sets are contained in $[l^m_1, r^m_1]$ and $[l^m_2, r^m_2]$, respectively. Indeed, recall that $o(l^m_2) = \langle m, \alpha\rangle$. By 4, r^m_1 and r^m_2 appear for the first time not earlier than in $A^m_{\alpha+1}$. By 2, $r^m_1 \geq \sup(A^m_{\alpha+1} \cap I_1)$ and $r^m_2 \geq \sup(A^m_{\alpha+1} \cap I_2)$, from where the inclusions follow. By 3, $c_f(y) = i_1$ and $c_f(z) = i_2$. Hence $c_f \in U_S$. Since c_f coincides with f on A, f is covered by $\mathcal{U}(A^m_{\alpha+1})$ as well.

Case II ($x_1, x_2 \in J_1$): In this case $i_1 = i_2$. So put $S = \{g : g([l^m_1, r_1]_X) = \{i_1\}, g(x_3) = i_3\}$. The rest of the argument is as in Case I. □

3. Corollaries and related questions

Note that if for each $i \in \omega$, X_i satisfies the conditions of Theorem 2.9 then so does $\oplus_{i\in\omega} X_i$.

Corollary 3.1. *Let X and X_i be subspaces of some ordinals. If X and X_i are first countable and of countable extent, then $(C_p(X,2))^\omega$, $C_p(X,2^\omega)$, and $\Pi_{i\in\omega} C_p(X_i,2)$ are Lindelöf.*

Theorem 2.9 gives a sufficient condition for a subspace of an ordinal to have Lindelöf $C_p(\cdot,2)$, but not a criterion. Indeed, let X be uncountable discrete. Then X is a first-countable subspace of an ordinal. Clearly, the extent of X is uncountable, nevertheless, $C_p(X,2)$ is Lindelöf being homeomorphic to 2^X. Yet, the following theorem holds.

Theorem 3.2. *Let X be a subspace of an ordinal and let $C_p(X,2)$ be Lindelöf. Then X is first-countable and the derived set X' has countable extent.*

Proof: First countability is obvious. Indeed, X has countable tightness (the argument is the same as in Asanov's theorem [ASA]). Any countably tight GO is first countable.

For the second condition, assume the contrary, and fix a closed discrete $\{x_\alpha : \alpha < \omega_1\} \subset X'$. Since any GO is collectionwise Hausdorff and normal, there exists an uncountable discrete family $\{J_\alpha : \alpha < \omega_1\}$ of mutually disjoint open convex sets such that each J_α contains x_α. Since $\operatorname{ind} X = 0$, each J_α can be chosen clopen. Therefore, $\bigcup_\alpha J_\alpha$ is clopen and $C_p(X,2)$ contains a closed copy of $C_p(\bigcup_\alpha J_\alpha, 2)$. Since J_α's form a discrete family of disjoint closed sets, we have $\bigcup_\alpha J_\alpha$ is homeomorphic to $\oplus_\alpha J_\alpha$. Therefore, $C_p(\bigcup_\alpha J_\alpha, 2)$ is homeomorphic to $\Pi_\alpha C_p(J_\alpha, 2)$. Since the interior of every J_α meets X', J_α is not discrete, whence $C_p(J_\alpha, 2)$ is not compact. Thus, $C_p(\bigcup_\alpha J_\alpha, 2)$ is not Lindelöf since it is homeomorphic to the product of uncountably many non-compact spaces. □

Question 3.3. *Let X be a first-countable subspace of an ordinal and let the derived set X' have countable extent. Is $C_p(X,2)$ Lindelöf?*

Although, first countable subspaces of ordinals are mainly non-metrizable (some of course are metrizable), they all share one property that makes them a little closer to metric spaces. Every first countable subspace of ordinals have a base of countable order. This concept was introduced by Arhangelskii in [AR2]: a base $\mathcal{B}$ for the topology of X is a *base of countable order* if any sequence $B_1 \supset B_2 \supset \dots$ of distinct members of $\mathcal{B}$, all of which contain a point x, forms a local base at x. Let us show that a first countable subspace X of an ordinal has a base of countable order. For each $x \in X$, fix a countable nested local base $\mathcal{B}_x$ at x whose elements are in form $[\alpha, x]_X$. Let us show that $\mathcal{B} = \bigcup_{x\in X} \mathcal{B}_x$ has the required property. Let $[\alpha_1, x_1]_X \supset [\alpha_2, x_2]_X \supset \dots$ be a sequence of distinct members of $\mathcal{B}$, all of which contain x. Since no sequence of ordinals is strictly decreasing, there exists $y \in X$ such that $x_n = y$ for almost all n. Then $[\alpha_n, x_n] \in \mathcal{B}_y$ for almost all n. Hence, the sequence in question forms a local base at y. Since each member contains x, we have $x = y$.

Question 3.4. *Let X be a countably compact GO-space (or LOTS) with Lindelöf $C_p(X)$. Does X have a base of countable order?*

Question 3.5. *Let X be a GO-space (or LOTS) with a base of countable order and countable extent. Is $C_p(X)$ Lindelöf? What if X is countably compact?*

Surprisingly, it seems to be an open question if Nahmanson's theorem can be generalized to Lindelöf LOTS.

Question 3.6. *Let X be a Lindelöf LOTS (or GO-space) with Lindelöf $C_p(X)$. Is X metrizable?*

And let us finish with a question which is an unaccomplished goal of this paper.

Question 3.7. *Let X be first countable, of countable extent, and a subspace of an ordinal. Is $C_p(X)$ Lindelöf?*

References

[AR1] Arhangelskii A., *Topological Function Spaces*, Math. Appl., vol. 78, Kluwer Academic Publisher, Dordrecht, 1992.

[AR2] Arhangelskii A., *Some Metrization Theorems* (Russian), Uspehi Mat. Nauk **18** (1963), no. 5 (113), 139–145.

[ASA] Asanov M.O., *On cardinal invariants of function spaces*, Modern Topology and Set Theory, Igevsk, (2), 1979, pp. 8–12.

[BUZ] Buzyakova R.Z., *In search for Lindelöf C_p's*, Comment. Math. Univ. Carolin. **45** (2004), no. 1, 145–151.

[ENG] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann, Berlin, revised ed., 1989.

[NAH] Nahmanson L.B., *Lindelöfness in function spaces*, Fifth Teraspol Simposium on Topology and its Applications, Kishinev, 1985, 183.

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