Regularity of weak solutions to certain degenerate elliptic equations

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Abstract. In this article we establish the existence of higher order weak derivatives of weak solutions of Dirichlet problem for a class of degenerate elliptic equations.

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1. Introduction

In this paper we study the existence of higher order weak derivatives (see Theorem 3.8) of weak solutions of degenerate elliptic equations $Lu = g - \operatorname{div}\vec{f}$, where L is an elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu)(x) - \sum_{i=1}^{n} b_i(x)D_iu(x) \tag{1.1}$$

whose coefficients a_{ij} and b_i are measurable, real-valued functions, and whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$\lambda\omega(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda\omega(x)|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and almost all $x \in \Omega\subset\mathbb{R}^n$ on a bounded open set Ω, ω is a weight function, λ and Λ are positive constants.

2. Definitions and basic results

By a *weight* we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus $\omega(E) = \int_E \omega\, dx$ for measurable sets $E\subset\mathbb{R}^n$.

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Definition 2.1. Let $\Omega\subset\mathbb{R}^n$ be open and let ω be a weight. For $1 < p < \infty$, we define $L^p(\Omega,\omega)$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\, dx\right)^{1/p} < \infty.$$

Definition 2.2. Let $1 \le p < \infty$. A weight ω belongs to the Muckenhoupt class A_p if there is a constant $\mathbf{C} = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega\, dx\right)\left(\frac{1}{|B|}\int_B \omega^{-1/(p-1)}\, dx\right)^{p-1} \le \mathbf{C} \quad (\text{if } \ 1 < p < \infty)$$
$$\left(\frac{1}{|B|}\int_B \omega\, dx\right)\left(\operatorname*{ess\,sup}_B \frac{1}{\omega}\right) \le \mathbf{C} \quad (\text{if } \ p = 1),$$

for every ball $B\subset\mathbb{R}^n$, where $|B|$ is the n-dimensional Lebesgue measure of B. The infimum over all constants $\mathbf{C}$ is called " A_p-constant of ω".

Example 2.3. The function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is a weight A_p if and only if $-n < \alpha < n(p-1)$ (see [6, Chapter 15]).

Remark 2.4. If $\omega \in A_p$, $1 \le p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable when $p > 1$, and $1/\omega$ is locally bounded, when $p = 1$, we have $L^p(\Omega,\omega)\subset L^1_{\text{loc}}(\Omega)$ for every open set Ω. If Ω is bounded, one obtains in the same way that $L^p(\Omega,\omega) \subset L^1(\Omega)$. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$. □

Definition 2.5. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set, $1 \le p < \infty$ and k be a nonnegative integer. Suppose that the weight $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$ for $|\alpha| \le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is given by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\sum_{0\le|\alpha|\le k}\int_\Omega |D^\alpha u|^p \omega\, dx\right)^{1/p}.$$

We also define $W_0^{k,p}(\Omega,\omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega,\omega)$.
If $\Omega \subset \mathbb{R}^n$ is open, $k \ge 1$, $1 \le p < \infty$ and $\omega \in A_p$ then $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega,\omega)$ (see Corollary 2.1.6 in [8]). The spaces $W^{k,p}(\Omega,\omega)$ are Banach spaces.

In this paper we use frequently the following two theorems.

Theorem 2.6 (Muckenhoupt Theorem). *Let ω be a weight in $\mathbb{R}^n$ and*

$$[M(f)](x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$$

be the Hardy-Littlewood maximal function. Then for $p > 1$, $M : L^p(\mathbb{R}^n, \omega) \longrightarrow L^p(\mathbb{R}^n, \omega)$ is continuous (that is, $\|Mf\|_{L^p(\mathbb{R}^n,\omega)} \leq C_M \|f\|_{L^p(\mathbb{R}^n,\omega)}$) if and only if $\omega \in A_p$. The constant C_M is called Muckenhoupt constant and C_M depends only on n, p and the A_p-constant of ω.

Proof: See [7] or [4, Corollary 4.3]. □

Theorem 2.7 (Weighted Sobolev inequality). *Let Ω be a bounded open set in $\mathbb{R}^n$, $1 < p < \infty$ and $\omega \in A_p$. Then there exist constants C_Ω and δ positive such that for all $\varphi \in C_0^\infty(\Omega)$ and k satisfying $1 \leq k \leq \frac{n}{n-1} + \delta$,*

$$\|\varphi\|_{L^{kp}(\Omega,\omega)} \leq C_\Omega \| |\nabla \varphi| \|_{L^p(\Omega,\omega)}$$

where C_Ω may be taken to depend only on n, the A_p constant of ω, p and the diameter of Ω.

Proof: See Theorem 1.3 of [2]. □

Definition 2.8. We say that $u \in W^{1,2}(\Omega, \omega)$ is a weak solution of the equation

$$Lu = g - \sum_{i=1}^n D_i f_i, \quad \text{with } \frac{g}{\omega}, \frac{f_i}{\omega} \in L^2(\Omega, \omega)$$

if

$$\mathcal{B}(u, \varphi) = \sum_{i=1}^n \int_\Omega f_i D_i \varphi + \int_\Omega g \varphi \, dx, \quad \forall\, \varphi \in W_0^{1,2}(\Omega, \omega)$$

where

$$\mathcal{B}(u, \varphi) = \int_\Omega \Big[\sum_{i,j=1}^n a_{ij} D_i u D_j \varphi - \sum_{i=1}^n b_i \varphi D_i u \Big] dx.$$

Theorem 2.9. *Let L be the operator (1.1) satisfying (1.2) and $|b_i(x)| \leq C_1 \omega(x)$ in Ω. Assume that $\psi \in W^{1,2}(\Omega, \omega)$, $g/\omega \in L^2(\Omega, \omega)$, $f_i/\omega \in L^2(\Omega, \omega)$ and $\omega \in A_2$. Then the Dirichlet problem*

$$Lu = g - \sum_{i=1}^n D_i f_i$$

$$u - \psi \in W_0^{1,2}(\Omega, \omega)$$

has a unique solution $u \in W^{1,2}(\Omega, \omega)$ and

$$\|u\|_{W^{1,2}(\Omega,\omega)} \leq C \left(\|g/\omega\|_{L^2(\Omega,\omega)} + \|f_j/\omega\|_{L^2(\Omega,\omega)} + \|\psi\|_{W^{1,2}(\Omega,\omega)} \right).$$

Proof: It is consequence of the Lax-Milgram Theorem and the proof follows the lines of Theorem 2.2. of [2]. □

3. Differentiability of weak solutions

In this section we prove that weak solutions $u \in W^{1,2}(\Omega, \omega)$ of the equation $Lu = g$ are twice weakly differentiable and $D_{ij}u \in L^2(\Omega', \omega)$ (that is, $u \in W^{2,2}(\Omega', \omega)$, $\forall\, \Omega' \subset\subset \Omega$).

Definition 3.1. Let u be a function on a bounded open set $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient of u at x in the direction e_i by

$$\Delta_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}, \quad (0 < |h| < \operatorname{dist}(x, \partial\Omega)). \tag{3.1}$$

Lemma 3.2. *Let $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$. If $u, v \in L^2_{\mathrm{loc}}(\Omega, \omega)$, $\operatorname{supp}(v) \subset \Omega'$ and g is a measurable function with $|g(x)| \le C\omega(x)$, then*

- (a) $\Delta_k^h(uv)(x) = u(x + he_k)\Delta_k^h v(x) + v(x)\Delta_k^h u(x)$, *with* $1 \le k \le n$;
- (b) $\int_\Omega g(x)u(x)\Delta_k^{-h} v(x)\, dx = -\int_\Omega v(x)\Delta_k^h (gu)(x)\, dx$;
- (c) $\Delta_k^h(D_j v)(x) = D_j(\Delta_k^h v)(x)$.

Proof: The proof of this lemma follows trivially from Definition 3.1. □

Definition 3.3. Let ω be a weight in $\mathbb{R}^n$. We say that ω is uniformly A_p in each coordinate if

- (a) $\omega \in A_p(\mathbb{R}^n)$;
- (b) $\omega_i(t) = \omega(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$ is in $A_p(\mathbb{R})$, for $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ a.e., $1 \le i \le n$, with A_p constant of ω_i bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

Example 3.4. Let $\omega(x, y) = \omega_1(x)\omega_2(y)$, with $\omega_1(x) = |x|^{1/2}$ and $\omega_2(y) = |y|^{1/2}$. We have that ω is uniformly A_2 in each coordinate.

Lemma 3.5. *Let $u \in W^{1,p}(\Omega, \omega)$, $p > 1$, and let ω be a weight uniformly A_p in each coordinate. Then for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, we have*

$$\|\Delta_k^h u\|_{L^p(\Omega', \omega)} \le C \|D_k u\|_{L^p(\Omega, \omega)} \tag{3.2}$$

where $C = 2C_M$, and C_M is the Muckenhoupt constant.

Proof: Case 1. Let us suppose initially that $u \in C^\infty(\Omega)$. We have,

$$\begin{aligned} \Delta_k^h u(x) &= \frac{u(x + he_k) - u(x)}{h} = \frac{1}{h}\int_0^h D_k(x + \zeta e_k)\, d\zeta \\ &= \frac{1}{h}\int_0^h D_k u(x_1, \dots, x_{k-1}, x_k + \zeta, x_{k+1}, \dots, x_n)\, d\zeta. \end{aligned}$$

For $1 \le k \le n$, we define the functions

$$G_k(x) = \begin{cases} D_k u(x), & \text{if } \ x \in \Omega \\ 0, & \text{if } \ x \notin \Omega. \end{cases}$$

We have for $x \in \Omega' \subset\subset \Omega$ and h satisfying $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$,

$$\begin{aligned} |\Delta_k^h u(x)| &\le \frac{1}{|h|}\left| \int_0^h |D_k u(x_1, \ldots, x_{k-1}, x_k + \zeta, x_{k+1}, \ldots, x_n)| \, d\zeta \right| \\ &= \frac{1}{|h|}\left| \int_{x_k}^{x_k+h} |D_k u(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)| \, dt \right| \\ &= \frac{1}{|h|}\left| \int_{x_k}^{x_k+h} |G_k(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)| \, dt \right| \\ &\le \frac{1}{|h|}\left| \int_{x_k-h}^{x_k+h} |G_k(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_n)| \, dt \right| \\ &\le \sup_{h\neq 0} \ \frac{1}{|h|}\left| \int_{x_k-h}^{x_k+h} |G_k(x_1, \ldots x_{k-1}, t, x_{k+1}, \ldots, x_n)| \, dt \right| \\ &\le 2M(G_k^{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n})(\mathbf{x_k}), \end{aligned}$$

where $G_k^{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n}(\mathbf{x_k}) = G_k(x_1, \ldots, \mathbf{x_k}, \ldots, x_n)$. Consequently, using the notation $\widehat{dx_k} = dx_1 \ldots dx_{k-1} dx_{k+1} \ldots dx_n$ (where the hat indicates the term that must be omitted in the product) and by Theorem 2.6, we obtain

$$\begin{aligned} &\int_{\Omega'} |\Delta_k^h u(x)|^p \omega(x) \, dx \\ &\le 2^p \int_{\Omega'} [M(G_k^{x_1,\ldots,x_{k-1}x_{k+1},\ldots,x_n})]^p(\mathbf{x_k})\omega(x_1, \ldots, x_k, \ldots, x_n) \, dx \\ &\le 2^p \int_{\mathbb{R}^n} [M(G_k^{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n})]^p(\mathbf{x_k})\omega(x_1, \ldots, x_k, \ldots, x_n) \, dx_1 \ldots dx_k \ldots dx_n \\ &= 2^p \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} [M(G_k^{x_1,\ldots,x_{k-1}x_{k+1},\ldots,x_n})]^p(\mathbf{x_k})\omega(x_1, \ldots, \mathbf{x_k}, \ldots x_n) dx_k \right) \widehat{dx_k} \\ &\le 2^p \int_{\mathbb{R}^{n-1}} \left(C_M^p \int_{\mathbb{R}} |G_k^{x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n}(\mathbf{x_k})|^p \omega(x_1, \ldots, \mathbf{x_k}, \ldots, x_n) \, dx_k \right) \widehat{dx_k} \\ &= 2^p C_M^p \int_{\mathbb{R}^n} |G_k(x)|^p \omega(x) \, dx \\ &= 2^p C_M^p \int_{\Omega} |D_k u(x)|^p \omega(x) \, dx, \end{aligned}$$

where C_M is independent of $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n$ because ω is uniformly A_p in each coordinate. Therefore

$$\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \le C\|D_k u\|_{L^p(\Omega,\omega)}, \quad \text{where} \ \ C = 2C_M.$$

Case 2. If $u \in W^{1,p}(\Omega,\omega)$ then there exists a sequence $\{u_m\}$, $u_m \in C^\infty(\Omega)$, Cauchy sequence in the norm $\|.\|_{W^{1,p}(\Omega,\omega)}$. By Definition 2.5, we have that

$$u_m \longrightarrow u \text{ and } D_k u_m \longrightarrow D_k u \text{ in } L^p(\Omega,\omega).$$

Consequently, since $\omega \in A_p$, there exists a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \longrightarrow u$ a.e. This implies, for $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, that

$$\Delta_k^h u_{m_j} \longrightarrow \Delta_k^h u \text{ a.e.}$$

We have that $\{\Delta_k^h u_{m_j}\}$ is a Cauchy sequence in $L^p(\Omega',\omega)$, for any $\Omega' \subset\subset \Omega$. In fact, using the first case, we have

$$\begin{aligned}
\|\Delta_k^h u_{m_r} - \Delta_k^h u_{m_s}\|_{L^p(\Omega',\omega)} &= \|\Delta_k^h (u_{m_r} - u_{m_s})\|_{L^p(\Omega',\omega)} \\
&\leq C\|D_k(u_{m_r} - u_{m_s})\|_{L^p(\Omega,\omega)} \\
&= C\|D_k u_{m_r} - D_k u_{m_s}\|_{L^p(\Omega,\omega)} \\
&\longrightarrow 0, \quad \text{as } m_r, m_s \longrightarrow \infty.
\end{aligned}$$

Therefore, there exists $g \in L^p(\Omega',\omega)$ such that $\Delta_k^h u_{m_j} \longrightarrow g$ in $L^p(\Omega',\omega)$. Consequently, there exists a subsequence $\Delta_k^h u_{m_{j_r}} \longrightarrow g$ a.e. We can conclude that $\Delta_k^h u = g$ a.e. Hence

$$\Delta_k^h u_{m_j} \longrightarrow \Delta_k^h u \text{ in } L^p(\Omega',\omega).$$

This implies that

$$\begin{aligned}
\|\Delta_k^h u\|_{L^p(\Omega',\omega)} &= \lim_{m_j \to \infty} \|\Delta_k^h u_{m_j}\|_{L^p(\Omega',\omega)} \\
&\leq C \lim_{m_j \to \infty} \|D_k u_{m_j}\|_{L^p(\Omega,\omega)} \\
&= C\|D_k u\|_{L^p(\Omega,\omega)},
\end{aligned}$$

that is, $\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \leq C\|D_k u\|_{L^p(\Omega,\omega)}$. □

Lemma 3.6. *Let $u \in L^p(\Omega,\omega)$, $1 < p < \infty$, $\omega \in A_p$ and suppose there exists a constant C such that*

$$\|\Delta_k^h u\|_{L^p(\Omega',\omega)} \leq C, \ k = 1, 2, \ldots, n \tag{3.3}$$

for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$ (with C independent of h). Then there exists $v \in L^p(\Omega,\omega)$ such that $D_k u = v$ in the weak sense, that is, $u \in W^{1,p}(\Omega,\omega)$ and $\|D_k u\|_{L^p(\Omega,\omega)} \leq C$.

Proof: The proof of this lemma follows the lines of Lemma 7.24 in [5]. □

Remark 3.7. We use the notation

$$\mathcal{D}^k(\Omega,\omega)=\left\{g\in W^k(\Omega)\ :\ \frac{D^\alpha g}{\omega}\in L^2(\Omega,\omega),\ 0\le|\alpha|\le k\right\},\quad\text{for}\quad k=0,1,2\dots,$$

where $W^k(\Omega)$ denotes the linear space of k times weakly derivative functions. For $k=0$, we have $g\in\mathcal{D}^0(\Omega,\omega)$ if $g/\omega\in L^2(\Omega,\omega)$. □

We are able now to prove the main result of this paper.

Theorem 3.8. *Let $u\in W^{1,2}(\Omega,\omega)$ be a weak solution of the equation $Lu=g$ in Ω, and assume that*

(a) $g\in\mathcal{D}^0(\Omega,\omega)$;
(b) *ω is a weight uniformly A_2 in each coordinate;*
(c) $|\Delta_k^h a_{ij}(x)|\le C_1\omega(x)$, $x\in\Omega'\subset\subset\Omega$ *a.e.*, $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$, *with a constant C_1 independent of Ω' and h;*
(d) $|b(x)|\le C\omega(x)$ *a.e. in Ω, where $b=(b_1,\dots,b_n)$.*

Then for any subdomain $\Omega'\subset\subset\Omega$ we have $u\in W^{2,2}(\Omega',\omega)$ and

$$\|u\|_{W^{2,2}(\Omega',\omega)}\le\mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega)}+\|g/\omega\|_{L^2(\Omega,\omega)}\right)\tag{3.4}$$

for $\mathbf{C}=\mathbf{C}(n,C_M,\lambda,\Lambda,C_1,d')$, and $d'=\operatorname{dist}(\Omega',\partial\Omega)$.

Proof: Since $u\in W^{1,2}(\Omega,\omega)$ is a weak solution of the equation $Lu=g$, we have by

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^n a_{ij}(x)D_iu(x)D_jv(x)\,dx-\int_\Omega\sum_{i=1}^n b_i(x)D_iu(x)v(x)\,dx\\&=\int_\Omega g(x)v(x)\,dx\end{aligned}\tag{3.5}$$

for all $v\in W_0^{1,2}(\Omega,\omega)$ (in particular for $v\in C_0^\infty(\Omega)$). Hence

$$\begin{aligned}&\int_\Omega\sum_{i,j=1}^n a_{ij}(x)D_iu(x)D_jv(x)\,dx\\&=\int_\Omega[g(x)v(x)+\sum_{i=1}^n b_i(x)D_iu(x)v(x)]\,dx.\end{aligned}$$

In (3.5) let us replace v by $\Delta_k^{-h}v$ ($1\le k\le n$), with $v\in C_0^\infty(\Omega)$, $\operatorname{supp}v\subset\subset\Omega$ and let $|2h| < \operatorname{dist}(\operatorname{supp} v, \partial\Omega)$. We then obtain

$$\begin{aligned}
&-\left(\int_\Omega [g(x)+b_i(x)D_iu(x)]\Delta_k^{-h}v(x)\,dx\right)\\
&= -\int_\Omega a_{ij}(x)D_iu(x)D_j(\Delta_k^{-h}v(x))\,dx\\
&= -\int_\Omega a_{ij}(x)D_iu(x)\Delta_k^{-h}D_jv(x),dx \qquad \text{(by Lemma 3.2(b))}\\
&= \int_\Omega \Delta_k^h(a_{ij}D_iu)(x)D_jv(x)\,dx \qquad \text{(by Lemma 3.2(a))}\\
&= \int_\Omega \Big(a_{ij}(x+he_k)\Delta_k^hD_iu(x)+D_iu(x)\Delta_k^ha_{ij}(x)\Big)D_jv(x)\,dx\\
&= \int_\Omega \Big([h\Delta_k^ha_{ij}(x)+a_{ij}(x)]\Delta_k^hD_iu(x)+D_iu(x)\Delta_k^ha_{ij}(x)\Big)D_jv(x)\,dx.
\end{aligned}$$

Consequently,

(3.6)

$$\begin{aligned}
&\int_\Omega a_{ij}(x)D_jv(x)\Delta_k^hD_iu(x)\,dx = -\Bigg(\int_\Omega [g(x)+b_i(x)D_iu(x)]\Delta_k^{-h}v(x)\,dx\\
&\quad+\int_\Omega \Delta_k^ha_{ij}(x)D_iu(x)D_jv(x)\,dx+\int_\Omega h\Delta_k^ha_{ij}(x)\Delta_k^hD_iu(x)D_jv(x)\,dx\Bigg)\\
&\le \int_\Omega |g(x)+b_i(x)D_iu(x)||\Delta_k^{-h}v(x)|\,dx+\int_\Omega |\Delta_k^ha_{ij}(x)||D_iu(x)||D_jv(x)|\,dx\\
&\quad+|h|\int_\Omega |\Delta_k^ha_{ij}(x)||\Delta_k^hD_iu(x)||D_jv(x)|\,dx\\
&= \mathrm{I}+\mathrm{II}+C_1|h|\int_\Omega \omega(x)|\Delta_k^hD_iu(x)||D_jv(x)|\,dx.
\end{aligned}$$

Let us estimate the integrals I and II. Considering $f = g + b_iD_iu$, by (a) and (d), we have

$$\begin{aligned}
\mathrm{I} &= \int_\Omega |f||\Delta_k^{-h}v|\,dx\\
&= \int_\Omega \left(\frac{|f|}{\omega}\right)\omega^{1/2}|\Delta_k^{-h}v|\omega^{1/2}\,dx\\
&\le \left(\int_\Omega\left(\frac{f}{\omega}\right)^2\omega\,dx\right)^{1/2}\left(\int_{\operatorname{supp}(v)}|\Delta_k^{-h}v|^2\omega\,dx\right)^{1/2}\\
&\le C_M\|f/\omega\|_{L^2(\Omega,\omega)}\left(\int_\Omega |D_kv|^2\omega\,dx\right)^{1/2} \quad \text{(by Lemma 3.5)}\\
&= C_M\Big(\|g/\omega\|_{L^2(\Omega,\omega)}+C_1\|u\|_{W^{1,2}(\Omega,\omega)}\Big)\|D_kv\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

$$
\begin{aligned}
\mathrm{II} &= \int_\Omega |\Delta_k^h a_{ij}||D_i u||D_j v|\,dx \le \int_\Omega C_1\omega|D_i u||D_j v|\,dx\\
&= C_1\int_\Omega |D_i u|\omega^{1/2}|D_j v|\omega^{1/2}\,dx\\
&\le C_1\Big(\int_\Omega |D_i u|^2\omega\,dx\Big)^{1/2}\Big(\int_\Omega |D_j v|^2\omega dx\Big)^{1/2}\\
&\le C_1\|u\|_{W^{1,2}(\Omega,\omega)}\|D_j v\|_{L^2(\Omega,\omega)}.
\end{aligned}
$$

Replacing the estimates of I and II in (3.6), we get the estimate

$$
\begin{aligned}
&\int_\Omega a_{ij}(x)\Delta_k^h D_i u(x) D_j v(x)\,dx\\
&\le C\Big(\|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}\Big)\|Dv\|_{L^2(\Omega,\omega)}\\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||D_j v(x)|\,dx.
\end{aligned}
\tag{3.7}
$$

We denote by $a = \|u\|_{W^{1,2}(\Omega,\omega)} + \|g/\omega\|_{L^2(\Omega,\omega)}$.
Let $\Omega' \subset\subset \Omega$. To proceed further let us take a function $\psi \in C_0^\infty(\Omega)$, satisfying $0 \le \psi \le 1$, $\psi \equiv 1$ in Ω' and with $\|D\psi\|_\infty \le 2/d'$, where $d' = \operatorname{dist}(\Omega', \partial\Omega)$, and set $v = \psi^2\Delta_k^h u$ (with $|2h| < \operatorname{dist}(\operatorname{supp}\psi, \partial\Omega)$). We have

$$D_j v = (2\psi D_j\psi)\Delta_k^h u + \psi^2 D_j(\Delta_k^h u).$$

Then we obtain

$$
\begin{aligned}
&\int_\Omega \Big(a_{ij}(x)\psi^2 D_j(\Delta_k^h u)D_i(\Delta_k^h u) + 2a_{ij}(x)\psi D_j\psi\Delta_k^h u\Delta_k^h D_i u\Big)\,dx\\
&\le Ca\|2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx\\
&\le Ca\Big(\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\Big)\\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_i(\Delta_k^h u)][\psi D_j(\Delta_k^h u)]\,dx\\
&\le Ca\Big(\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\Big)\\
&\quad + 2\int_\Omega |a_{ij}(x)||\psi D_i\Delta_k^h u||D_j\psi\Delta_k^h u|\,dx\\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned}
$$

By (1.2) we have $|a_{ij}(x)| \leq C\omega(x)$, and we can estimate the integral on the right hand side by

$$\begin{aligned}
&\int_\Omega |a_{ij}(x)||\psi D_i(\Delta_k^h u)||D_j\psi\Delta_k^h u|\,dx \\
&\leq C\int_\Omega |\psi D_i(\Delta_k^h u)||D_j\psi\Delta_k^h u|\omega\,dx \\
&\leq C\Big(\int_\Omega |\psi D_i(\Delta_k^h u)|^2\omega\,dx\Big)^{1/2}\Big(\int_\Omega |D_j\psi\Delta_k^h u|^2\omega\,dx\Big)^{1/2} \\
&= C\|\psi D_i(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_j(\Delta_k^h u)][\psi D_i(\Delta_k^h u)]\,dx \\
&\leq Ca\|\psi D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} \\
&\quad + Ca\|\psi^2 D_j(\Delta_k^h u)\|_{L^2(\Omega,\omega)} \\
&\quad + 2C\|\psi D_i(\Delta_k^h u)\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} \\
&\quad + C_1|h|\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx.
\end{aligned} \tag{3.8}$$

Finally, the integral on the right hand side in (3.8) can be estimated

$$\begin{aligned}
&\int_\Omega \omega(x)|\Delta_k^h D_i u(x)||2\psi D_j\psi\Delta_k^h u + \psi^2 D_j(\Delta_k^h u)|\,dx \\
&\leq \int_\Omega 2\omega(x)|\Delta_k^h D_i u(x)||\psi D_j\psi\Delta_k^h u| + \int_\Omega \omega(x)|\Delta_k^h D_i u||\psi^2 D_j(\Delta_k^h u)|\,dx \\
&= 2\int_\Omega \omega(x)|\psi\Delta_k^h D_i u||D_j\psi\Delta_k^h u|\,dx + \int_\Omega \omega(x)|\psi\Delta_k^h D_i u||\psi D_j(\Delta_k^h u)|\,dx \\
&\leq 2\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} \\
&\quad + \|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|\psi\Delta_k^h D_j u\|_{L^2(\Omega,\omega)}.
\end{aligned}$$

Applying this result in (3.8), we obtain

$$\begin{aligned}
&\int_\Omega a_{ij}(x)[\psi D_j\Delta_k^h u][\psi D_i\Delta_k^h u]\,dx \\
&\leq Ca\|\psi D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} \\
&\quad + Ca\|\psi^2 D_j\Delta_k^h u\|_{L^2(\Omega,\omega)}
\end{aligned}$$

$$+ 2C\|\psi D_i\Delta_k^h u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}$$
$$+ 2C_1|h|\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}$$
$$+ C_1|h|\|\psi\Delta_k^h D_i u\|_{L^2(\Omega,\omega)}\|\psi\Delta_k^h D_j u\|_{L^2(\Omega,\omega)}.$$

Consequently, by condition (1.2), we then have

$$\int_\Omega a_{ij}(x)[\psi D_j(\Delta_k^h u)][\psi D_i(\Delta_k^h u)]\,dx \geq \lambda\int_\Omega |\psi D(\Delta_k^h u)|^2\omega\,dx.$$

Denoting $b = \|\psi D(\Delta_k^h u)\|_{L^2(\Omega,\omega)}$, we have

$$\begin{aligned}\lambda b^2 \leq{}& Ca\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + Cab + 2Cb\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}\\ &+ 2C_1|h|b\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} + C_1|h|b^2.\end{aligned}$$

Using the Young's inequality

$$AB = (\varepsilon^{-1}A)(\varepsilon B) \leq \frac{1}{2}[(\varepsilon^{-1}A)^2 + (\varepsilon B)^2], \quad \forall\,\varepsilon > 0$$

to estimate ab and $b\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)}$, we obtain

$$\begin{aligned}\lambda b^2 \leq{}& Ca\ \|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + \frac{C}{2}\varepsilon^{-2}a^2 + \frac{C}{2}\varepsilon^2 b^2\\ &+ 2C\varepsilon^2 b^2 + C\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\Omega,\omega)}\\ &+ 2C_1|h|b\|D_j\psi\Delta_k^h u\|_{L^2(\Omega,\omega)} + C_1|h|b^2\\ \leq{}& Ca\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)} + \frac{C\varepsilon^{-2}}{2}a^2 + \frac{C\varepsilon^2}{2}b^2\\ &+ 2C^2\varepsilon^2 b^2 + C\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)}\\ &+ C_1\varepsilon^2|h|^2 b^2 + C_1\frac{\varepsilon^{-2}}{2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)} + C_1|h|b^2.\end{aligned}$$

Choose $\varepsilon > 0$ and h such that

$$\frac{C}{2}\varepsilon^2 + 2C\varepsilon^2 \leq \lambda/4 \quad \text{and} \quad |h| < \lambda/8C_1.$$

Then

$$\left(\frac{C}{2}\varepsilon^2 + 2C\varepsilon^2 + C_1|h|^2 + C_1|h|\right) \leq \frac{\lambda}{2}$$

and we can use Lemma 3.5 to get

$$\begin{aligned}\lambda b^2 &\le Ca\|D_j\psi\Delta_k^h u\|_{L^2(\operatorname{supp}\psi,\omega)}\\&\quad+\frac{C}{2}\varepsilon^{-2}a^2+\frac{\lambda}{2}b^2+C\varepsilon^{-2}\|D_j\psi\Delta_k^h u\|^2_{L^2(\operatorname{supp}\psi,\omega)}\\&\le Ca\|D_j\psi\|_\infty\|\Delta_k^h u\|_{L^2(\operatorname{supp}v,\omega)}+\frac{C}{2}\varepsilon^{-2}a^2\\&\quad+\frac{\lambda}{2}b^2+C\varepsilon^{-2}\|D_j\psi\|_\infty^2\|\Delta_k^h u\|^2_{L^2(\operatorname{supp}v,\omega)}\\&\le Ca\|D_j\psi\|_\infty\|D_k u\|_{L^2(\Omega,\omega)}+\frac{C}{2}\varepsilon^{-2}a^2+\frac{\lambda}{2}b^2\\&\quad+C\varepsilon^{-2}\|D_j\psi\|_\infty^2\|D_k u\|^2_{L^2(\Omega,\omega)}.\end{aligned}$$

Since $\|D_k u\|_{L^2(\Omega,\omega)}\le a$, we have

$$\begin{aligned}\frac{\lambda}{2}b^2 &\le C\|D_j\psi\|_\infty a^2+\frac{C}{2}\varepsilon^{-2}a^2+C\varepsilon^{-2}\|D_j\psi\|_\infty^2 a^2\\&\le\left(C\|D_j\psi\|_\infty+\frac{C}{2}\varepsilon^{-2}+C\varepsilon^{-2}\|D_j\psi\|_\infty^2\right)a^2\\&=Ca^2.\end{aligned}$$

Consequently, we obtain

$$b\le\left(\frac{2C}{\lambda}\right)^{1/2}a.$$

Using $\psi\equiv 1$ in Ω', we conclude that

$$\|\Delta_k^h(Du)\|_{L^2(\Omega',\omega)}\le\mathbf{C}a,\ \forall\,k,\ 1\le k\le n,\forall\,\Omega'\subset\subset\Omega$$

with $0<|h|<\operatorname{dist}(\Omega',\partial\Omega)$ and $h<\lambda/8C_1$. By Lemma 3.6 we obtain $Du\in W^{1,2}(\Omega',\omega)$. Therefore we have that $u\in W^{2,2}(\Omega',\omega)$ and

$$\|u\|_{W^{2,2}(\Omega',\omega)}\le\mathbf{C}a=\mathbf{C}\left(\|u\|_{W^{1,2}(\Omega,\omega)}+\|g/\omega\|_{L^2(\Omega,\omega)}\right).$$

□

By a straightforward induction argument, we can then conclude the following extension of Theorem 3.8.

Theorem 3.9. *Let $u\in W^{1,2}(\Omega,\omega)$ be a weak solution of the equation $Lu=g$ in Ω, and assume that*

(a) *ω is a weight uniformly A_2 in each coordinate;*

(b) $g \in \mathcal{D}^k(\Omega, \omega)$, $k \in \mathbb{N}$, $k \geq 1$;
(c) *there exist* $D^\alpha a_{ij}$ *a.e. and* $|\Delta_p^h D^\alpha a_{ij}(x)| \leq C_1\omega(x)$, $x \in \Omega' \subset\subset \Omega$, $0 \leq |\alpha| \leq k$, $1 \leq p \leq n$, $0 < |h| < \operatorname{dist}(\Omega', \partial\Omega)$, *with constant* C_1 *independent of* Ω' *and* h;
(d) *there exist* $D^\alpha b_i$ *a.e.,* $0 \leq |\alpha| \leq k-1$, *and* $|D^\alpha b_i(x)| \leq C_2\omega(x)$, $x \in \Omega' \subset\subset \Omega$.

Then for any subdomain $\Omega' \subset\subset \Omega$, *we have* $u \in W^{k+2,2}(\Omega', \omega)$ *and*

$$\|u\|_{W^{k+2,2}(\Omega',\omega)} \leq \mathbf{C}\bigg(\|u\|_{W^{1,2}(\Omega,\omega)} + \sum_{0\leq|\alpha|\leq k} \|D^\alpha g/\omega\|_{L^2(\Omega,\omega)}\bigg)$$

for $\mathbf{C} = \mathbf{C}(n, \lambda, \Lambda, C_M, C_1, C_2, d')$, *and* $d' = \operatorname{dist}(\Omega', \partial\Omega)$.

References

[1] Fabes E., Jerison D., Kenig C., *The Wiener test for degenerate elliptic equations*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 3, 151–182.
[2] Fabes E., Kenig C., Serapioni R., *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.
[3] Franchi B., Serapioni R., *Pointwise estimates for a class of strongly degenerate elliptic operators: a geometrical approach*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), no. 4, 527–568.
[4] Garcia-Cuerva J., Rubio de Francia J., *Weighted Norm Inequalities and Related Topics*, North-Holland Mathematics Studies 116, North-Holland, Amsterdam, 1985.
[5] Gilbarg D., Trudinger N., *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin-New York, 1977.
[6] Heinonen J., Kilpeläinen T., Martio O., *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford University Press, New York, 1993.
[7] Muckenhoupt B., *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[8] Turesson B.O., *Nonlinear potential theory and weighted Sobolev spaces*, Lecture Notes in Math. **1736**, Springer, Berlin, 2000.

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