Comment.Math.Univ.Carolin. 47,4 (2006)549–560

On large selforthogonal modules

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Dedicated to Claus Michael Ringel on the occasion of his 60^{th} birthday.

Abstract. We construct non faithful direct summands of tilting (resp. cotilting) modules large enough to inherit a functorial tilting (resp. cotilting) behaviour.

Keywords: partial tilting and partial cotilting modules, sincere and selforthogonal modules

Classification: 16E10, 16G70

Introduction

The aim of this note is to describe and explain the presence, in some sense the omnipresence, of very special large selforthogonal modules. Roughly speaking, the modules considered in the sequel are quite close to tilting and cotilting modules from the functorial point of view, but not necessarily faithful. More precisely, they satisfy a word-for-word generalization of one of the equivalent definitions of tilting (resp. cotilting) modules of projective (resp. injective) dimension ≤ 1. As in many definitions of tilting/cotilting objects, local and global conditions seem to be quite different. First of all, the local properties satisfied by our modules are inherited by their direct summands. Secondly, the only property of our modules which seems to be of global type is a functorial $\mathrm{Hom}-\mathrm{Ext}$ property, concerning the kernels of certain functors, namely their intersection. However, we may obtain the same intersection by dealing with the Hom and Ext functors associated to a big faithful module (for instance an injective cogenerator, or a projective generator) and one of its sincere summands of the smallest possible "size" (Examples 4 and 7).

This paper is organized as follows. In Section 1 we fix the notation and the conventions used in the sequel. In Section 2 we collect some lemmas on "superfluous" summands with respect to our functorial $\mathrm{Hom}-\mathrm{Ext}$ property. Finally, in Section 3, we show that any natural number $n \geq 2$ occurs as the projective (resp. injective) dimension of an injective (resp. projective) A-module M, where A is a finite dimensional K-algebra, and the following conditions hold:

- M is a sincere and selforthogonal module of minimal dimension over K, but M is not faithful;

This work was partially supported by G.N.S.A.G.A., Istituto Nazionale di Alta Matematica "Francesco Severi", Italy.

- the intersection of the kernels of all Hom and Ext covariant (resp. contravariant) functors associated to M is equal to 0.

1. Preliminaries

Throughout the paper, given any ring R, we denote by R–Mod the class of all left R-modules. Next for every $M \in R$–Mod, we denote by $\operatorname{Add} M$ (resp. $\operatorname{Prod} M$) the class of all modules isomorphic to summands of direct sums (resp. products) of copies of M.

In particular, for every cardinal λ, we denote by $M^{(\lambda)}$ (resp. M^{λ}) the direct sum (resp. product) of λ copies of M. Finally, as in [B1] and [B2], we denote by $M^{\perp_\infty}$ and ${}^{\perp_\infty}M$ the following classes:

$$M^{\perp_\infty} := \Big\{X \in R\text{–Mod} \,/\, \operatorname{Ext}^i_R(M, X) = 0 \ \text{ for all } \ i \geq 1\Big\},$$
$${}^{\perp_\infty}M := \Big\{X \in R\text{–Mod} \,/\, \operatorname{Ext}^i_R(X, M) = 0 \ \text{ for all } \ i \geq 1\Big\}.$$

In the following we say that an R-module T is an *n-tilting module* if the following conditions hold:

(T1) the projective dimension of T is at most n;
(T2) $\operatorname{Ext}^i_R(T, T^{(\lambda)}) = 0$ for every $i \geq 1$ and every cardinal λ;
(T3) there is a long exact sequence of the form:

$$0 \to R \to T_0 \to T_1 \to \cdots \to T_n \to 0,$$

where $T_i \in \operatorname{Add} T$ for every $i = 0, 1, \ldots, n$.

Dually, we say that an R-module C is an *n-cotilting module*, if the following conditions hold:

(C1) the injective dimension of C is at most n;
(C2) $\operatorname{Ext}^i(C^{\lambda}, C) = 0$ for every $i \geq 1$ and every cardinal λ;
(C3) there is a long exact sequence of the form:

$$0 \to C_n \to \cdots \to C_1 \to C_0 \to E \to 0,$$

where E is an injective cogenerator of R–Mod and $C_i \in \operatorname{Prod} C$ for every $i = 0, 1, \ldots, n$.

We also say that a module T (resp. C) is a *partial n-tilting* (resp. *n-cotilting*) *module*, if T (resp. C) satisfies conditions (T1) and (T2) (resp. (C1) and (C2)). Finally, we denote by (T3′) and (C3′) the following functorial properties of two modules T and C:

(T3′) $\operatorname{Ker} \operatorname{Hom}(T, -) \cap T^{\perp_\infty} = 0$,
(C3′) $\operatorname{Ker} \operatorname{Hom}(-, C) \cap {}^{\perp_\infty}C = 0$.

Keeping the above notation, we recall some Hom − Ext conditions, which are strong enough to characterize “global” tilting/cotilting objects in the larger worlds of “local” tilting/cotilting objects.

- A finitely presented module T is a 1-tilting module iff T satisfies conditions (T1), (T2) and (T3′) [C, Theorem 1] (also see [CbF, Theorem 3.2.1 and Section 3.1]).
- A module C is a 1-cotilting module iff C satisfies conditions (C1), (C2) and (C3′) ([AnTT, Proposition 2.3], [CDT1, Theorem 1.7], [CDT2]) and [CbF, Section 2.5].
- A tilting complex (over a ring R) in the sense of Rickard [Rk] is a right bounded complex, say $T^\bullet$, of finitely generated projective R-modules, which satisfies some selforthogonal conditions, involving the i-th translates $T^\bullet[i]$'s, together with the following global condition (i.e. condition (iii)′ in [Mi, p. 184]):

  For each non-zero right bounded complex $X^\bullet$, of projective R-modules, there is some $i \in \mathbb{Z}$ such that $\mathrm{Hom}_{KR\text{–}\mathrm{Mod}}(T^\bullet, X^\bullet[i]) \neq 0$, i.e. some complex morphism $T^\bullet \to X^\bullet[i]$ is not homotopic to zero.

- A tilting object, say T, in an abelian category $\mathcal{A}$, in the sense of Happel-Reiten-Smalø [HReS] satisfies several conditions of local type, together with the following condition:

$$\mathrm{Ext}^i_{\mathcal{A}}(T, X) = 0 \ \text{ for all } \ i \geq 0 \ \text{ implies } \ X = 0.$$

- For every $n \geq 2$, every n-tilting (resp. n-cotilting) module satisfies condition (T3′) (resp. (C3′)) [B1, p. 371].

However, we know from [D1] that a partial 2-tilting (resp. 2-cotilting) module satisfying condition (T3′) (resp. (C3′)) is not necessarily a 2-tilting (resp. 2-cotilting) module. As we shall see, the examples constructed in [D1] do not describe “sporadic” or “pathological” modules. Indeed a similar result holds, by replacing 2 with every natural number $n > 2$. From now on, we shall say, for brevity, that a module T (resp. C) is a *large partial n-tilting* (resp. *n-cotilting*) *module*, if T (resp. C) satisfies conditions (T1), (T2), (T3′) (resp. (C1), (C2), (C3′)).

In the following, K always denotes an algebraically closed field, and we always identify indecomposable modules with their isomorphism classes. In particular, we often replace indecomposable finite dimensional modules, defined over a K-algebra A given by a quiver Q, according to [R], by some pictures describing their composition factors in an obvious way. Over a representation-finite algebra given by a quiver Q, we often denote by $S(x)$ the simple module corresponding to the vertex x, and by $P(x)$ and $I(x)$ the projective cover and the injective envelope of $S(x)$. If there exist only finitely many simple modules, we say that a module

M of finite length is *sincere* ([AuReS, p. 317]), if every simple module appears as a composition factor of M. Next, we say that a module M is *selforthogonal* ([H2]), if $\mathrm{Ext}^i(M, M) = 0$ for every $i \geq 1$. Finally, if M is a module of finite length of the form $\bigoplus_{i=1}^m M_i^{d_i}$, where $d_i > 0$ for every i and $M_1, \ldots, M_m$ are indecomposable and pairwise non isomorphic, then we denote m by $\delta(M)$. Under the same hypotheses, we say that M is *multiplicity-free* ([HR] and [H2]) if $d_i = 1$ for every i.

For unexplained representation-theoretic terminology, we refer to [AuReS] and [R].

2. Proofs

In the next statement we collect some results often used in the sequel.

Lemma 0. *Let A be a representation-finite algebra of finite global dimension m. The following facts hold.*

(i) *The regular module ${}_AA$ is an n-cotilting module for some $n \leq m$.*
(ii) *The injective cogenerator ${}_AD = \mathrm{Hom}_K(A_A, K)$ is an n-tilting module for some $n \leq m$.*
(iii) *Let M be a module of finite length satisfying either ($T3'$) or ($C3'$). Then M is sincere.*

Proof: See [D1, Lemmas 1 and 2]. □

Under suitable hypotheses (for instance, the hypotheses of Lemma 0), every indecomposable projective-injective module occurs as a direct summand of every n-tilting or n-cotilting module for every n. However, an indecomposable projective-injective module is not necessarily a direct summand of a large partial 2-tilting (resp. 2-cotilting) module ([D1, Example 3]), defined over a representation-finite algebra of global dimension 2. More generally, the following lemmas indicate that condition (T3$'$) (resp. (C3$'$)) does not depend on indecomposable projective (resp. injective) summands with a very special structure.

Lemma 1. *Let P be an indecomposable projective module with the following properties:*

(1) $\mathrm{soc}\, P$ *is a simple and essential submodule of* P;
(2) $\mathrm{Hom}(P/\mathrm{soc}\, P, P) \neq 0$.

Suppose $P/\mathrm{soc}\, P$ is a summand of a module M, and let L denote the module $M \oplus P$. Then the following conditions are equivalent:

(i) $\mathrm{Ker}\,\mathrm{Hom}(M, -) \cap M^{\perp\infty} = 0$;
(ii) $\mathrm{Ker}\,\mathrm{Hom}(L, -) \cap L^{\perp\infty} = 0$.

Proof: (i)⇒(ii) This immediately follows from the remark that $L^{\perp\infty} = M^{\perp\infty}$ and $\mathrm{Ker}\,\mathrm{Hom}(L, -) \subseteq \mathrm{Ker}\,\mathrm{Hom}(M, -)$.

(ii)$\Rightarrow$(i) Let X be a non-zero module such that $X \in M^{\perp_\infty}$. We claim that $\mathrm{Hom}(M,X) \neq 0$. Indeed, since (ii) holds and $L^{\perp_\infty} = M^{\perp_\infty}$, there is a non-zero morphism $f : L = M \oplus P \to X$. If $f(M) \neq 0$, then we have $\mathrm{Hom}(M,X) \neq 0$, as claimed. Next assume $f(P) \neq 0$. If $\mathrm{Ker}\, f \cap P = 0$, then X has a submodule isomorphic to P. Since $\mathrm{Hom}(P/\operatorname{soc} P, P) \neq 0$ by (2), it follows that $\mathrm{Hom}(P/\operatorname{soc} P, X) \neq 0$. If $\mathrm{Ker}\, f \cap P \neq 0$, then we deduce from (1) that $\operatorname{soc} P \subseteq \mathrm{Ker}\, f$. Thus, also in this case, we have $\mathrm{Hom}(P/\operatorname{soc} P, X) \neq 0$. Since $P/\operatorname{soc} P$ is a summand of M, we conclude that $\mathrm{Hom}(M,X) = 0$. Hence (i) holds. □

Note that any uniserial projective module of finite length ≥ 2 with isomorphic socle and top obviously satisfies conditions (1) and (2) of Lemma 1. Dually, any uniserial injective module with the same properties satisfies conditions (1) and (2) of the following lemma.

Lemma 2. *Let I be an indecomposable injective module with the following properties:*

(1) *I has a unique maximal submodule, say* $\operatorname{rad} I$*;*
(2) $\mathrm{Hom}(I, \operatorname{rad} I) \neq 0$.

Suppose $\operatorname{rad} I$ *is a summand of a module M, and let L denote the module $M \oplus I$. Then the following conditions are equivalent:*

(i) $\mathrm{Ker}\,\mathrm{Hom}(-,M) \cap {}^{\perp_\infty}M = 0$*;*
(ii) $\mathrm{Ker}\,\mathrm{Hom}(-,L) \cap {}^{\perp_\infty}L = 0$.

Proof: (i)$\Rightarrow$(ii) This is an obvious consequence of the fact that ${}^{\perp_\infty}L = {}^{\perp_\infty}M$ and $\mathrm{Ker}\,\mathrm{Hom}(-,L) \subseteq \mathrm{Ker}\,\mathrm{Hom}(-,M)$.

(ii)$\Rightarrow$(i) Let X be a non-zero module such that $X \in {}^{\perp_\infty}M$. We claim that $\mathrm{Hom}(X,M) \neq 0$. Indeed, since (ii) holds and ${}^{\perp_\infty}L = {}^{\perp_\infty}M$, there is a non zero morphism $f : X \to L = M \oplus I$. If either $f(X) \neq I$ or $f(X) \subseteq \operatorname{rad} I$ then we have $\mathrm{Hom}(X,M) \neq 0$, as desired. Finally, suppose $f(X) = I$. This assumption and (2) imply that $\mathrm{Hom}(X, \operatorname{rad} I) \neq 0$. Since $\operatorname{rad} I$ is a summand of M, also in this case we have $\mathrm{Hom}(X,M) \neq 0$, as claimed. □

The proof of Lemmas 1 and 2 suggests the following result.

Theorem 3. *Let L and M be modules, and let P* (*resp. I*) *be an indecomposable projective* (*resp. injective*) *module. The following facts hold.*

(i) *Suppose $L = M \oplus P$ and* $\mathrm{Hom}(M,X) \neq 0$ *for every module X such that $X \in M^{\perp_\infty}$ and* $\mathrm{Hom}(P,X) \neq 0$*. Then we have* $\mathrm{Ker}\,\mathrm{Hom}(M,-) \cap M^{\perp_\infty} = 0$ *if and only if* $\mathrm{Ker}\,\mathrm{Hom}(L,-) \cap L^{\perp_\infty} = 0$.
(ii) *Suppose $L = M \oplus I$ and* $\mathrm{Hom}(X,M) \neq 0$ *for every module X such that $X \in {}^{\perp_\infty}M$ and* $\mathrm{Hom}(X,I) \neq 0$*. Then we have* $\mathrm{Ker}\,\mathrm{Hom}(-,M) \cap {}^{\perp_\infty}M = 0$ *if and only if* $\mathrm{Ker}\,\mathrm{Hom}(-,L) \cap {}^{\perp_\infty}L = 0$.

PROOF: (i) The first part of the proof is similar to the proof of Lemma 1. To end the proof, we proceed as follows. Let $g : P \to X$ be a non-zero morphism with $X \in M^{\perp_\infty}$. Then our assumptions imply that $\operatorname{Hom}(M, X) \neq 0$.

(ii) To see this, we proceed as in the first part of the proof of Lemma 2. To complete the proof, we continue as follows. Let $g : X \to I$ be a non-zero morphism with $X \in {}^{\perp_\infty}M$. Then our hypotheses guarantee that $\operatorname{Hom}(X, M) \neq 0$. $\square$

In view of condition (2), the modules P in Lemma 1 and I in Lemma 2 admit non-zero endomorphisms which are not isomorphisms. However this restriction does not hold for the modules P and I in Theorem 3. Indeed, we shall prove Example 4 by means of a Nakayama [AuReS] algebra with the following property:

> Some indecomposable projective-injective modules U satisfy the hypotheses of P or I in Theorem 3, and the endomorphism ring of U is isomorphic to K.

3. Applications

The proof of the next result makes use of modules with an easy structure. Indeed, their *Loewy length* ([AF] or [AuReS]) is equal to two.

Example 4. Let n be a natural number ≥ 2. Then there exist a K-algebra A and two A-modules T and C with the following properties:

(i) A is a representation-finite algebra of finite global dimension, such that $\delta(A) = n + 1$;

(ii) T and C are non faithful modules, such that $\delta(T) = \delta(C) \leq (n+3)/2$ and $\dim_K(T) = \dim_K(C) \leq n + 2$;

(iii) T (resp. C) is a large partial n-tilting (resp. n-cotilting) module of projective (resp. injective) dimension n;

(iv) if M is a multiplicity-free sincere and selforthogonal module of projective (resp. injective) dimension n, then we have $\delta(M) \geq \delta(T)$ and $\dim_K(M) \geq \dim_K(T)$.

Construction. Let A denote the K-algebra given by the quiver

$$\underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \cdots\cdots\cdots \underset{n}{\bullet} \xrightarrow{\alpha_n} \underset{n+1}{\bullet}$$

with relations $\alpha_{i+1}\alpha_i = 0$ for $i = 1, \ldots, n-1$. Then condition (i) clearly holds. Next, let T denote the following injective module:

$$T = \begin{cases} \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 1 & \text{if } n \text{ is even} \\[2ex] \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \begin{smallmatrix} n-1 \\ n \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 1 & \text{if } n \text{ is odd} \end{cases}.$$

Finally, let C denote the following projective module:

$$C = \begin{cases} n{+}1 \oplus \begin{smallmatrix} n-1 \\ n \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} & \text{if } n \text{ is even} \\ n{+}1 \oplus \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} & \text{if } n \text{ is odd} \end{cases}.$$

Then condition (ii) holds, and n is the projective (resp. injective) dimension of T (resp. C). Moreover, by Lemma 0, T (resp. C) is a partial n-tilting (resp. n-cotilting) module. The definition of T and C also implies that $T^{\perp_\infty}$ (resp. ${}^{\perp_\infty}C$) is the class of all injective (resp. projective) modules. Consequently, it is easy to check (or to deduce from Theorem 3) that $\operatorname{Ker}\operatorname{Hom}(T,-) \cap T^{\perp_\infty} = 0$ and $\operatorname{Ker}\operatorname{Hom}(-,C) \cap {}^{\perp_\infty}C = 0$. Hence T (resp. C) is a large partial n-tilting (resp. n-cotilting) module. This remark completes the proof of (iii).

Let M be as in the hypotheses of (iv). Then M has a decomposition of the form $S \oplus L$, where S is a simple module and L is a projective-injective module. We also note that $2\delta(L) = \dim_K(L) \geq n$. Thus the following facts hold:

(1) $\delta(M) = 1 + \delta(L) \geq 1 + \dfrac{n}{2}$;

(2) $\dim_K(M)$ is an odd natural number $\geq n+1$.

On the other hand, T has the following properties:

(3) $\delta(T) \leq \dfrac{n+3}{2}$ and $\dim_K(T)$ is the largest odd natural number $\leq n+2$.

Putting (1), (2) and (3) together, we obtain $\delta(M) \geq \delta(T)$ and $\dim_K(M) \geq \dim_K(T)$. Hence (iv) holds, and the proof is complete. $\square$

Remark 5. Let A be the algebra constructed in Example 4. If n is even, then conditions (ii) and (iii) uniquely determine the modules T and C. On the other hand, if n is odd, then also the following modules T' and C' satisfy conditions (ii) and (iii):

$$T' = \begin{smallmatrix} n \\ n+1 \end{smallmatrix} \oplus \begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1\,,$$

$$C' = n{+}1 \oplus \begin{smallmatrix} n-1 \\ n \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}.$$

Consequently, the indecomposable summands of both T and T' (resp. C and C'), that is the modules $\begin{smallmatrix} n \\ n+1 \end{smallmatrix}$ and 1 (resp. $n{+}1$ and $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$), are also direct summands of any sincere and selforthogonal module M of projective (resp. injective) dimension n. Hence T and T' (resp. C and C') are as different as possible. Moreover, $T \oplus T'$ is a cogenerator (resp. $C \oplus C'$ is a generator) of A-Mod.

In the next corollary we compare large partial n-tilting modules and maximal summands of multiplicity free n-tilting modules, that is *almost complete tilting modules* (see, for instance, [H1], [Mt], [HU], [CoHU] and [BS]).

Corollary 6. *Any natural number $n \geq 2$ is the projective dimension of a module M, such that M is a large partial n-tilting module, but M is not an almost complete tilting module.*

Proof: Assume first $n \geq 4$. Next, let A and T be as in Example 4. Since $n \geq 4$ and condition (ii) holds, we have

$$\delta(A) - \delta(T) \geq n + 1 - \frac{n+3}{2} > 1.$$

This means that T is not an almost complete tilting module. On the other hand, by Example 4(iii), T is a large partial n-tilting A-module of projective dimension n. Hence we may choose $M = T$. To complete the proof, let S and R be the K-algebras given by the following quivers:

$$\begin{tikzcd} & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \\ 2\,\bullet \arrow[dr] & & \bullet\,3 \arrow[dl] \\ & \underset{4}{\bullet} & \end{tikzcd} \quad , \qquad \begin{tikzcd} & \overset{1}{\bullet} \arrow[dl] \arrow[dr] & \\ 2\,\bullet \arrow[dr] & & \bullet\,3 \arrow[dl] \\ & \bullet\,4 \arrow[d] & \\ & \bullet\,5 & \end{tikzcd} \quad ,$$

with relations $\alpha_i \alpha_j = 0$ for all arrows α_i and α_j. Then the R-module $M = \begin{smallmatrix} 2 & & 3 \\ & 4 & \end{smallmatrix} \oplus 1$ satisfies the hypotheses of the corollary with $n = 2$ ([D1, Corollary 7]). Finally, it is easy to check ([D1, Proposition 8]) that the S-module $M = \begin{smallmatrix} 4 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 2 & & 3 \\ & 4 & \end{smallmatrix} \oplus 1$ satisfies the hypotheses of the corollary with $n = 3$. The proof is complete. ☐

The next result shows that many large partial m-tilting and m-cotilting modules (with $m \geq 2$ and m even) are indecomposable. As we shall see, they are actually uniserial modules with a very rigid structure. Indeed, they are bricks ([R, p. 52]), that is their endomorphism ring is isomorphic to K.

Example 7. Let n be a natural number ≥ 2, and let $m = 2(n-1)$. Then there exist a K-algebra Λ and two Λ-modules T and C with the following properties:

- (i) Λ is a representation-finite algebra of finite global dimension such that $\delta(\Lambda) = n$;
- (ii) T and C are indecomposable non faithful Λ-modules such that $\dim_K(T) = \dim_K(C) = n$;
- (iii) T (resp. C) is a large partial m-tilting (resp. m-cotilting) module of projective (resp. injective) dimension m.

Construction. Let Λ denote the Nakayama algebra, considered in [M, Example 3.2], given by the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, "\alpha_1"] & \underset{2}{\bullet} \cdots\cdots & \underset{n-1}{\bullet} \arrow[r, "\alpha_{n-1}"] & \underset{n}{\bullet} \arrow[lll, bend right, "\alpha_n"'] \end{tikzcd}$$

with relation $\alpha_n \cdots \alpha_1 = 0$. Next, let T and C denote the modules $I(1)$ and $P(1)$ respectively. Then conditions (i) and (ii) obviously hold. Suppose first $n = 2$. Then also condition (iii) holds ([D1, Example 3]). Assume now $n > 2$. Then the indecomposable projective modules $P(1), P(2), \ldots, P(n)$ are of the form

$$\begin{matrix} 1 \\ 2 \\ \vdots \\ n \end{matrix}\ ,\quad \begin{matrix} 2 \\ 3 \\ \vdots \\ 1 \\ 2 \end{matrix}\ ,\ \ldots\ ,\quad \begin{matrix} n-1 \\ n \\ \vdots \\ n-2 \\ n-1 \end{matrix}\ ,\quad \begin{matrix} n \\ 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{matrix}\ .$$

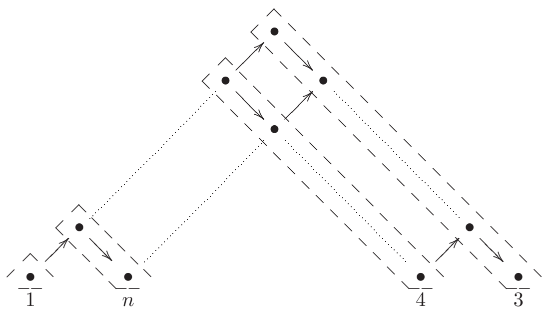
As observed in [M, Example 3.2], there is a long exact sequence (i.e. the *sequence of complements* of a projective-injective almost complete tilting module) of the form

(1) $\quad 0 \to C = P(1) \to P(n) \to P(n) \to \cdots \to P(2) \to P(2) \to T = I(1) \to 0.$

Since the modules $P(2), \ldots, P(n)$ are injective, we deduce from (1) that T (resp. C) is a partial m-tilting (resp. m-cotilting) module of projective (resp. injective) dimension m. To complete the proof of (iii), it remains to check that T (resp. C) satisfies condition (T3′) (resp. (C3′)). To this end, we first note that

(2) A module M of finite length belongs to $\operatorname{Ker}\operatorname{Hom}_\Lambda(T,-)$ (resp. $\operatorname{Ker}\operatorname{Hom}_\Lambda(-,C)$) if and only if the simple module $S(2)$ (resp. $S(n)$) is not a composition factor of M.

Hence the following triangle, contained in the Auslander-Reiten quiver of Λ, describes the indecomposable modules X such that $\operatorname{Hom}_\Lambda(T,X) = 0$.



By looking at the top of these modules (or by looking at the slices depicted in the picture), we observe that

(3) If X is an indecomposable module such that $\mathrm{Hom}_A(T, X) = 0$, then $\mathrm{Ext}^1_A(S(i), X) \neq 0$ for some $i = 2, \dots, n$.

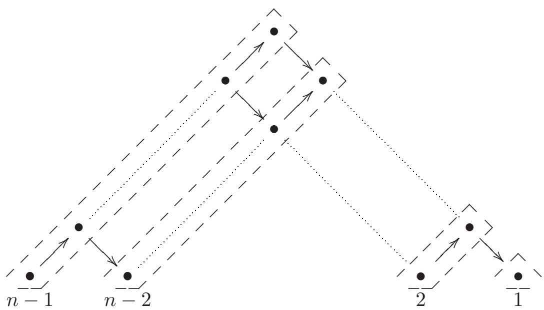
On the other hand, by dimension shifting, we deduce from (1) that

$$\text{(4)} \quad \mathrm{Ext}^2_\Lambda(T, -) \simeq \mathrm{Ext}^1_\Lambda(S(2), -), \quad \mathrm{Ext}^4_\Lambda(T, -) \simeq \mathrm{Ext}^1_\Lambda(S(3), -), \dots \\ \dots, \ \mathrm{Ext}^m_\Lambda(T, -) \simeq \mathrm{Ext}^1_\Lambda(S(n), -).$$

Putting (3) and (4) together, we obtain

$$\text{(5)} \qquad \mathrm{Ker}\,\mathrm{Hom}_\Lambda(T, -) \cap T^{\perp_\infty} = 0.$$

Dually, a triangle of the form



describes the indecomposable modules Y such that $\mathrm{Hom}_\Lambda(Y, C) = 0$. In this case, by looking at the socle of these modules (or by looking at the slices depicted in the above picture), we note that

(6) If Y is an indecomposable module such that $\mathrm{Hom}_\Lambda(Y, C) = 0$, then $\mathrm{Ext}^1_\Lambda(Y, S(i)) \neq 0$ for some $i = 2, \dots, n$.

Finally, we deduce from (1) that

$$\text{(7)} \quad \mathrm{Ext}^2_\Lambda(-, C) \simeq \mathrm{Ext}^1_\Lambda(-, S(n)), \quad \mathrm{Ext}^4_\Lambda(-, C) \simeq \mathrm{Ext}^1_\Lambda(-, S(n-1)), \dots \\ \dots, \ \mathrm{Ext}^m_\Lambda(-, C) \simeq \mathrm{Ext}^1_\Lambda(-, S(2)).$$

Comparing (6) and (7), we obtain

$$\text{(8)} \qquad \mathrm{Ker}\,\mathrm{Hom}_\Lambda(-, C) \cap {}^{\perp_\infty}C = 0.$$

By the previous remarks, (5) and (8) complete the proof of condition (iii). $\square$

The big gap between n-tilting modules and large partial n-tilting modules (with $n \geq 2$) also appears by comparing tilting complexes in the sense of Rickard [Rk] and bounded complexes (of finitely generated projective modules) obtained as projective resolutions of large partial n-tilting modules. For instance, all the modules T, used in Example 4, correspond to complexes $T^\bullet$ which behave quite differently form Rickard's tilting complexes, also in the world of complexes (of finitely generated modules) bounded on both sides ([D2]).

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(*Received* November 19, 2005)