Representation of bilinear forms in non-Archimedean Hilbert space by linear operators

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This paper is dedicated to the memory of Tosio Kato.

Abstract. The paper considers representing symmetric, non-degenerate, bilinear forms on some non-Archimedean Hilbert spaces by linear operators. Namely, upon making some assumptions it will be shown that if ϕ is a symmetric, non-degenerate bilinear form on a non-Archimedean Hilbert space, then ϕ is representable by a unique self-adjoint (possibly unbounded) operator A.

Keywords: non-Archimedean Hilbert space, non-Archimedean bilinear form, unbounded operator, unbounded bilinear form, bounded bilinear form, self-adjoint operator

Classification: 47S10, 46S10

1. Introduction

Representing bounded or unbounded, symmetric, bilinear forms by linear operators is among the most attractive topics in representation theory due to its significance and its possible applications. Applications include those arising in quantum mechanics through the study of the form sum associated with the Hamiltonians, mathematical physics, symplectic geometry, variational methods through the study of weak solutions to some partial differential equations, and many others, see, e.g., [3], [7], [10], [11]. This paper considers representing symmetric, non-degenerate, bilinear forms defined over the so-called non-Archimedean Hilbert spaces $\mathbb{E}_\omega$ by linear operators as it had been done for closed, positive, symmetric, bilinear forms in the classical setting, see, e.g., Kato [11, Chapter VI, Theorem 2.23, p. 331]. Namely, upon making some assumptions it will be shown that if $\phi : D(\phi) \times D(\phi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ ($\mathbb{K}$ being the ground field) is a symmetric, non-degenerate, bilinear form, then there exists a unique self-adjoint (possibly unbounded) operator A such that

$$\phi(u, v) = \langle Au, v\rangle, \quad \forall\, u \in D(A), v \in D(\phi) \tag{1.1}$$

where $D(A)$ and $D(\phi)$ denote the domains of A and ϕ, respectively.

Note that a bilinear form ϕ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ satisfying (1.1) will be called *representable*.

Among other things, when the bilinear form ϕ is bounded, it will be shown that the norm of ϕ coincides with that of both A and its adjoint A^*. In contrast with the classical setting, we give a direct proof of the existence of those self-adjoint operators rather than using a non-Archimedean Riesz representation theorem that the author is unaware of. Moreover, because of non-positiveness in the non-Archimedean world, we do not require that the form ϕ be "positive", as it had been required in the classical setting. One should also mention that the closedness of bilinear forms in the sense of quadratic forms ([6]) will not be required here although that was an important argument in the proof of both the first and second representation theorems in the classical setting, see [11].

To deal with the above-mentioned issues we shall make extensive use of the formalism of *unbounded* linear operators on non-Archimedean Hilbert spaces $\mathbb{E}_\omega$ ([4], [5]) and that of (un)bounded, symmetric, bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, recently introduced and studied in [6]. The general case, that is, representing general bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, not necessarily symmetric will be left as an open question.

2. Preliminaries

Let $\mathbb{K}$ be a complete non-Archimedean valued field. Classical examples of such a field include $(\mathbb{Q}_p, |\cdot|)$, the field of p-adic numbers equipped with the p-adic absolute value, where $p \geq 2$ is a prime, $\mathbb{C}_p$ the field of complex p-adic numbers, and the field of formal Laurent series, see, e.g., [8], [9].

A non-Archimedean Banach space $\mathbb{E}$ over $\mathbb{K}$ is said to be a *free* Banach space ([2], [4], [8], [9]) if there exists a family $(e_i)_{i\in I}$ (I being an index set) of elements of $\mathbb{E}$ such that each element $x \in \mathbb{E}$ can be written in a unique fashion as

$$x = \sum_{i\in I} x_i e_i, \quad \lim_{i\in I} x_i e_i = 0, \quad \text{and} \quad \|x\| = \sup_{i\in I} |x_i| \|e_i\|.$$

The family $(e_i)_{i\in I}$ is then called an *orthogonal base* for $\mathbb{E}$, and if $\|e_i\| = 1$, for all $i \in I$, the family $(e_i)_{i\in I}$ is then called an *orthonormal base*. From now on, we suppose that the index set I is $\mathbb{N}$, the set of all natural integers.

For a free Banach space $\mathbb{E}$, let $\mathbb{E}^*$ denote its (topological) dual and $B(\mathbb{E})$ the Banach algebra of all bounded linear operators on $\mathbb{E}$ ([2], [8], [9]). Both $\mathbb{E}^*$ and $B(\mathbb{E})$ are equipped with their respective natural norms. For $(u, v) \in \mathbb{E}\times\mathbb{E}^*$, define the linear operator $(v \otimes u)$ by setting

$$\forall\, x \in \mathbb{E},\ (v \otimes u)(x) := v(x)u = \langle v, x\rangle\, u.$$

Let $(e_i)_{i\in\mathbb{N}}$ be an orthogonal base for $\mathbb{E}$. We then define $e'_i \in \mathbb{E}^*$ by setting

$$x = \sum_{i\in\mathbb{N}} x_i e_i, \quad e'_i(x) = x_i.$$

It turns out that $\|e'_i\| = \frac{1}{\|e_i\|}$. Furthermore, every $x' \in \mathbb{E}^*$ can be expressed as a pointwise convergent series $x' = \sum_{i\in\mathbb{N}} \langle x', e_i\rangle e'_i$, and $\|x'\| := \sup_{i\in\mathbb{N}} \frac{|\langle x', e_i\rangle|}{\|e_i\|}$.

Recall that every bounded linear operator A on $\mathbb{E}$ can be expressed as a pointwise convergent series ([8], [9]), that is, there exists an infinite matrix $(a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{ij} a_{ij}(e'_j \otimes e_i), \quad \text{and for any } \; j \in \mathbb{N}, \quad \lim_{i\to\infty} |a_{ij}| \, \|e_i\| = 0. \tag{2.1}$$

Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i\in\mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\|A\| := \sup_{i,j} \frac{|a_{ij}| \, \|e_i\|}{\|e_j\|}. \tag{2.2}$$

In this paper we shall make extensive use of the non-Archimedean Hilbert space $\mathbb{E}_\omega$ whose definition is given below. Again, for details, see, e.g., [2], [4], [8], [9], and [2]. Let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of non-zero elements in a complete nontrivial non-Archimedean field $\mathbb{K}$. Define the space $\mathbb{E}_\omega$ by

$$\mathbb{E}_\omega := \left\{ u = (u_i)_{i\in\mathbb{N}} \mid \forall\, i, \; u_i \in \mathbb{K} \;\text{ and }\; \lim_{i\to\infty} |u_i| \, |\omega_i|^{1/2} = 0 \right\}.$$

Clearly, $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ if and only if $\lim_{i\to\infty} u_i^2 \omega_i = 0$. Actually $\mathbb{E}_\omega$ is a non-Archimedean Banach space over $\mathbb{K}$ with the norm given by

$$u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega, \quad \|u\| = \sup_{i\in\mathbb{N}} |u_i| \, |\omega_i|^{1/2}. \tag{2.3}$$

Clearly, $\mathbb{E}_\omega$ is a free Banach space and it has a canonical orthogonal base. Namely, $(e_i)_{i\in\mathbb{N}}$, where e_i is the sequence all of whose terms are 0 except the i-th term which is 1, in other words, $e_i = \left(\delta_{ij}\right)_{j\in\mathbb{N}}$, where δ_{ij} is the usual Kronecker symbol. We shall make extensive use of such a canonical orthogonal base throughout the paper. It should be mentioned that for each i, $\|e_i\| = |\omega_i|^{1/2}$. Now if $|\omega_i| = 1$ we shall refer to $(e_i)_{i\in\mathbb{N}}$ as the canonical orthonormal base.

Let $\langle\cdot,\cdot\rangle : \mathbb{E}_\omega \times \mathbb{E}_\omega \to \mathbb{K}$ be the $\mathbb{K}$-bilinear form defined by

$$\forall\, u, v \in \mathbb{E}_\omega, \; u = (u_i)_{i\in\mathbb{N}}, \; v = (v_i)_{i\in\mathbb{N}}, \;\; \langle u, v\rangle := \sum_{i\in\mathbb{N}} \omega_i \, u_i v_i. \tag{2.4}$$

Clearly, $\langle\cdot,\cdot\rangle$ is a symmetric, non-degenerate form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ with value in $\mathbb{K}$, and it satisfies the Cauchy-Schwarz inequality

$$|\langle u, v\rangle| \le \|u\| \, . \, \|v\|,$$

for all $u, v \in \mathbb{E}_\omega$.

In addition to the above, note that the vectors $(e_i)_{i\in\mathbb{N}}$, of the canonical orthogonal base satisfy the following: $\langle e_i, e_j\rangle = \omega_i \delta_{ij}$ for all $i, j \in \mathbb{K}$.

In the next sections, we shall be studying (general) symmetric, bilinear forms, which have some common properties as the $\mathbb{K}$-form, $\langle\cdot,\cdot\rangle$, given by (2.4). However, most of those forms do not necessarily satisfy the Cauchy- Schwarz inequality.

Definition 2.2 ([4], [8], [9])**.** The space $(\mathbb{E}_\omega, \|\cdot\|, \langle\cdot,\cdot\rangle)$ is called a non-Archimedean (or p-adic) Hilbert space.

Remark 2.2. In contrast with the classical context, the norm given in (2.3) is not deduced from the inner product given in (2.4).

Let $\omega = (\omega_i)_{i\in\mathbb{N}}$, $\varpi = (\varpi_i)_{i\in\mathbb{N}}$ be sequences of nonzero elements in a complete non-Archimedean field $\mathbb{K}$, and let $\mathbb{E}_\omega$ and $\mathbb{E}_\varpi$ denote their corresponding non-Archimedean Hilbert spaces, respectively. Let $(e_i)_{i\in\mathbb{N}}$ and $(h_j)_{j\in\mathbb{N}}$ denote the canonical orthogonal bases associated with $\mathbb{E}_\omega$ and $\mathbb{E}_\varpi$, respectively.

Definition 2.3 ([4], [5])**.** An unbounded linear operator A from $\mathbb{E}_\omega$ into $\mathbb{E}_\varpi$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}_\omega$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\varpi$. Namely, the domain $D(A)$ contains the basis $(e_i)_{i\in\mathbb{N}}$ and consists of all $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ such $Au = \sum_{i\in\mathbb{N}} u_i Ae_i$ converges in $\mathbb{E}_\varpi$, that is,

$$(2.5)\qquad \begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\}, \\[2ex] A = \displaystyle\sum_{i,j\in\mathbb{N}} a_{ij}\ e'_j \otimes h_i, \quad \forall\, j \in \mathbb{N},\ \lim_{i\to\infty}\ |a_{i,j}|\ \|h_i\| = 0. \end{cases}$$

Let $U(\mathbb{E}_\omega, \mathbb{E}_\varpi)$ denote the collection of those unbounded linear operators from $\mathbb{E}_\omega$ into $\mathbb{E}_\varpi$. Note that if A is a bounded linear operator from $\mathbb{E}_\omega$ into $\mathbb{E}_\varpi$ then $D(A) = \mathbb{E}_\omega$. Without loss of generality, throughout the rest of the paper we suppose that $\mathbb{E}_\omega = \mathbb{E}_\varpi$. We then denote $U(\mathbb{E}_\omega, \mathbb{E}_\omega)$ by $U(\mathbb{E}_\omega)$.

Definition 2.4 ([4])**.** A linear operator

$$\begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\}, \\[2ex] A = \displaystyle\sum_{i,j\in\mathbb{N}} a_{ij}\ e'_j \otimes e_i, \quad \forall\, j \in \mathbb{N},\ \lim_{i\to\infty}\ |a_{ij}|\ \|e_i\| = 0 \end{cases}$$

is said to have an adjoint $A^* \in U(\mathbb{E}_\omega)$ if and only if

$$(2.6)\qquad \lim_{j\to\infty} \left(\frac{|a_{ij}|}{|\omega_j|^{1/2}}\right) = 0, \quad \forall\, i \in \mathbb{N}.$$

In this event the adjoint A^* of A is uniquely expressed by

$$\begin{cases} D(A^*) := \{v = (v_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |v_i|\ \|A^* e_i\| = 0\}, \\[2ex] A^* = \displaystyle\sum_{i,j\in\mathbb{N}} a^*_{ij}\ e'_j \otimes e_i, \quad \forall\, j \in \mathbb{N},\ \lim_{i\to\infty} |a^*_{ij}|\ |\omega_i|^{\frac{1}{2}} = 0, \end{cases}$$

where $a^*_{ij} = \omega_i^{-1}\omega_j a_{ji}$.

Let $U_0(\mathbb{E}_\omega)$ denote the collection of linear operators in $U(\mathbb{E}_\omega)$ whose adjoint operators exist.

Remark 2.5. In contrast with the classical setting: (1) There are linear operators which do not have adjoint operators and, (2) If $A \in U_0(\mathbb{E}_\omega)$, then $A^{**} = A$, rather $\bar{A}$, the closure of A.

Throughout the rest of the paper $\mathbb{K}$ denotes a complete non-Archimedean field. If $\omega = (\omega_i)_{i\in\mathbb{N}}$ is a sequence of nonzero elements in $\mathbb{K}$, we then let $\mathbb{E}_\omega$ denote its corresponding non-Archimedean Hilbert space and $(e_i)_{i\in\mathbb{N}}$ denotes the canonical orthogonal base for $\mathbb{E}_\omega$.

3. Non-Archimedean bilinear forms

Definition 3.1. A (symmetric) mapping $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be a bilinear form whenever $u \mapsto \phi(u, v)$ is linear for each $v \in \mathbb{E}_\omega$ and $v \mapsto \phi(u, v)$ linear for each $u \in \mathbb{E}_\omega$.

One can easily check that if $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is a well-defined (symmetric) bilinear form over $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then, for all $u = (u_i)_{i\in\mathbb{N}}$, $v = (v_j)_{j\in\mathbb{N}} \in \mathbb{E}_\omega$,

$$\phi(u, v) = \sum_{i,j=0}^{\infty} \sigma_{ij}\ u_i v_j, \quad \text{and} \quad \forall\, j \in \mathbb{N}, \quad \lim_{i\to\infty} \left\{ |u_i| \,.\, |\sigma_{ij}|^{1/2} \right\} = 0, \tag{3.1}$$

where $\sigma_{ij} = \phi(e_i, e_j)$ for all $i, j \in \mathbb{N}$ with $\sigma_{ij} = \sigma_{ji}$ for all $i, j \in \mathbb{N}$.

3.1 Bounded bilinear forms.

Definition 3.2. A non-Archimedean bilinear form $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be bounded if there exists $M \geq 0$ such that

$$|\phi(u, v)| \leq M \,.\, \|u\| \,.\, \|v\|, \quad u, v \in \mathbb{E}_\omega. \tag{3.2}$$

The smallest M such that (3.2) holds is called the norm of the bilinear form ϕ and is defined by

$$\|\phi\| = \sup_{u,v\neq 0} \left\{ \frac{|\phi(u, v)|}{\|u\| \,.\, \|v\|} \right\}.$$

Proposition 3.3. *Let $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a bounded bilinear form. Then its norm $\|\phi\|$ can be explicitly expressed as*

$$\|\phi\| = \sup_{i,j\in\mathbb{N}} \left\{ \frac{|\phi(e_i, e_j)|}{\|e_i\| \,.\, \|e_j\|} \right\}.$$

Proof: The inequality, $\|\phi\| \geq \sup_{i,j\in\mathbb{N}}\left\{\frac{|\phi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right\}$, is a straightforward consequence of the definition of the norm $\|\phi\|$ of ϕ.

Now using the ultrametric inequality in $\mathbb{K}$ it easily follows that for all $u = (u_i)_{i\in\mathbb{N}}$, $v = (v_j)_{j\in\mathbb{N}} \in \mathbb{E}_\omega$ and $n, m \in \mathbb{N}$,

$$\begin{aligned}\left|\sum_{i=0}^{n}\sum_{j=0}^{m}\phi(e_i,e_j)\ u_iv_j\right| &\leq \max_{0\leq i\leq n}\left|\sum_{j=0}^{m}\phi(e_i,e_j)\ u_iv_j\right|\\ &\leq \max_{0\leq i\leq n}\left(\max_{0\leq j\leq m}\left|\phi(e_i,e_j)\ u_iv_j\right|\right)\\ &\leq \sup_{i,j\in\mathbb{N}}\left(\left|\phi(e_i,e_j)u_iv_j\right|\right).\end{aligned}$$

One should point out that $\sup_{i,j\in\mathbb{N}}\left(|\phi(e_i,e_j)u_iv_j|\right) < \infty$ since $\phi(e_i,e_j)\ u_iv_j \to 0$ in $\mathbb{K}$ as $i, j \to \infty$.

Passing to the limit in the previous inequality, as $n, m \to \infty$, one has

$$\left|\sum_{i,j=0}^{\infty}\phi(e_i,e_j)\ u_iv_j\right| \leq \sup_{i,j\in\mathbb{N}}\left(|\phi(e_i,e_j)|\,.\,|u_i|\,.\,|v_j|\right).$$

Now suppose $u, v \neq 0$. In view of the above, one has

$$\begin{aligned}|\phi(u,v)| &= \left|\sum_{i,j=0}^{\infty}\phi(e_i,e_j)\ u_iv_j\right|\\ &\leq \sup_{i,j\in\mathbb{N}}\left(|\phi(e_i,e_j)|\,.\,|u_i|\,.\,|v_j|\right)\\ &= \sup_{i,j\in\mathbb{N}}\left\{\frac{|\phi(e_i,e_j)|(|u_i|\,.\,\|e_i\|)\,(|v_j|\,.\,\|e_j\|)}{\|e_i\|\,.\,\|e_j\|}\right\}\\ &\leq \|u\|\,.\,\|v\|\,.\,\sup_{i,j\in\mathbb{N}}\left\{\frac{|\phi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right\},\end{aligned}$$

and hence

$$\|\phi\| \leq \sup_{i,j\in\mathbb{N}}\left\{\frac{|\phi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right\}.$$

One completes the proof by combining the first and the latest inequalities. □

Theorem 3.4. *Let $\phi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a (symmetric) non-degenerate, bounded bilinear form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$. Suppose that*

$$\forall\, j \in \mathbb{N}, \quad \lim_{i\to\infty}\frac{|\phi(e_i,e_j)|}{\|e_i\|} = 0. \tag{3.3}$$

Then there exists a unique bounded self-adjoint operator A with $\|A^*\| = \|A\| = \|\phi\|$ and such that

$$\phi(u,v) = \langle Au, v\rangle$$

for all $u, v \in \mathbb{E}_\omega$.

Proof: From the expression of ϕ in (3.1), define (formally) the linear operator A on $\mathbb{E}_\omega$ by setting

$$A := \sum_{i,j\in\mathbb{N}} \left[\frac{\phi(e_i,e_j)}{\omega_i}\right] (e'_j \otimes e_i).$$

We first show that the operator A given above is well-defined on $\mathbb{E}_\omega$. Indeed, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left|\frac{\phi(e_i,e_j)}{\omega_i}\right| \|e_i\| = \lim_{i\to\infty} \frac{|\phi(e_i,e_j)|}{\|e_i\|} = 0,$$

by using assumption (3.3).

Moreover, it is not hard to see that $\phi(u,v) = \langle Au, v\rangle$ for all $u, v \in \mathbb{E}_\omega$. Now, the uniqueness of A is guaranteed by the fact that ϕ is non-degenerate. It remains to show that A^*, the adjoint of A exists and that $A^* = A$. Indeed,

$$\begin{aligned}\lim_{j\to\infty} \left(\frac{\left|\frac{\phi(e_i,e_j)}{\omega_i}\right|}{\|e_j\|}\right) &= \frac{1}{|\omega_i|} \cdot \lim_{j\to\infty} \left(\frac{|\phi(e_i,e_j)|}{\|e_j\|}\right)\\ &= \frac{1}{|\omega_i|} \cdot \lim_{j\to\infty} \left(\frac{|\phi(e_j,e_i)|}{\|e_j\|}\right)\\ &= 0, \quad \forall\, i \in \mathbb{N},\end{aligned}$$

by using assumption (3.3), and hence the adjoint A^* of A exists.

Now, writing $A^* = \sum_{i,j\in\mathbb{N}} a^*_{ij}(e'_j \otimes e_i)$ it is clear that the coefficients a^*_{ij} of A^* can be expressed in terms of those of A as follows:

$$a^*_{ij} = \omega_i^{-1}\omega_j \left[\frac{\phi(e_j,e_i)}{\omega_j}\right] = \left[\frac{\phi(e_i,e_j)}{\omega_i}\right],$$

that is $A = A^*$.

Now

$$\|A^*\| = \|A\| := \sup_{i,j} \left(\frac{\left|\frac{\phi(e_i,e_j)}{\omega_i}\right| \|e_i\|}{\|e_j\|}\right) = \sup_{i,j\in\mathbb{N}} \left(\frac{|\phi(e_i,e_j)|}{\|e_i\| \,.\, \|e_j\|}\right) = \|\phi\|.$$

□

Example 3.5. Let $\mathbb{K} = \mathbb{Q}_p$ equipped with the p-adic absolute value and let $\omega_i = p^{-i}$ for each $i \in \mathbb{N}$. If $m \in \mathbb{N}$ with $m \geq 1$ (fixed), then set

$$Q(\omega_i, \omega_j) = 1 + \frac{1}{\omega_i \omega_j} + \frac{1}{\omega_i^2 \omega_j^2} + \cdots + \frac{1}{\omega_i^m \omega_j^m}$$

for all $i, j \in \mathbb{N}$.

Clearly, $\forall\, j \in \mathbb{N}$, $\lim_{i\to\infty} \frac{|Q(\omega_i,\omega_j)|}{\|e_i\|} = 0$, since $|Q(\omega_i, \omega_j)| = 1$ and $\|e_i\| = p^{i/2}$ for all $i \in \mathbb{N}$. For all $u = (u_i)_{i\in\mathbb{N}}, v = (v_j)_{j\in\mathbb{N}} \in \mathbb{E}_\omega$, define the (symmetric) bilinear form $\phi(u, v) = \sum_{i,j=0}^{\infty} Q(\omega_i, \omega_j)\ u_i v_j$. Clearly, ϕ is well-defined since, $\forall\, j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left(|u_i| \,.\, |Q(\omega_i, \omega_j)|^{1/2}\right) \leq \|u\| \,.\, \lim_{i\to\infty} \left(\frac{|Q(\omega_i, \omega_j)|}{|\omega_i|}\right)^{1/2} = 0.$$

Moreover ϕ is non-degenerate, and its norm, $\|\phi\| = 1$. Therefore, the only bounded self-adjoint operator on $\mathbb{E}_\omega$ associated with ϕ is the one defined by

$$A = \sum_{i,j\in\mathbb{N}} \left[\frac{Q(\omega_i, \omega_j)}{\omega_i}\right] (e'_j \otimes e_i)$$

with $\|A\| = \|\phi\| = 1$.

3.2 Unbounded symmetric bilinear forms. In this subsection we prove an unbounded version of Theorem 3.4. For that, we will make use of the definition of an unbounded bilinear form that was introduced by the author in [6].

Definition 3.6 ([6])**.** A (symmetric) mapping $\phi : D(\phi) \times D(\phi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is called a non-Archimedean (unbounded) bilinear form if $u \mapsto \phi(u, v)$ is linear for each $v \in D(\phi)$ and $v \mapsto \phi(u, v)$ linear for each $u \in D(\phi)$, where $D(\phi)$ contains the basis $(e_i)_{i\in\mathbb{N}}$ and

$$\begin{cases} D(\phi) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \ \lim\limits_{i\to\infty} \left(|u_i|\ |\phi(e_i, e_i)|^{1/2}\right) = 0\}, \\ \\ \phi(u, v) = \sum\limits_{i,j=0}^{\infty} \sigma_{ij}\ u_i v_j, \ \ \text{and} \ \ \forall\, j \in \mathbb{N}, \ \ \lim\limits_{i\to\infty} \left(|u_i| \,.\, |\sigma_{ij}|^{1/2}\right) = 0 \end{cases}$$

for all $u, v \in D(\phi)$, where $\sigma_{ij} = \phi(e_i, e_j)$.

The subspace $D(\phi) \subset \mathbb{E}_\omega$ defined above is called the *domain* of the bilinear form ϕ.

Theorem 3.7. *Let $\phi : D(\phi) \times D(\phi) \mapsto \mathbb{K}$ be an unbounded, symmetric, non-degenerate, bilinear form such that* (3.3) *holds. Then there exists a unique unbounded linear operator A such that*

$$\phi(u, v) = \langle Au, v\rangle, \quad \forall\, u \in D(A),\ v \in D(\phi).$$

Moreover, the adjoint A^* exists and $A = A^*$.

Proof: Although the proof is similar to that of Theorem 3.4, the domains $D(\phi)$ and $D(A)$ should be watched with care.

For all $u = (u_i)_{i\in\mathbb{N}}, v = (v_j)_{j\in\mathbb{N}} \in D(\phi)$, write

$$\phi(u, v) = \sum_{i,j=0}^{\infty} \phi(e_i, e_j)\ u_i v_j, \quad \text{with} \quad \forall\, j \in \mathbb{N}, \quad \lim_{i\to\infty} \left(|u_i| \,.\, |\phi(e_i, e_j)|^{1/2}\right) = 0.$$

Define the linear operator A on $\mathbb{E}_\omega$ by setting

$$\begin{cases} D(A) := \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\}, \\ Au = \sum\limits_{i,j\in\mathbb{N}} \left[\dfrac{\phi(e_i, e_j)}{\omega_i}\right] (e'_j \otimes e_i)u, \quad \forall\, u = (u_i)_{i\in\mathbb{N}} \in D(A). \end{cases}$$

Clearly, A is well-defined, since, $\forall\, j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left|\frac{\phi(e_i, e_j)}{\omega_i}\right| \|e_i\| = \lim_{i\to\infty} \frac{|\phi(e_i, e_j)|}{\|e_i\|} = 0,$$

by using assumption (3.3). And,

$$Au = \sum_{j\in\mathbb{N}} \frac{1}{\omega_j} \left(\sum_{i\in\mathbb{N}} u_i \phi(e_i, e_j)\right) e_j, \quad \forall\, u = (u_i)_{i\in\mathbb{N}} \in D(A).$$

First of all, note that $D(A) \subset D(\phi)$. Indeed, if $u = (u_i)_{i\in\mathbb{N}} \in D(A)$, then, $\forall\, i \in \mathbb{N}$,

$$\begin{aligned} |u_i|^2\ |\phi(e_i, e_i)| &= |u_i|^2 \|e_i\| \left(\frac{|\phi(e_i, e_i)|}{\|e_i\|}\right) \\ &\le |u_i|^2 \|e_i\| \,.\, \|Ae_i\| \\ &= (|u_i| \|e_i|) \,.\, (|u_i| \|Ae_i\|), \end{aligned}$$

and hence $\lim_{i\to\infty} \left(|u_i| \,.\, |\phi(e_i, e_i)|^{1/2}\right) = 0$, that is, $u \in D(\phi)$.

Now

$$\begin{aligned}\langle Au, v\rangle &= \sum_{k\in\mathbb{N}} \omega_k v_k \frac{1}{\omega_k}\left(\sum_{i\in\mathbb{N}} u_i \phi(e_i, e_k)\right)\\ &= \sum_{k\in\mathbb{N}} v_k \left(\sum_{i\in\mathbb{N}} u_i \phi(e_i, e_k)\right)\\ &= \sum_{i,k\in\mathbb{N}} \phi(e_i, e_k) u_i v_k\\ &= \phi(u, v)\end{aligned}$$

for all $u = (u_i)_{i\in\mathbb{N}} \in D(A) \subset D(\phi)$ and $v = (v_k)_{k\in\mathbb{N}} \in D(\phi)$.

To justify the above equalities, note that $u_i v_k \phi(e_i, e_k) \to 0$ as $i, k \to \infty$, by using the fact that ($u \in D(A) \subset D(\phi)$ and $v \in D(\phi)$):

$$|u_i v_k \phi(e_i, e_k)| = \left(|u_i||\phi(e_i, e_k)|^{1/2}\right) . \left(|\phi(e_k, e_i)|^{1/2}|v_k|\right) \to 0, \quad i, k \to \infty.$$

And hence

$$\sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}} u_i v_k \phi(e_i, e_k) = \sum_{i\in\mathbb{N}}\sum_{k\in\mathbb{N}} u_i v_k \phi(e_i, e_k),$$

according to a result by Cassels [1].

Furthermore, the uniqueness of A is guaranteed by the fact that ϕ is non-degenerate. It remains to show that A^*, the adjoint of A exists and that $A^* = A$; this can be done as in the bounded case.

Now, writing $A^* = \sum_{i,j\in\mathbb{N}} a^*_{ij}(e'_j \otimes e_i)$ it is clear that the coefficients a^*_{ij} of A^* can be expressed in terms of that of A as follows:

$$a^*_{ij} = \omega_i^{-1}\omega_j \left[\frac{\phi(e_j, e_i)}{\omega_j}\right] = \left[\frac{\phi(e_i, e_j)}{\omega_i}\right],$$

that is $A = A^*$. □

Example 3.8. We consider a non-Archimedean version of an example considered by Kato [11, Example 1.24, p. 317] consisting of the bilinear form defined by

$$\phi(u, v) = \sum_{i\in\mathbb{N}} a_i u_i v_i, \quad \forall\, u = (u_i)_{i\in\mathbb{N}}, v = (v_i)_{i\in\mathbb{N}} \in D(\phi)$$

where $a = (a_i)_{i\in\mathbb{N}} \subset \mathbb{K}$ is a sequence of nonzero elements and $D(\phi)$ is defined by

$$D(\phi) = \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty} |a_i||u_i| = 0\}.$$

Here, $\phi(e_i, e_j) = \delta_{ij} a_i$ where δ_{ij} is the classical Kronecker symbol. And an equivalent of (3.3) is given by

$$\lim_{i\to\infty} \frac{|a_i|}{\|e_i\|} = 0.$$

Upon making the previous assumption, the unique self-adjoint operator associated with ϕ is given by

$$Au = \sum_{i\in\mathbb{N}} \frac{a_i}{\omega_i} u_i e_i, \quad \forall\, u = (u_i)_{i\in\mathbb{N}} \in D(A)$$

where $D(A) = \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty} \frac{|a_i|}{\|e_i\|} |u_i| = 0\}$.

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