On the functorial prolongations of principal bundles

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Abstract. We describe the fundamental properties of the infinitesimal actions related with functorial prolongations of principal and associated bundles with respect to fiber product preserving bundle functors. Our approach is essentially based on the Weil algebra technique and an original concept of weak principal bundle.

Keywords: fiber product preserving bundle functor, weak principal bundle, Lie algebroid, functorial prolongation, Weil algebra

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It is well known that the r-th jet prolongation J^rP of a principal bundle $P(M,G)$ is not a principal bundle. Ehresmann studied the groupoidal form of this problem. He constructed the r-th prolongation Φ^r of a Lie groupoid Φ and deduced that if Φ acts on a fibered manifold Y, then Φ^r acts canonically on the r-th jet prolongation J^rY, [4]. The principal bundle form of this construction can be found e.g. in [10]. If P^rM denotes the r-th order frame bundle of M, then $W^rP = P^rM \times_M J^rP$ is a principal bundle and the r-th jet prolongation of every fiber bundle $P[S]$ associated to P is canonically associated to W^rP.

We present a conceptual explanation of this fact by generalizing the problem: we replace J^r by an arbitrary fiber product preserving bundle functor F on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps. In Section 3 we recall the concept of weak principal bundle $Q \to M$, which was introduced by the first author in [6] when investigating certain problems concerning connections. The weak principal bundle has a structure group bundle $K \to M$. If K is the product $M \times H$, where H is a group, then Q is a classical principal bundle. Next we describe a general situation, in which the fiber product $P \times_M Q$ is a principal bundle whose structure group is a semidirect product of the groups G and H. In Section 5 we clarify that this situation appears for every F: if r is the base order of the functor F, we have to take P^rM for P and FP for Q. In Section 6 we describe the Lie algebroid of $P \times_M Q$ by using a general construction of a Lie algebroid $E \times_M D$ from a Lie algebroid E acting by derivations on a Lie algebra bundle D.

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In Section 4 we characterize the functor F in terms of Weil algebras. We find very attractive even from the practical point of view that every product preserving bundle functor on the category $\mathcal{M}f$ of all manifolds and all smooth maps is a Weil functor T^A, [10]. The multiplication in the Weil algebra A plays an important role in many concrete considerations. In Section 1 we underline that every Weil algebra A is a factor algebra of the Weil algebra $\mathbb{D}_k^r$ corresponding to the classical functor T_k^r of k-dimensional velocities of order r. Then we present a velocity-like definition of the bundle $T^A M$. In Section 2 we collect several formulae concerning the T^A-prolongations of infinitesimal actions that are very simple thanks to the use of the multiplication in A. Finally, in Section 7 we clarify how our previous procedures and results can be applied to the geometry of the principal bundle $W^F P$ constructed from a principal bundle P by means of the functor F and to certain associated bundles.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from [10].

1. Velocity-like approach to Weil bundles

A Weil algebra A is a finite dimensional, commutative, associative and unitary algebra, in which there exists an ideal N such that the factor space A/N is one-dimensional and $N^{r+1} = 0$ for some integer r.

The distinguished element $1 + N$ in A/N identities this factor space with $\mathbb{R}$. This defines a product decomposition $A = \mathbb{R} \times N$. Clearly, N is the ideal of all nilpotent elements of A. We say r is the order of A and the dimension of the vector space N/N^2 is called the width of A, [10].

A typical example of a Weil algebra is $\mathbb{D}_k^r = J_0^r(\mathbb{R}^k, \mathbb{R})$ with the addition and multiplication induced by the addition and multiplication of reals. A simple algebraic consideration shows that every Weil algebra of order r and width k is a factor algebra

$$\mathbb{D}_k^r \xrightarrow{\pi} A. \tag{1}$$

If $\widetilde{\pi}$ is another such epimorphism, then $\pi = \widetilde{\pi} \circ \sigma$, where σ is an algebra isomorphism $\mathbb{D}_k^r \to \mathbb{D}_k^r$.

Let M be a manifold.

Definition 1. Two maps $\gamma, \delta : \mathbb{R}^k \to M$ are said to determine the same A-velocity $j^A\gamma = j^A\delta$, if for every smooth function $\varphi : M \to \mathbb{R}$,

$$\pi\big(j_0^r(\varphi \circ \gamma)\big) = \pi\big(j_0^r(\varphi \circ \delta)\big). \tag{2}$$

This condition is independent of the choice of π. We also say that γ and δ determine the same A-jet.

Every smooth map $f : M \to N$ induces $T^A f : T^A M \to T^A N$ by

$$T^A f(j^A\gamma) = j^A(f \circ \gamma). \tag{3}$$

In this way A defines the Weil functor T^A of A-velocities on $\mathcal{M}f$.

If B is another Weil algebra, then the natural transformations $T^A \to T^B$ are in bijection with the algebra homomorphisms $h : A \to B$. If we assume B is of order r and width l and consider it as a factor algebra $\mathbb{D}^r_l \xrightarrow{\varrho} B$, we construct easily an algebra homomorphism $\chi : \mathbb{D}^r_l \to \mathbb{D}^r_k$ satisfying $\chi \circ \varrho = h \circ \pi$. Since χ is determined by its values on the canonical basis $x_1, \dots, x_l$ of $\mathbb{D}^r_l$, we can view it as an r-jet $\chi = j^r_0\psi \in J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0$. Then we have

$$h_M(j^A\gamma) = j^B(\gamma \circ \psi)\,, \tag{4}$$

where $h_M : T^A M \to T^B M$ is the value of the natural transformation determined by h on M. Thus, under the velocity-like approach the natural transformations $T^A \to T^B$ can be interpreted as a special kind of reparametrization.

Clearly, the functor T^A preserves products. Conversely, if G is a product preserving bundle functor on $\mathcal{M}f$, then $G\mathbb{R}$ is an algebra with respect to the G-prolongation of the addition and multiplication of reals. The main theorem reads that $G\mathbb{R} = A$ is a Weil algebra and G coincides with the Weil functor T^A, [10].

For every vector space V, $T^A V$ is also a vector space. Consider the map $\otimes : V \times A \to T^A V$,

$$\otimes\big(v, j^A\varphi(\tau)\big) = j^A\big(\varphi(\tau)v\big), \quad v \in V,\ \varphi : \mathbb{R}^k \to \mathbb{R},\ \tau \in \mathbb{R}^k. \tag{5}$$

If $(v_1 \dots, v_n)$ are some linear coordinates on V, then (5) is of the form

$$\otimes\big((v_1, \dots, v_n), a\big) = (v_1 a, \dots, v_n a), \qquad v_i \in \mathbb{R},\ a \in A.$$

This implies $T^A V = V \otimes A$. For an algebra homomorphism $h : A \to B$, we obtain in the same way that $h_V : V \otimes A \to V \otimes B$ coincides with $\mathrm{id}_V \otimes h$, i.e.

$$h_V(v \otimes a) = v \otimes h(a)\,, \qquad v \in V,\ a \in A. \tag{6}$$

If W is another vector space and $f : V \to W$ is a linear map, then

$$T^A f(v \otimes a) = T^A f\big(j^A(\varphi(\tau)v)\big) = j^A\big(\varphi(\tau)f(v)\big) = f(v) \otimes a. \tag{7}$$

Hence $T^A f = f \otimes \mathrm{id}_A : V \otimes A \to W \otimes A$.

If $f : V_1 \times V_2 \to W$ is a bilinear map, we have

$$T^A f\big(j^A(\varphi_1(\tau)v_1), j^A(\varphi_2(\tau)v_2)\big) = j^A\big(\varphi_1(\tau)\varphi_2(\tau)\big)\, f(v_1, v_2).$$

Hence $T^A f : V_1 \otimes A \times V_2 \otimes A \to W \otimes A$ is a bilinear map characterized by

$$T^A f(v_1 \otimes a_1, v_2 \otimes a_2) = f(v_1, v_2) \otimes a_1 a_2, \tag{8}$$

the product $a_1 a_2$ being in A.

2. Prolongation of infinitesimal actions

The following results can be deduced directly from the corresponding diagrams, [5], [10]. If G is a Lie group with the composition $\mu : G \times G \to G$, then

$$T^A\mu : T^AG \times T^AG \to T^AG$$

is also a Lie group. For every algebra homomorphism $\mu : A \to B$, the natural transformation

$$h_G : T^AG \to T^BG \tag{9}$$

is a group homomorphism. For every left action $l : G \times M \to M$,

$$T^Al : T^AG \times T^AM \to T^AM$$

is also a left action. If $\mathfrak{g}$ is the Lie algebra of G, then the Lie algebra of T^AG is $T^A\mathfrak{g} = \mathfrak{g} \otimes A$ and the bracket in $\mathfrak{g} \otimes A$ is the T^A-prolongation of the bracket in $\mathfrak{g}$, [10]. So (8) implies

$$[v_1 \otimes a_1, v_2 \otimes a_2]_{\mathfrak{g}\otimes A} = [v_1, v_2]_{\mathfrak{g}} \otimes a_1a_2. \tag{10}$$

The Weil algebra corresponding to the tangent functor T is $\mathbb{D}^1_1 = \mathbb{D}$. The exchange algebra isomorphism $\varkappa^A : A \otimes \mathbb{D} \to \mathbb{D} \otimes A$ induces a natural equivalence

$$\varkappa^A_M : T^ATM \to TT^AM$$

with the following property. For every vector field $\xi : M \to TM$, its functorial prolongation $T^A\xi : T^AM \to T^ATM$ and the flow prolongation $\mathcal{T}^A\xi : T^AM \to TT^AM$ satisfy

$$\varkappa^A_M \circ T^A\xi = \mathcal{T}^A\xi. \tag{11}$$

The infinitesimal action $\lambda : \mathfrak{g} \times M \to TM$ of l is defined by

$$\lambda = Tl \circ (i_G \times 0_M), \tag{12}$$

where $i_G : \mathfrak{g} \to TG$ or $0_M : M \to TM$ is the canonical injection or the zero section, respectively. We write $\lambda(v) = \lambda(v, -) : M \to TM$ for the fundamental vector field on M determined by $v \in \mathfrak{g}$. The infinitesimal action of T^Al, which will we denoted by

$$\mathcal{T}^A\lambda : T^A\mathfrak{g} \times T^AM \to TT^AM\,, \tag{13}$$

is of the form, [7],

$$\mathcal{T}^A\lambda = \varkappa_M^A \circ T^A\lambda. \tag{14}$$

Consider the scalar multiplication

$$m : \mathbb{R} \times TM \to TM$$

of the tangent bundle. Applying T^A, using $T^A\mathbb{R} = A$ and adding the exchange map $\varkappa_M^A$, we obtain an action

$$\mathcal{T}^A m : A \times TT^AM \to TT^AM$$

of A on the tangent bundle of T^AM, [10]. To simplify the notation, we shall write

$$\mathcal{T}^A m(a, Z) = aZ\,, \qquad a \in A,\ Z \in TT^AM. \tag{15}$$

Every $v \otimes a \in \mathfrak{g} \otimes A$ defines the fundamental vector field $\mathcal{T}^A\lambda(v \otimes a)$ on T^AM. On the other hand, $\lambda(v)$ is a vector field on M and we can construct its flow prolongation $\mathcal{T}^A\big(\lambda(v)\big)$. In [7] it is deduced

$$\mathcal{T}^A\lambda(v \otimes a) = a\mathcal{T}^A\big(\lambda(v)\big). \tag{16}$$

Consider the case $M = V$ is a vector space. Then $TV = V \times V$ and the first component of $\lambda : \mathfrak{g} \times V \to V \times V$ is the product projection $\mathfrak{g} \times V \to V$. The second component will be denoted by

$$\widetilde{\lambda} : \mathfrak{g} \times V \to V.$$

Since $T^AV = V \otimes A$ is also a vector space, we have $T^ATV = V \otimes A \times V \otimes A$ and $TT^AV = V \otimes A \times V \otimes A$. Under these identifications, $\varkappa_V^A$ is the identity of $V \otimes A \times V \otimes A$.

For the infinitesimal action $\mathcal{T}^A\lambda : T^A\mathfrak{g} \times T^AV \to TT^AV$, we have introduced

$$\widetilde{\mathcal{T}^A\lambda} : T^A\mathfrak{g} \times T^AV \to T^AV.$$

Then our previous results yield

Proposition 1. *We have*

$$\widetilde{\mathcal{T}^A\lambda} = T^A\widetilde{\lambda} : T^A\mathfrak{g} \times T^AV \to T^AV.$$

In particular, let l be a linear action of G on vector space V, so that $\widetilde{\lambda}$ is the classical representation of Lie algebra $\mathfrak{g}$ on V. Hence $\widetilde{\lambda}$ is a bilinear map and (8) implies that $\widetilde{\mathcal{T}^A\lambda}$ is of the form

$$\widetilde{\mathcal{T}^A\lambda}(v \otimes a, z \otimes b) = \widetilde{\lambda}(v, z) \otimes ab, \tag{17}$$

where the product ab is in A.

Next we present a general construction of a semidirect group product and of a related action, whose special cases will be used later. Let K be a Lie group and $D : K \to \mathrm{Aut}\, A$ be a group homomorphism, where $\mathrm{Aut}\, A$ is the group of all algebra isomorphisms of A. For every group G, we have the semidirect product $W_D^A G = K \rtimes_D T^A G$ with the multiplication

$$(k_1, U_1)(k_2, U_2) = \big(k_1 k_2, T^A\mu(D(k_2^{-1})_G(U_1), U_2)\big),$$

$k_1, k_2 \in K$, $U_1, U_2 \in T^A G$. For every action $l : G \times M \to M$, we define $W_D^A l : W_D^A G \times T^A M \to T^A M$ by

$$W_D^A l\big((k, U), Z\big) = \big(D(k)\big)_M \big(T^A l(U, Z)\big), \tag{18}$$

$k \in K$, $U \in T^A G$, $Z \in T^A M$.

Proposition 2. (18) *is a left action.*

Proof: We have

$$\begin{aligned}
&W_D^A l\big((k_1, U_1), W_D^A l\big((k_2, U_2), Z\big)\big) \\
&= D(k_1)_M \big(T^A l(U_1, D(k_2)_M (T^A l(U_2, Z)))\big) \\
&= D(k_1)_M \big(T^A l(T^A\mu(U_1, D(k_2)_G(U_2), D(k_2)_M(Z)))\big) \\
&= D(k_1 k_2)_M \big(T^A l(T^A\mu(D(k_2^{-1})(U_1), U_2), Z\big).
\end{aligned}$$

□

As a vector space, the Lie algebra of $W_D^A G$ is

$$\mathrm{Lie}\,(W_D^A G) = \mathfrak{k} \times \mathfrak{g} \otimes A\,,$$

where $\mathfrak{k}$ is the Lie algebra of K. The infinitesimal action of $W_D^A l$ will be denoted by

$$\mathrm{Inf}\,(W_D^A l) : (\mathfrak{k} \times \mathfrak{g} \otimes A) \times T^A M \to TT^A M. \tag{19}$$

By linearity of (19) in $\mathrm{Lie}\,(W^A_D G)$, we can consider separately the cases

(20) $$\text{(a) } X_2 \in \mathfrak{g} \otimes A \quad \text{and} \quad \text{(b) } X_1 \in \mathfrak{k}.$$

The case (a) corresponds to the action $T^A l$, so that

$$\mathrm{Inf}\,(W^A_D l)\big((0, X_2), Z\big) = \mathcal{T}^A \lambda(X_2, Z).$$

Before discussing the case (b), we recall that every vector C of the Lie algebra $\mathfrak{Aut}\,A$ of Aut A determines a vertical vector field

$$C_M : T^A M \to TT^A M$$

as follows. We have $C = \frac{d}{dt}\big|_0 \gamma(t)$, $\gamma : \mathbb{R} \to \mathrm{Aut}\ A$ and for $Z \in T^A M$ we define

(21) $$C_M(Z) = \frac{d}{dt}\Big|_0 \big(\gamma(t)_M(Z)\big),$$

where $\gamma(t)_M : T^A M \to T^A M$ is the natural transformation determined by $\gamma(t) \in \mathrm{Aut}\ A$, [10].

Let $d : \mathfrak{k} \to \mathfrak{Aut}\,A$ be the tangent map of $D : K \to \mathrm{Aut}\ A$. Then (18) with $X_2 = 0$, $\frac{dk(0)}{dt} = X_1$ implies

$$\mathrm{Inf}\,(W^A_D l)\big((X_1, 0), Z\big) = h(X_1)_M(Z).$$

Combining (a) and (b), we deduce

Proposition 3. *We have*

$$\mathrm{Inf}\,(W^A_D l)\big((X_1, X_2), Z\big) = \mathcal{T}^A \lambda(X_2, Z) + d(X_1)_M(Z),$$

$X_1 \in \mathfrak{k}$, $X_2 \in \mathfrak{g} \otimes A$, $Z \in T^A M$.

Let $M = V$ be a vector space, so that $T^A V = V \otimes A$. Then (6) implies

(22) $$d(X)_V(z \otimes a) = z \otimes d(X)(a)\,, \qquad z \in V,\ a \in A,\ X \in \mathfrak{k}.$$

3. Weak principal bundles

A fibered manifold $p : K \to M$ is called a group bundle, if each fiber K_x, $x \in M$, is a Lie group and K is locally trivial in the following sense. There is a Lie group H and a neighbourhood U of every $x \in M$ such that $p^{-1}(U) \approx U \times H$.

Hence the group compositions form a base preserving morphism $\nu : K \times_M K \to K$. The product $M \times H$ is called the product group bundle.

Let Aut H be the Lie group of all automorphisms of the group H. Consider a principal bundle $P(M, G)$ and a group homomorphism $\varphi : G \to \mathrm{Aut}\ H$. This defines a left action $(g, h) \mapsto \varphi(g^{-1})(h)$ of G on H, so that we can construct the associated bundle $P[H]$. This is a group bundle, if we define

$$\{u, h_1\}\{u, h_2\} = \{u, h_1 h_2\}\,, \qquad u \in P,\ h_1, h_2 \in H.$$

Indeed, this definition is correct, for $\{ug, \varphi(g^{-1})(h_1)\}\{ug, \varphi(g^{-1})(h_2)\} = \{ug, \varphi(g^{-1})(h_1 h_2)\} = \{u, h_1 h_2\}$.

The following concept, which was introduced in [6], is very important for our prolongation theory.

Definition 2. A fibered manifold $Q \to M$ is called a weak principal bundle with structure group bundle $K \to M$, if we are given a base-preserving morphism $\varrho : Q \times_M K \to Q$ such that each group K_x acts simply transitively on the right on Q_x.

In particular, every principal bundle $P(M,G)$ is a weak principal bundle with respect to the product group bundle $M \times G$.

Consider a weak principal bundle $Q \to M$ with structure group bundle $P[H]$. We write simply

$$\varrho\big(v, \{u,h\}\big) = v\{u,h\}\,, \qquad v \in Q_x,\ u \in P_x,\ h \in H.$$

So we have

$$\big(v, \{u,h_1\}\big)\{u,h_2\} = v\{u,h_1h_2\}.$$

Consider the semidirect group product $G \rtimes_\varphi H$, i.e. $G \times H$ with the composition

$$(g_1,h_1)(g_2,h_2) = \big(g_1g_2, \varphi(g_2^{-1})(h_1)h_2\big).$$

Proposition 4. *If we define*

$$(u,v)(g,h) = \big(ug, v\{ug,h\}\big), \tag{23}$$

$(u,v) \in P \times_M Q$, $g \in G$, $h \in H$, then $P \times_M Q$ is a principal bundle with structure group $G \rtimes_\varphi H$.

Proof: We have

$$\begin{aligned}\big((u,v)(g_1,h_1)\big)(g_2,h_2) &= \big(ug_1, v\{ug_1,h_1\}\big)(g_2,h_2)\\ &= \big(ug_1g_2, v\{ug_1,h_1\}\{ug_1g_2,h_2\}\big)\\ &= \big(ug_1g_2, v\{ug_1g_2, \varphi(g_2^{-1})(h_1)h_2\}\big).\end{aligned}$$

□

4. The functor $F = (A, H, t)$

All fiber product preserving bundle functors on $\mathcal{FM}_m$ are characterized in [9]. We write $\underline{f}$ for the base map of an $\mathcal{FM}_m$-morphism f. The order of a bundle functor F on $\mathcal{FM}_m$ is said to be (q,s,r), $s \geq q \leq r$, if for every two morphisms f, g from $p : Y \to M$ into $\overline{Y}$,

$$j^q_y f = j^q_y g,\ j^s_y(f|Y_x) = j^s_y(g|Y_x),\ j^r_x \underline{f} = j^r_x \underline{g},\ x = p(y), y \in Y,$$

implies $Ff|F_yY = Fg|F_zY$. We say that r is the base order of F.

According to [9], every fiber product preserving bundle functor F on $\mathcal{FM}_m$ of base order r is identified with a triple (A, H, t) such that A is a Weil algebra, $H : G^r_m \to \mathrm{Aut}\ A$ is a group homomorphism and $t : \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism under the identification $G^r_m \approx \mathrm{Aut}\ \mathbb{D}^r_m$, $(g, X) \mapsto X \circ g^{-1}$, $g \in G^r_m$, $X \in \mathbb{D}^r_m$. For example, in the case $F = J^r$ we have $A = \mathbb{D}^r_m$, $H = \mathrm{id}_{G^r_m}$, $t = \mathrm{id}_{\mathbb{D}^r_m}$.

The functor $F = (A, H, t)$ can be reconstructed from these data as follows. In the product case of $M \times N \to M$,

$$F(M \times N) = P^r M[T^A N, H_N] \tag{24}$$

is the fiber bundle associated to $P^r M$ with standard fiber $T^A N$ and the action H_N of G^r_m on $T^A N$ determined by the natural transformations $H(g)_N : T^A N \to T^A N$, $g \in G^r_m$. In the general case of $p : Y \to M$, FY is the subset of $P^r M[T^A Y]$ formed by the equivalence classes $\{u, Z\}$ satisfying

$$t_M(u) = T^A p(Z) \in T^A M\,, \qquad u \in P^r M,\ Z \in T^A Y, \tag{25}$$

where $P^r M \subset T^r_m M$ and $t_M : T^r_m M \to T^A M$.

For an $\mathcal{FM}_m$-morphism $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$, we have

$$Ff = \{P^r \underline{f}, T^A f\},$$

where $P^r \underline{f} : P^r M \to P^r \overline{M}$ and $T^A f : T^A Y \to T^A \overline{Y}$ form a morphism of associated bundles. In particular, if $M = \mathbb{R}^m$, then $P^r \mathbb{R}^m = \mathbb{R}^m \times G^r_m$, so that

$$F(\mathbb{R}^m \times N) \approx \mathbb{R}^m \times T^A N. \tag{26}$$

Moreover, if $\overline{Y} = \mathbb{R}^m \times \overline{N}$ and $f = \mathrm{id}_{\mathbb{R}^m} \times \varphi$, $\varphi : N \to \overline{N}$, then

$$Ff = \mathrm{id}_{\mathbb{R}^m} \times T^A \varphi : \mathbb{R}^m \times T^A N \to \mathbb{R}^m \times T^A \overline{N}. \tag{27}$$

5. F-prolongation of some kinds of bundles

If $K \to M$ is a group bundle with composition $\nu : K \times_M K \to K$, then

$$F\nu : FK \times_M FK \to FK$$

is also a group bundle. This can be deduced by discussing the diagrams in question. But we can also use Section 4. We have locally $K = \mathbb{R}^m \times H$ and $FK = \mathbb{R}^m \times T^A H$, where $T^A H$ is a group. However, if $K = M \times H$ is a product group bundle, then (24) yields

$$F(M \times H) = P^r M[T^A H], \tag{28}$$

which is not the product group bundle in general.

Consider a weak principal bundle $\varrho : Q \times_M K \to Q$ with structure group bundle $K \to M$.

Proposition 5. *$F\varrho : FQ \times_M FK \to FQ$ is a weak principal bundle with structure group bundle $FK \to M$.*

Proof: This can be deduced by discussing the diagrams in question. But even here we can use local trivializations. Locally, $Q = \mathbb{R}^m \times H$ and we have the previous situation. □

Clearly, if Q is a principal bundle, then FQ need not be principal. The simplest example is the first jet prolongation J^1P of a principal bundle P.

If $D \to M$ is a vector bundle, we have already deduced that $FD \to M$ is a bundle of Abelian groups. Further, the multiplication by scalars can be interpreted as a base preserving morphism $m : (M \times \mathbb{R}) \times_M D \to D$. Applying F, we obtain

$$Fm : P^r M[A, H] \times FD \to FD. \tag{29}$$

But $\mathbb{R} \subset A$ in an H-invariant subspace, so that we can restrict (29) to a map $\mathbb{R} \times FD \to FD$, which is the multiplication by scalars of the vector bundle $FD \to M$.

We recall that a Lie algebra bundle $p : D \to M$ is a bundle of Lie algebras locally trivial in that sense that there exists a Lie algebra C and a neighbourhood U of every point $x \in M$ such that $p^{-1}(U) \approx U \times C$. Clearly, if $D \to M$ is a Lie algebra bundle with the bracket $b : D \times_M D \to D$, then $FD \to M$ is also a Lie algebra bundle with the bracket Fb.

Every group bundle $K \to M$ induces fiberwise the Lie algebra bundle $\mathrm{Lie}\,(K)$. We have

$$F\big(\mathrm{Lie}\,(K)\big) = \mathrm{Lie}\,(FK). \tag{30}$$

Indeed, locally $K \approx \mathbb{R}^m \times G$ and $F\big(\mathrm{Lie}\,(K)\big) = \mathbb{R}^m \times T^A\mathfrak{g} = \mathrm{Lie}\,(FK)$.

6. The Lie algebroid version of $P \times_M Q$

We start with a general result. Consider a Lie algebroid $(E, q) \to M$ with the anchor map $q : E \to TM$ and the bracket $[\![\ ,\ ]\!] : C^\infty E \times C^\infty E \to C^\infty E$, [11]. The action φ of E on a vector bundle $D \to M$ is an $\mathbb{R}$-bilinear map $\varphi : C^\infty E \times C^\infty D \to C^\infty D$ satisfying, for every $\varrho \in C^\infty E$, $\sigma \in C^\infty D$ and $f : M \to \mathbb{R}$,

(i) $\varphi(f\varrho, \sigma) = f\varphi(\varrho, \sigma)$,
(ii) $\varphi(\varrho, f\sigma) = f\varphi(\varrho, \sigma) + (q\varrho)(f)\sigma$,
(iii) $\varphi([\![\varrho_1, \varrho_2]\!], \sigma) = \varphi\big(\varrho_1, \varphi(\varrho_2, \sigma)\big) - \varphi\big(\varrho_2, \varphi(\varrho_1, \sigma)\big)$.

Assume D is a Lie algebra bundle, whose bracket will be denoted by $[\ ,\ ] : C^\infty D \times C^\infty D \to C^\infty D$.

Definition 3. We say that φ is an action by derivations, if the following condition holds

$$\text{(iv)} \qquad \varphi\big(\varrho, [\sigma_1, \sigma_2]\big) = \big[\varphi(\varrho, \sigma_1), \sigma_2\big] + \big[\sigma_1, \varphi(\varrho, \sigma_2)\big].$$

On $E \times_M D$ we define the anchor map $\overline{q}(x, y) = q(x)$ and the bracket $\{\ ,\ \} : C^\infty(E \times_M D) \times C^\infty(E \times_M D) \to C^\infty(E \times_M D)$ by

$$\{(\varrho_1, \sigma_1), (\varrho_2, \sigma_2)\} = \big([\![\varrho_1, \varrho_2]\!], \varphi(\varrho_1, \sigma_2) - \varphi(\varrho_2, \sigma_1) + [\sigma_1, \sigma_2]\big). \tag{31}$$

Proposition 6. *$(E \times_M D, \overline{q}) \to M$ with the bracket $\{\ ,\ \}$ is a Lie algebroid.*

Proof: First we deduce the Jacobi identity. We have

$$\begin{aligned}\big\{\{(\varrho_1, \sigma_1), (\varrho_2, \sigma_2)\}, (\varrho_3, \sigma_3)\big\} = \big([\![\, [\![\varrho_1, \varrho_2]\!], \varrho_3]\!], \varphi\big([\![\varrho_1, \varrho_2]\!], \sigma_3\big) \\ -\varphi\big(\varrho_3, \varphi(\varrho_1, \sigma_2)\big) - \varphi(\varrho_2, \sigma_1) + [\sigma_1, \sigma_2]\big) + \big[\varphi(\varrho_1, \sigma_2) - \varphi(\varrho_2, \sigma_1) \\ +[\sigma_1, \sigma_2], \sigma_3\big]\big).\end{aligned}$$

The sum of the cyclic permutations in 1, 2, 3 should vanish. The terms corresponding to $[\![\, [\![\varrho_1, \varrho_2]\!], \varrho_3]\!]$ and $[\,[\sigma_1, \sigma_2], \sigma_3]$ vanish by the Jacobi identities for E and D. The rest is

$$\begin{aligned}\varphi\big(\varrho_1, \varphi(\varrho_2, \varrho_3)\big) - \varphi\big(\varrho_2, \varphi(\varrho_1, \sigma_3)\big) - \varphi\big(\varrho_3, \varphi(\varrho_1, \sigma_2)\big) + \varphi\big(\varrho_3, \varphi(\varrho_2, \sigma_1)\big) \\ -\big[\varphi(\varrho_3, \sigma_1), \sigma_2\big] - \big[\sigma_1, \varphi(\varrho_3, \sigma_2)\big] + \big[\varphi(\varrho_1, \sigma_2), \sigma_3\big] - \big[\varphi(\varrho_2, \sigma_1), \sigma_3\big]\end{aligned}$$

and the sum of the cyclic permutations vanishes algebraically. Further we have

$$\begin{aligned}\big\{(\varrho_1, \sigma_1), (f\varrho_2, f\sigma_2)\big\} &= \big(f[\![\varrho_1, \varrho_2]\!] + (q\varrho_1)(f)\varrho_2, f\varphi(\varrho_1, \sigma_2) \\ &\quad + (q\varrho_1)(f)\sigma_2 - f\varphi(\varrho_2, \sigma_1) + [\sigma_1, f\sigma_2]\big) \\ &= f\big\{(\varrho_1, \sigma_1), (\varrho_2, \sigma_2)\big\} + (q\varrho_1)(f)(\varrho_2, \sigma_2).\end{aligned}$$ □

The Lie algebroid LP of a principal bundle P is defined by the right-invariant vector fields on P. Every vector bundle associated to P is endowed with a canonical action of LP, [8].

Consider the group homomorphism $\varphi : G \to \operatorname{Aut} H$ from Section 3 and the tangent action $\Phi : G \to \operatorname{Aut} \mathfrak{h}$. Hence $D = P[\mathfrak{h}, \Phi]$ is a Lie algebra bundle. Using the local description of the canonical action, [8], and the standard theory of semidirect products of Lie groups and Lie algebras, [1], we obtain

Lemma 1. *The canonical action of LP on D is by derivations.* □

Consider the principal bundle P and a weak principal bundle Q with associated group bundle $P[H]$. Then $P \times_M Q$ is a principal bundle and $LP \times_M D$ is a Lie algebroid. Using local trivializations, we deduce directly

Proposition 7. *The Lie algebroid of $P \times_M Q$ is $LP \times_M D$.* □

7. The principal bundle $W^F P$ and the associated bundles

We know that FP is a weak principal bundle with structure group bundle $F(M\times G)=P^rM[T^AG,H_G]$. By Sections 2 and 3,

$$W^F P := P^r M \times_M FP$$

is a principal bundle with structure group $W^A_H G = G^r_m \rtimes T^A G$. This bundle was constructed in a direct way in [3]. In particular, Proposition 6 determines the Lie algebroid of $W^F P$, which was described in another way in [8].

The most important property of $W^F P$ is that for every associated bundle $Y=P[S,l]$, FY is an associated bundle

$$FY = W^F P[T^A S, W^A_H l],$$

where $W^A_H l$ is a special case of (18), i.e.

$$W^A_H l\big((g,U),Z\big) = H(g)_S\big(T^A l(U,Z)\big), \tag{32}$$

$g\in G^r_m$, $U\in T^AG$, $Z\in T^AS$. According to Section 2, its infinitesimal action

$$\mathrm{Inf}\,(W^A_H l) : (\mathfrak{g}^r_m \times \mathfrak{g}\otimes A)\times T^A S \to TT^A S$$

is of the form

$$\mathrm{Inf}\,(W^A_H l)\big((X,U),Z\big) = \mathcal{T}^A\lambda(U,Z) + h(X)_S(Z), \tag{33}$$

$X\in\mathfrak{g}^r_m$, $U\in\mathfrak{g}\otimes A$, $Z\in T^AS$.

If S is a vector space and l is a linear action, then (17) and (33) imply

$$\mathrm{Inf}\,(W^A_H l)(X, v\otimes a, z\otimes b) = \widetilde{\lambda}(v,z)\otimes ab + z\otimes h(X)(b).$$

This formula plays an interesting role in the theory of F-prolongations of Lie algebroids, [8].

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