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On free modes

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Abstract. We prove a theorem describing the equational theory of all modes of a fixed type. We use this result to show that a free mode with at least one basic operation of arity at least three, over a set of cardinality at least two, does not satisfy identities selected by Á. Szendrei in *Identities satisfied by convex linear forms*, Algebra Universalis **12** (1981), 103–122, that hold in any subreduct of a semimodule over a commutative semiring. This gives a negative answer to the question raised by A. Romanowska: Is it true that each mode is a subreduct of some semimodule over a commutative semiring?

Keywords: modes, Szendrei modes, subreducts, semimodules, equational theory

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1. Introduction

Modes are entropic and idempotent algebras (properties defined later). The class of these algebras is very wide. It contains such structures as affine spaces, semilattices or convex sets. Most of them are known to be subreducts of appropriate semimodules over commutative semirings. This suggests the following question raised by A. Romanowska (see [7, p. 543] and [6, Problem 8.11]):

Problem 1. Is it true that each mode is a subreduct of some semimodule over a commutative semiring?

A positive answer for groupoid modes was given in [4]. A slightly stronger result for Szendrei modes (modes satisfying the Szendrei identities) with one n-ary basic operation, for any $n \geq 2$, may easily be deduced from [5] (where in fact the authors consider a much more general situation). In this paper we show that not all modes are Szendrei modes. This ultimately gives a negative answer to Romanowska's question. We do it in two steps. First we prove Theorem 3: if a certain identity holds in the class of all modes of a fixed type, then there exists a proof of it, using entropicity and idempotency, having a certain quite transparent form. Next, using this result, we show that the Szendrei identities cannot be derived from entropicity and idempotency (Theorem 7).

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2. Basic concepts

Let Ω be a set of operation symbols, and let $\tau\colon \Omega \to \mathbb{Z}^+$ be a type function. We do not allow constants, because modes with constants are just trivial algebras. For $\omega, \mu, \nu \in \Omega$ we shall consider the *idempotent identities*

$$(\iota_\omega) \qquad \omega(x, \dots, x) = x,$$

the *entropic identities*

$$\begin{aligned}(\varepsilon_{\mu,\nu}) \quad & \mu(\nu(x_1^1, \dots, x_{\tau(\nu)}^1), \dots, \nu(x_1^{\tau(\mu)}, \dots, x_{\tau(\nu)}^{\tau(\mu)})) \\ & = \nu(\mu(x_1^1, \dots, x_1^{\tau(\mu)}), \dots, \mu(x_{\tau(\nu)}^1, \dots, x_{\tau(\nu)}^{\tau(\mu)}))\end{aligned}$$

and the *Szendrei identities*

$$\begin{aligned}(\sigma_\omega^{i,j}) \quad & \omega(\omega(x_1^1, \dots, x_{\tau(\omega)}^1), \dots, \omega(x_1^{\tau(\omega)}, \dots, x_{\tau(\omega)}^{\tau(\omega)})) \\ & = \omega(\omega(\pi_j^i(x_1^1), \dots, \pi_j^i(x_{\tau(\omega)}^1)), \dots, \omega(\pi_j^i(x_1^{\tau(\omega)}), \dots, \pi_j^i(x_{\tau(\omega)}^{\tau(\omega)}))),\end{aligned}$$

where $1 \le i < j \le \tau(\omega)$ and π_j^i is a permutation of the variables which transposes x_j^i with x_i^j and leaves the other variables fixed. As an example, $\sigma_\omega^{1,2}$, for $\tau(\omega) = 3$ is given by

$$\begin{aligned}& \omega(\omega(x_1^1, x_2^1, x_3^1), \omega(x_1^2, x_2^2, x_3^2), \omega(x_1^3, x_2^3, x_3^3)) \\ & = \omega(\omega(x_1^1, x_1^2, x_3^1), \omega(x_2^1, x_2^2, x_3^2), \omega(x_1^3, x_2^3, x_3^3)).\end{aligned}$$

Note that if $\tau(\omega) = 2$, then $\sigma_\omega^{1,2}$ and $\varepsilon_{\omega,\omega}$ coincide. Define $\iota = \{\, \iota_\omega \mid \omega \in \Omega \}$, $\varepsilon = \{\, \varepsilon_{\mu,\nu} \mid \mu, \nu \in \Omega \}$ and $\sigma = \{\, \sigma_\omega^{i,j} \mid \omega \in \Omega,\ 1 \le i < j \le \tau(\omega) \}$. An algebra (A, Ω) is called a *mode* if $(A, \Omega) \models \iota \cup \varepsilon$, and a *Szendrei mode* if additionally $(A, \Omega) \models \sigma$. The fundamental monograph on modes is [7] and the most resent survey is [6]. Szendrei modes were investigated in [8], and later in [2].

Semirings are intuitively "rings without subtraction." Similarly, semimodules are "modules without subtraction." They are like modules, but with a commutative monoid instead of an abelian group. The precise definitions are as follows: A *commutative semiring* is an algebra $(R, +, 0, \cdot, 1)$, where $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids, $0x = 0$ and $\cdot$ distributes over $+$. A *semimodule* over a commutative semiring $(R, +, 0, \cdot, 1)$ is an algebra $(M, +, 0, R)$ where the unary operations determined by elements of R are endomorphisms of the commutative monoid $(M, +, 0)$ and moreover

$$\begin{aligned} 1m &= m, \\ 0m &= 0, \\ (r_1 \cdot r_2)m &= r_1(r_2 m), \\ (r_1 + r_2)m &= r_1 m + r_2 m. \end{aligned}$$

For further information on semirings and semimodules, we refer the reader to [3].

A *reduct* of an algebra (A,Ω) is an algebra (A,Ψ) such that all the operations in Ψ are term operations of (A,Ω). A *subreduct* is a subalgebra of a reduct.

Now the question of A. Romanowska may be formulated in the following way: Is it true that for any mode (A,Ω) there exists a semimodule $(M,+,0,R)$ such that $A\subseteq M$ and for any $\omega\in\Omega$ there are $r_1^\omega,\dots,r_{\tau(\omega)}^\omega\in R$ such that

$$\omega(a_1,\dots,a_{\tau(\omega)})=\sum_{i=1}^{\tau(\omega)} r_i^\omega a_i \ \ ?$$

Note that all subreducts of a semimodule satisfy the Szendrei identities. Thus if the answer is "yes", all modes satisfy the Szendrei identities.

Let X be a countable infinite set of variables. The absolutely free algebra of type τ (free in the variety of all τ-algebras) over the set X is denoted by $(F_\tau(X),\Omega)$. Its elements are τ-terms with variables in X. A term is *unary* if only one variable occurs in it. A term is *linear* if each of its variables occurs exactly once in it. An identity $t=s$ is *linear* if both terms t and s are linear.

In this note we use a common convention and use the same symbol to denote a term and the term operation corresponding to it. We apply this convention mostly to the absolutely free algebra $(F_\tau(X),\Omega)$. In particular for any term $p\in F_\tau(X)$ we have $p(x_1,\dots,x_n)=p$, where $x_1,\dots,x_n$ are variables occurring in p. In this equality symbol p denotes on the right hand side simply a term and on the left hand side a term function.

Note that for any $p\in F_\tau(X)$ there are a linear term t and not necessarily mutually distinct variables $x_1,\dots,x_n$ such that $p=t(x_1,\dots,x_n)$. Obviously the set $\{x_1,\dots,x_n\}$ is equal to the set of variables occurring in p. In other words $p=f(t)$ for a certain endomorphism f of $(F_\tau(X),\Omega)$ such that $f(X)\subseteq X$. In such situation we say that *p is obtained from a linear term t.*

It is well known that equational theories with variables in the set X are just fully invariant congruences of $(F_\tau(X),\Omega)$. An equational theory over X with basis $E\subseteq F_\tau(X)^2$ is denoted by $\mathrm{Th}(E)$. An equational theory is *linear* if it has a basis consisting of linear identities. The following lemma, proved in [5, Theorem 1.1], will be useful in the sequel.

Lemma 2. *Let $T\subseteq F_\tau(X)^2$ be a linear equational theory. Then for any identity $t=s$ in T there is a linear identity $t'=s'$ in T and an endomorphism $f\colon(F_\tau(X),\Omega)\to(F_\tau(X),\Omega)$ such that $f(t')=t$, $f(s')=s$ and $f(X)\subseteq X$.*

For a relation $R\subseteq F_\tau(X)^2$ let us denote by R^∞ its transitive and reflexive closure and by $R^\dagger$ its converse. By $R\circ S$ we denote the composition of relations.

3. Equational theory of modes

For $p,q\in F_\tau(X)$ let $(p,q)\in I^{\rightarrow}$ if q results from p by replacing one occurrence of a variable x by $\omega(x,\dots,x)$. One may formalize this as follows. Let

$p = t(x_1, \dots, x_n)$ be obtained from a linear term t. Then $(p, q) \in I^{\rightarrow}$ if $q = t(x_1, \dots, x_{i-1}, \omega(x_i, \dots, x_i), x_{i+1}, \dots, x_n)$ for some $i \in \{1, \dots, n\}$. Moreover $(p, q) \in (I^{\rightarrow})^{\infty}$ whenever there are unary terms $u_1 = u_1(x_1), \dots, u_n = u_n(x_n)$ such that $q = t(u_1, \dots, u_n)$. Put $I^{\leftarrow} = (I^{\rightarrow})^{\dagger}$.

Theorem 3. $\mathrm{Th}(\iota \cup \varepsilon) = (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}$.

PROOF: Put $\Gamma = (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}$. The inclusion $\Gamma \subseteq \mathrm{Th}(\iota \cup \varepsilon)$ is obvious. In order to prove that these relations are equal it is sufficient to note that $\iota \cup \varepsilon \subseteq \Gamma$ and to show that Γ is a fully invariant congruence of $(F_\tau(X), \Omega)$. We do this in few steps.

First we prove that

$$(I^{\leftarrow})^{\infty} \circ (I^{\rightarrow})^{\infty} \subseteq (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}. \tag{1}$$

Let $p \, (I^{\leftarrow})^{\infty} \, q \, (I^{\rightarrow})^{\infty} \, r$. Assume that $q = t(x_1, \dots, x_n)$ is obtained from a linear term t. Let $u_1 = u_1(x_1), \dots, u_n = u_n(x_n), v_1 = v_1(x_1), \dots, v_n = v_n(x_n)$ be unary terms such that $p = t(u_1, \dots, u_n)$ and $r = t(v_1, \dots, v_n)$. Then

$$\begin{aligned} p = {} & t(u_1, \dots, u_n) \\ (I^{\rightarrow})^{\infty} \; & t(u_1(v_1), \dots, u_n(v_n)) \\ \mathrm{Th}(\varepsilon) \; & t(v_1(u_1), \dots, v_n(u_n)) \\ (I^{\leftarrow})^{\infty} \; & t(v_1, \dots, v_n) \\ = {} & r. \end{aligned}$$

Hence $(p, r) \in (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}$. Next assume that an identity $p = q$ belongs to $\mathrm{Th}(\varepsilon)$. Then, by Lemma 2, we have $p = t(x_1, \dots, x_n)$ and $q = s(x_1, \dots, x_n)$ for some linear identity $t = s \in \mathrm{Th}(\varepsilon)$. Hence

$$\begin{aligned} p \quad I^{\rightarrow} \quad & t(x_1, \dots, x_{i-1}, \omega(x_i, \dots, x_i), x_{i+1}, \dots, x_n) \\ & \qquad \mathrm{Th}(\epsilon) \quad s(x_1, \dots, x_{i-1}, \omega(x_i, \dots, x_i), x_{i+1}, \dots, x_n) \quad I^{\leftarrow} \quad q. \end{aligned}$$

This implies

$$\mathrm{Th}(\varepsilon) \circ I^{\rightarrow} \subseteq I^{\rightarrow} \circ \mathrm{Th}(\varepsilon). \tag{2}$$

Dually we have

$$I^{\leftarrow} \circ \mathrm{Th}(\varepsilon) \subseteq \mathrm{Th}(\varepsilon) \circ I^{\leftarrow}. \tag{3}$$

It follows first by (1) and then by (2) and (3) that

$$\begin{aligned} & \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty} \circ (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \\ & \qquad \subseteq \mathrm{Th}(\varepsilon) \circ (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \\ & \qquad\qquad \subseteq (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty} \end{aligned}$$

and hence

$$(I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty} \circ (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty} \subseteq (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}.$$

This means that Γ is transitive. Further

$$\begin{aligned}((I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty})^{\dagger} &= ((I^{\leftarrow})^{\infty})^{\dagger} \circ (\mathrm{Th}(\varepsilon))^{\dagger} \circ ((I^{\rightarrow})^{\infty})^{\dagger} \\ &= (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}.\end{aligned}$$

Thus Γ is symmetric. As Γ is obviously reflexive, it is an equivalence relation.

Now note that $(I^{\rightarrow})^{\infty}$ and $(I^{\leftarrow})^{\infty}$ are subalgebras of $(F_{\mathcal{T}}(X)^2, \Omega)$. This is also true for $\mathrm{Th}(\varepsilon)$, since it is a fully invariant congruence of $(F_{\mathcal{T}}(X), \Omega)$. Hence $\Gamma = (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(\varepsilon) \circ (I^{\leftarrow})^{\infty}$ is a subalgebra of $(F_{\mathcal{T}}(X)^2, \Omega)$, too. This means that Γ is a congruence of $(F_{\mathcal{T}}(X), \Omega)$.

It remains to show that Γ is fully invariant. So let $(p, q) \in \Gamma$ and f be an endomorphism of $(F_{\mathcal{T}}(X), \Omega)$. We would like to show that $(f(p), f(q)) \in \Gamma$. Let $p = t(x_1, \dots, x_n)$ be obtained from a linear term t and $q = s(x_1, \dots, x_n)$ be obtained from a linear term s. By the assumption there are unary terms $u_1 = u_1(x_1), \dots, u_n = u_1(x_n), v_n = v_1(x_1), \dots, v_n = v_n(x_n)$ such that the identity $t(u_1, \dots, u_n) = s(v_1, \dots, v_n)$ belongs to $\mathrm{Th}(\varepsilon)$. Put $f(x_i) = r_i = r_i(x_1^i, \dots, x_{k_i}^i)$. Then

$$\begin{array}{rl} f(p) = & t(r_1, \dots, r_n) \\ (I^{\rightarrow})^{\infty} & t(r_1(u_1(x_1^1), \dots, u_1(x_{k_1}^1)), \dots, r_n(u_n(x_1^n), \dots, u_n(x_{k_n}^n))) \\ \mathrm{Th}(\varepsilon) & t(u_1(r_1), \dots, u_n(r_n)) \\ \mathrm{Th}(\varepsilon) & s(v_1(r_1), \dots, v_n(r_n)) \\ \mathrm{Th}(\varepsilon) & s(r_1(v_1(x_1^1), \dots, v_1(x_{k_1}^1)), \dots, r_n(v_n(x_1^n), \dots, v_n(x_{k_n}^n))) \\ (I^{\leftarrow})^{\infty} & s(r_1, \dots, r_n) \\ = & f(q). \end{array}$$

The proof is finished. □

In fact our proof allows us to formulate a slightly stronger result. For an operation $\omega \in \Omega$ and an unary term t consider the identity

$$(\varepsilon_{t,\omega}) \qquad t(\omega(x_1, \dots, x_{\tau(\omega)})) = \omega(t(x_1), \dots, t(x_{\tau(\omega)})).$$

Put $\varepsilon^* = \{\varepsilon_{t,\omega} \mid t \text{ is an unary } \tau\text{-term and } \omega \in \Omega\}$.

Theorem 4. *Let $L \subseteq F_{\mathcal{T}}(X)^2$ be a set of linear identities such that $\varepsilon^* \subseteq \mathrm{Th}(L)$. Then $\mathrm{Th}(\iota \cup L) = (I^{\rightarrow})^{\infty} \circ \mathrm{Th}(L) \circ (I^{\leftarrow})^{\infty}$.*

But we shall not use this in the subsequent considerations.

4. Non-Szendrei modes

The main result of this section, Theorem 7, shows that any Szendrei identity $\sigma_\omega^{i,j}$, where $\tau(\omega) > 2$, is not a consequence of the idempotent and entropic identities. We start with some auxiliary results.

Lemma 5. *Let $\Psi \subseteq \Omega$ and $\rho = \tau|_\Psi$. If t and s are ρ-terms, and the identity $t = s$ is satisfied in every τ-mode, then it is satisfied in every ρ-mode.*

Proof: Let (A, Ψ) be a mode. We enrich this algebra by adding new operations from $\Omega - \Psi$. For $\mu \in \Omega - \Psi$ we put $\mu(a_1, \dots, a_{\tau(\mu)}) = a_1$. Then (A, Ω) is a τ-mode and by the assumption it must satisfy the identity $t = s$. Hence (A, Ψ) satisfies it as well. □

For the moment we switch our attention to linear identities involving only one operation symbol which can be derived from the corresponding entropic law. Assume that $\Omega = \{\omega\}$ and $\tau(\omega) = n > 1$. For a linear term, define the addresses of variables occurring in it as words over the alphabet $\{1, \dots, n\}$, given recursively by $a(x, x) = \epsilon$, the empty word, and further by $a(\omega(t_1, \dots, t_n), x) = ia(t_i, x)$ if x is a variable of t_i. A linear term t is *full* if all its variables have addresses of the same length. This length is called the *depth* of t.

For a natural number m let T_m be the set of all linear full terms with depth m over the set $A_m = \{1, \dots, n\}^m = \{\alpha_1, \dots, \alpha_N\}$, where $N = n^m$. We identify the set A_m with the set of words over the alphabet $\{1, \dots, n\}$ of length m. Obviously there is a bijection from T_m onto the set $A_m!$ of all permutations of A_m given by

$$t(\alpha_1, \dots, \alpha_N) \mapsto (\alpha_i \mapsto a(t, \alpha_i)).$$

For a word $\beta \in A_k$, where $0 \le k < m - 1$, we define a permutation e^β of A_m by

$$e^\beta :\ \alpha \mapsto \begin{cases} \beta i j \gamma & \text{if } \alpha = \beta j i \gamma, \\ \alpha & \text{otherwise,} \end{cases}$$

where $i, j \in \{1, \dots, n\}$ and γ is a word of length $n - k - 2$. Note that each such permutation is an involution. Let $(G_m, \circ, {}^{-1}, 1)$ be the subgroup of the group $(A_m!, \circ, {}^{-1}, 1)$ generated by all permutations e^β, where $\beta \in A_k$ and $k < m - 1$.

Lemma 6. *Let $s \in T_m$ be a term such that for all $\alpha \in A_m$ we have $a(s, \alpha) = \alpha$. For $t \in T_m$ the identity $s = t$ is a consequence of $\varepsilon_{\omega,\omega}$ if and only if φ: $\alpha \mapsto a(t, \alpha)$ belongs to G_m.*

Theorem 7. *A free mode of type τ over a set of cardinality at least two does not satisfy the Szendrei identity $\sigma_\omega^{i,j}$, for $\tau(\omega) = n > 2$.*

Proof: Let $x, y \in X$ be two distinct variables and let

$$p = \omega(\omega(x, \dots, x), \omega(y, x, \dots, x), \omega(y, x, \dots, x), \omega(x, \dots, x), \dots, \omega(x, \dots, x)),$$
$$q = \omega(\omega(x, y, x, \dots, x), \omega(x, \dots, x), \omega(y, x, \dots, x), \omega(x, \dots, x), \dots, \omega(x, \dots, x)).$$

Suppose that $(p,q)\in\mathrm{Th}(\iota\cup\varepsilon)$. Let t be a linear term such that $p=t(y,y,x,x,\dots,x)$ and $q=t(y,x,y,x,\dots,x)$. By Theorem 3 there are unary terms $u_1=u_1(y), u_2=u_2(y), u_3=u_3(x),\dots,u_{n^2}=u_{n^2}(x)$ and $v_1=v_1(y), v_2=v_2(x), v_3=v_3(y), v_4=v_4(x),\dots,v_{n^2}=v_{n^2}(x)$, such that the identity $t(u_1,\dots,u_{n^2})=t(v_1,\dots,v_{n^2})$ follows from entropicity. By Lemma 5 we may assume that all u_i and v_j contain only the operation symbol ω. Further by Lemma 2 we may assume that all these terms are full and have the same depth. This means that they are equal to $w(x)$ or to $w(y)$, where w is a full unary term of depth $m-2$, for certain $m>1$, involving only operation ω.

The theory $\mathrm{Th}(\varepsilon)$ is linear, hence by Lemma 2 there exist terms $s'(\alpha_1,\dots,\alpha_N)$ and $t'(\alpha_1,\dots,\alpha_N)$ in the set T_m and a mapping $f\colon\{\alpha_1,\dots,\alpha_N\}\to\{x,y\}$ such that

$$
\begin{aligned}
p' &:= t(w(y),w(y),w(x),w(x),\dots,w(x)) = s'(f(\alpha_1),\dots,f(\alpha_N)),\\
q' &:= t(w(y),w(x),w(y),w(x),\dots,w(x)) = t'(f(\alpha_1),\dots,f(\alpha_N))
\end{aligned}
$$

and $(t',s')\in\mathrm{Th}(\varepsilon)$. Without loss of generality we may assume that for all $\alpha\in A_m$ we have $a(s',\alpha)=\alpha$. Let $\varphi\in A_m!$ be the corresponding bijection of Lemma 6. Then

$$t'(\alpha_1,\dots,\alpha_N)=s'(\varphi^{-1}(\alpha_1),\dots,\varphi^{-1}(\alpha_N)).$$

Let $t[\gamma]$ denote a subterm of the term t under the address γ. We know that $p'[21\dots1]=y$, whence $q'[\varphi(21\dots1)]=y$. So

$$\varphi(21\dots1)\in\{12\gamma\mid\gamma\in A_{m-2}\}\cup\{31\gamma\mid\gamma\in A_{m-2}\}.$$

On the other hand

$$\varphi(21\dots1)\in\{21\dots1,121\dots1,\dots,1\dots12\}.$$

Thus we get

$$\varphi(21\dots1)=121\dots1.$$

Further note that for any permutation $\beta\in A_{m-2}$ and $0\le k\le m-1$ if

$$e^{\beta}(\overbrace{1\dots1}^{k\text{ times}}\ 21\dots1)=\overbrace{1\dots1}^{k'\text{ times}}\ 21\dots1,$$

then

$$e^{\beta}(\overbrace{1\dots1}^{k\text{ times}}\ 31\dots1)=\overbrace{1\dots1}^{k'\text{ times}}\ 31\dots1.$$

Thus, because φ is a composition of such permutations, we get

$$\varphi(31\dots1)=131\dots1.$$

This would mean that $q'[131\dots1]=y$, but we have $q'[131\dots1]=x$, a contradiction.

We have shown that $\sigma_\omega^{1,2}$ cannot hold in any free mode over the set of cardinality at least two. Similar considerations show the same for any other Szendrei identity $\sigma_\omega^{i,j}$. $\square$

Corollary 8. *No nontrivial free mode having at least one basic operation of arity at least* 3 *is a subreduct of a semimodule.*

Remark 9. After reading a preliminary version of this article, David Stanovský from Charles University in Prague was able to construct a finite example of a non-Szendrei mode. With his permission, we present it here.

Let ω be a ternary operation on the set $\{0, 1, 2\}$ given by

$$\omega(x, y, z) = \begin{cases} 2 - z & \text{if } \ x = y = 1, \\ z & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} &\omega(\omega(x_1, x_2, x_3), \omega(y_1, y_2, y_3), \omega(z_1, z_2, z_3)) \\ &\qquad = \begin{cases} 2 - z_3 & \text{if } \ x_3 = y_3 = 1 \ \text{ and } \ (z_1, z_2) \neq (1, 1) \\ & \text{or } \ z_1 = z_2 = 1 \ \text{ and } \ (x_3, y_3) \neq (1, 1), \\ z_3 & \text{otherwise.} \end{cases} \end{aligned}$$

So the algebra $(\{0, 1, 2\}, \omega)$ is a mode. But

$$\omega(\omega(0, 0, 1), \omega(0, 0, 0), \omega(0, 1, z)) = z \neq 2 - z = \omega(\omega(0, 0, 0), \omega(0, 0, 0), \omega(1, 1, z))$$

for $z \neq 1$. Thus $(\{0, 1, 2\}, \omega)$ is not a Szendrei mode.

Problem 10. Is the equational theory of all modes of a fixed type decidable?

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