Supremum properties of Galois–type connections

Árpád Száz

Abstract. In a former paper, motivated by a recent theory of relators (families of relations), we have investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections and residuated mappings of partially ordered sets.

A function f of one preordered set X into another Y has been called

(1) increasingly g-normal, for some function g of Y into X, if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly φ-regular, for some function φ of X into itself, if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the present paper, we shall prove that if f is an increasingly regular function of X onto Y, or f is an increasingly normal function of X into Y, then $f[\sup(A)] \subset \sup(f[A])$ for all $A \subset X$. Moreover, we shall also prove some more delicate, but less important supremum properties of such functions.

Keywords: preordered sets, Galois connections (residuated mappings), supremum properties

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Introduction

In a former paper [14], motivated by a recent theory of relators (see [10] and [7]), we have investigated increasingly regular and normal functions of one preordered set into another instead of Galois connections [5, p. 155] and residuated mappings [2, p. 11] of partially ordered sets.

A function f of one preordered set X into another Y has been called

(1) increasingly g-normal, for some function g of Y into X, if for any $x \in X$ and $y \in Y$ we have $f(x) \leq y$ if and only if $x \leq g(y)$;

(2) increasingly φ-regular, for some function φ of X into itself, if for any $x_1, x_2 \in X$ we have $x_1 \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.

In the first part of the present paper, we shall prove that if f is an increasingly regular function of X onto Y, or f is an increasingly normal function of X into Y, then

$$f[\sup(A)] \subset \sup(f[A])$$

for all $A \subset X$. Moreover, we shall also show that under some completeness properties of X the converse statements are also true.

In the second part of present paper, we shall prove that if f is an increasingly φ-regular function of X onto a partially ordered set Y, then

$$\sup\big(f[A]\big) \subset f\big[\min\big(\mathrm{ub}(A) \cap \varphi[X]\big)\big]$$

for all $A \subset X$. Moreover, by using a similar proof, we shall also show that if f is an increasingly g-normal function of X into Y, then

$$g\big[\sup(f[A])\big] \subset \min\big(\mathrm{ub}(A) \cap g[Y]\big)$$

for all $A \subset X$.

Actually, we shall prove the same inclusions for the relations g_f of Y into X and φ_f of X into itself defined by $g_f(y) = \max\{x \in X : f(x) \leq y\}$ and $\varphi_f(x) = g_f(f(x))$ for all $y \in Y$ and $x \in X$. Moreover, we shall establish some immediate consequences of these inclusions.

1. A few basic facts on relations

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $F \subset X^2$, then we may simply say that F is a relation on X. Thus, $\Delta_X = \{(x, x) : x \in X\}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{y \in Y : (x, y) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of x and A under F, respectively.

Moreover, the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $D_F = X$ ($R_F = Y$), then we say that F is a relation of X to Y (on X onto Y).

In particular, a relation f on X to Y is called a function if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{y\}$. In this case, by identifying singletons with their elements, we may usually write $f(x) = y$ in place of $f(x) = \{y\}$.

If F is a relation on X to Y, then a function f of D_F to Y is called a selection of F if $f \subset F$, i.e., $f(x) \in F(x)$ for all $x \in D_F$. Thus, the Axiom of Choice can be briefly expressed by saying that every relation has a selection.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse relation F^{-1} can be defined such that $F^{-1}(y) = \{x \in X : y \in F(x)\}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the composition relation $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset X$.

A relation R on X is called reflexive, antisymmetric, and transitive if $\Delta_X \subset R$, $R \cap R^{-1} \subset \Delta_X$, and $R \circ R \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder. And an antisymmetric preorder is called a partial order.

2. A few basic facts on ordered sets

If $\leq$ is a relation on a nonvoid set X, then having in mind the terminology of Birkhoff [1, p. 2] the ordered pair $X(\leq) = (X, \leq)$ is called a goset (generalized ordered set). And we usually write X in place of $X(\leq)$.

If $X(\leq)$ is a goset, then by taking $X^* = X$ and $\leq^* = \leq^{-1}$ we can form a new goset $X^*(\leq^*)$. This is called the dual of $X(\leq)$. And we usually write $\geq$ in place of $\leq^*$.

The goset X is called reflexive, transitive, and antisymmetric if the inequality relation $\leq$ in it has the corresponding property. Moreover, for instance, X is called preordered if it is reflexive and transitive.

In particular, a preordered set will be called a proset and a partially ordered set will be called a poset. The usual definitions on posets can be naturally extended to gosets [12]. (See also [11].)

For instance, for any subset A of a goset X, the members of the families

$$\mathrm{lb}(A) = \big\{ x \in X : \forall\, a \in A : x \leq a \big\}$$

and

$$\mathrm{ub}(A) = \big\{ x \in X : \forall\, a \in A : a \leq x \big\}$$

are called the lower and upper bounds of A in X, respectively.

Moreover, the members of the families

$$\begin{aligned} \min(A) &= A \cap \mathrm{lb}(A), & \max(A) &= A \cap \mathrm{ub}(A), \\ \inf(A) &= \max\big(\mathrm{lb}(A)\big), & \sup(A) &= \min\big(\mathrm{ub}(A)\big) \end{aligned}$$

are called the minima, maxima, infima and suprema of A in X, respectively.

Thus, for any $A, B \subset X$, we have $A \subset \mathrm{lb}(B)$ if and only if $B \subset \mathrm{ub}(A)$. Moreover, in [13], we have proved that a reflexive goset X is antisymmetric if and only if $\mathrm{card}(\max(A)) \leq 1$ (resp. $\mathrm{card}(\sup(A)) \leq 1$) for all $A \subset X$.

Now, the goset X may, for instance, be naturally called

(1) sup-complete if $\sup(A) \neq \emptyset$ for all $A \subset X$;
(2) quasi-sup-complete if $\sup(A) \neq \emptyset$ for all $A \subset X$ with $A \neq \emptyset$.

In [3], we have proved that X is quasi-sup-complete if and only if it is pseudo-inf-complete in the sense that $\inf(A) \neq \emptyset$ for all $A \subset X$ with $\mathrm{lb}(A) \neq \emptyset$.

3. Closure operations and regular structures

Definition 3.1. A function φ of a proset X into itself is called an unary operation on X. More generally, a function f of X into another proset Y is called a structure on X.

Remark 3.2. The latter terminology has been mainly motivated by the various structures derived from relators. (See [9] and [11].)

Definition 3.3. An operation φ on X is called

(1) expansive if $\Delta_X \leq \varphi$;
(2) quasi-idempotent if $\varphi^2 \leq \varphi$.

Moreover, a structure f on X is called increasing if for any $x_1, x_2 \in X$, with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.

Remark 3.4. Note that if (1) holds, then we also have $\varphi = \Delta_X \circ \varphi \leq \varphi \circ \varphi = \varphi^2$. Therefore, if both (1) and (2) hold and X is a poset, then φ is actually idempotent.

Thus, according to [1, p. 111], we may also naturally have the following

Definition 3.5. An increasing, expansive and quasi-idempotent operation φ on X is called a closure operation on X.

Remark 3.6. Now, an operation φ on X may be naturally called an interior operation if it is a closure operation on X^*.

In [14], having in mind the ideas of [7], we have also introduced the following

Definition 3.7. A structure f on X is called increasingly φ-regular, for some operation φ on X, if for any $x_1, x_2 \in X$ we have

$$x_1 \leq \varphi(x_2) \iff f(x_1) \leq f(x_2).$$

Remark 3.8. Now, a structure f on X to Y may be naturally called decreasingly φ-regular if it is an increasingly φ-regular structure on X to Y^*.

If f is a φ-regular structure on X, then according to a recent definition of Galois connections [5, p. 155] we may also naturally say that the pair (f, φ) is a Pataki connection on X.

However, even instead of Galois connections, it has been more convenient to use residuated mappings ([2, p. 11]) in the following modified form.

Definition 3.9. A structure f on X to Y is called increasingly g-normal, for some structure g on Y to X, if for any $x \in X$ and $y \in Y$ we have

$$f(x) \leq y \iff x \leq g(y).$$

Remark 3.10. Now, a structure f on X to Y may be naturally decreasingly g-normal if it is an increasingly g-normal structure on X to Y^*.

The importance of the latter definition lies mainly in the fact that if X is a goset and $F(A) = \mathrm{ub}(A)$ and $G(A) = \mathrm{lb}(A)$ for all $A \subset X$, then F is a decreasingly G-normal structure on $\mathcal{P}(X)$. (See [5, 7.24 and 7.38].)

4. Relationships between closure operations and regular structures

By using the above definitions, in [14], we have proved the following theorems.

Theorem 4.1. *If f is an increasingly φ-regular structure on X, then*

(1) *φ is expansive;*
(2) *f is increasing;*
(3) *$f \leq f \circ \varphi \leq f$.*

Corollary 4.2. *If f is an increasingly φ-regular structure on X to a poset Y, then $f = f \circ \varphi$.*

Theorem 4.3. *If φ is an operation on X, then the following assertions are equivalent:*

(1) *φ is a closure operation;*
(2) *φ is increasingly φ-regular;*
(3) *there exists an increasingly φ-regular structure f on X.*

Corollary 4.4. *If f is a structure and φ is an operation on X, then f is increasingly φ-regular if and only if φ is a closure operation and for any $x_1, x_2 \in X$ we have $\varphi(x_1) \leq \varphi(x_2)$ if and only if $f(x_1) \leq f(x_2)$.*

Theorem 4.5. *If f is an increasingly g-normal structure on X to Y and φ is an operation on X such that $\varphi \leq g \circ f \leq \varphi$, then f is increasingly φ-regular.*

Hence, by using that now g is an increasingly f-normal structure on Y^* to X^*, we can also state

Theorem 4.6. *If f is an increasingly g-normal structure on X to Y, then f and g are increasing. Moreover, $\varphi = g \circ f$ is a closure operation on X and $\psi = f \circ g$ is an interior operation on Y.*

Moreover, we shall also need the following very particular results of [14].

Theorem 4.7. *If f is an increasingly φ-regular structure on one poset X to another Y, then f is injective if and only if $\varphi = \Delta_X$.*

Theorem 4.8. *If f is an increasingly g-normal structure on one poset X to another Y, then f is injective if and only if $R_g = X$.*

5. Characterizations of increasingly normal structures

Definition 5.1. For a structure f on X to Y, we define two relations Γ_f and g_f on Y to X such that

$$\Gamma_f(y) = \big\{ x \in X : f(x) \leq y \big\} \quad \text{and} \quad g_f(y) = \max\big(\Gamma_f(y)\big)$$

for all $y \in Y$.

Remark 5.2. Note that if in particular X is a poset, then g_f is already a function of a subset of Y into X.

Concerning the relation g_f, in [14], we have, for instance, proved the following

Theorem 5.3. *For any structures f on X to Y and g on Y to X, the following assertions are equivalent:*

(1) *f is increasingly g-normal;*
(2) *f is increasing and g is a selection of g_f.*

Definition 5.4. For a structure f on X to Y, we define

$$\mathcal{Q}_f = \big\{ g \in X^Y : f \ \text{ is increasingly } g\text{-normal} \big\}.$$

Moreover, if in particular $\mathcal{Q}_f \neq \emptyset$, then we say that f is increasingly normal.

Concerning increasingly normal structures, in [14], we have, for instance, proved the following theorems.

Theorem 5.5. *If f is a structure on X to Y, then the following assertions are equivalent:*

(1) *f is increasingly normal;*
(2) *f is increasing and Y is the domain of g_f.*

Theorem 5.6. *If f is an increasingly normal structure on X to Y, then*

$$g_f(y) = \big\{ g(y) : g \in \mathcal{Q}_f \big\}$$

for all $y \in Y$. Therefore, we actually have $g_f = \bigcup Q_f$.

Theorem 5.7. *If f is an increasingly normal structure on a poset X to Y, then g_f is an increasing structure on Y to X and $\mathcal{Q}_f = \{g_f\}$.*

6. Characterizations of increasingly regular structures

Definition 6.1. For a structure f on X, we define two relations Λ_f and φ_f on X such that

$$\Lambda_f(x) = \big\{ u \in X : f(u) \leq f(x) \big\} \qquad \text{and} \qquad \varphi_f(x) = \max\big(\Lambda_f(x)\big)$$

for all $x \in X$.

Remark 6.2. Note thus Λ_f is preorder relation on X. Moreover, we have $\Lambda_f = \Gamma_f \circ f$ and $\varphi_f = g_f \circ f$.

Concerning the relation φ_f, in [14], we have also proved the following

Theorem 6.3. *If φ is an operation and f is a structure on X, then the following assertions are equivalent:*

(1) *f is increasingly φ-regular;*
(2) *f is increasing and φ is a selection of φ_f.*

Definition 6.4. For a structure f on X, we define

$$\mathcal{O}_f = \{\varphi \in X^X : f \text{ is increasingly } \varphi\text{-regular}\}.$$

Moreover, if in particular $\mathcal{O}_f \neq \emptyset$, then we say that f is increasingly regular.

Concerning increasingly regular structures, in [14], we have, for instance, proved the following theorems.

Theorem 6.5. *If f is a structure on X, then the following assertions are equivalent:*

(1) *f is increasingly regular;*
(2) *f is increasing and X is the domain of φ_f.*

Theorem 6.6. *If f is an increasingly regular structure on X, then*

$$\varphi_f(x) = \{\varphi(x) : \varphi \in \mathcal{O}_f\}$$

for all $x \in X$. Therefore, we actually have $\varphi_f = \bigcup \mathcal{O}_f$.

Theorem 6.7. *If f is an increasingly regular structure on X onto Y, then f is already increasingly normal.*

Theorem 6.8. *If f is an increasingly regular structure on a poset X, then φ_f is a closure operation on X and $\mathcal{O}_f = \{\varphi_f\}$.*

7. Supremum properties of increasingly normal structures

As an extension of an observation of Pickert [8] and the first part of [5, Proposition 7.31], we can prove the following

Theorem 7.1. *If f is an increasingly normal structure on X to Y, then for any $A \subset X$ we have*

$$f[\sup(A)] \subset \sup\big(f[A]\big).$$

Proof: If $y \in f[\sup(A)]$, then there exists $x \in \sup(A)$ such that $y = f(x)$. Hence, we can see that

$$x \in \mathrm{ub}(A) \qquad \text{and} \qquad x \in \mathrm{lb}\big(\mathrm{ub}(A)\big).$$

Thus, in particular, for any $a \in A$, we have $a \leq x$. Hence, by using Theorem 4.6, we can infer that $f(a) \leq f(x) = y$. Therefore, $y \in \mathrm{ub}(f[A])$.

On the other hand, if $v \in \mathrm{ub}(f[A])$, then for any $a \in A$ we have $f(a) \leq v$. Hence, by choosing a $g \in \mathcal{Q}_f$, we can infer that $a \leq g(v)$. Therefore, $g(v) \in \mathrm{ub}(A)$. Hence, by using that $x \in \mathrm{lb}(\mathrm{ub}(A))$, we can infer that $x \leq g(v)$. This implies that $y = f(x) \leq v$. Therefore, $y \in \mathrm{lb}\big(\mathrm{ub}(f[A])\big)$, and thus

$$y \in \mathrm{ub}\big(f[A]\big) \cap \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big) = \sup\big(f[A]\big)$$

also holds. This proves the required inclusion. □

From the above theorem, it is clear that in particular we have

Corollary 7.2. *If f is an increasingly normal structure on a sup-complete proset X to a poset Y, then $f[\sup(A)] = \sup(f[A])$ for all $A \subset X$.*

Proof: Note that now, in addition to $f[\sup(A)] \subset \sup(f[A])$, we also have $f[\sup(A)] \neq \emptyset$ and $\mathrm{card}\big(\sup(f[A])\big) \leq 1$ for all $A \subset X$. Therefore, the required assertion is also true. □

Moreover, we can also prove the following partial converse to Theorem 7.1.

Theorem 7.3. *If f is a structure on a sup-complete proset X to Y such that*

$$f[\sup(A)] \subset \sup\big(f[A]\big)$$

for all $A \subset X$, then f is increasingly normal.

Proof: By Theorem 5.5, it is enough to show only that now f is increasing and $g_f(y) \neq \emptyset$ for all $y \in Y$.

For this, first note that if $x_1, x_2 \in X$ such that $x_1 \leq x_2$, then by taking $A = \{x_1, x_2\}$ we have

$$x_2 \in A \cap \mathrm{ub}(A) = \max(A) \subset \sup(A).$$

Hence, by using the assumed sup-preservingness of f, we can infer that

$$f(x_2) \in f[\sup(A)] \subset \sup\big(f[A]\big) \subset \mathrm{ub}\big(f[A]\big) = \mathrm{ub}\big(\{f(x_1), f(x_2)\}\big).$$

Therefore, $f(x_1) \leq f(x_2)$, and thus f is increasing.

Next, note that if $y \in Y$, then by the assumed sup-completeness of X there exists $x \in X$ such that $x \in \sup\big(\Gamma_f(y)\big)$. Hence, by using the assumed sup-preservingness of f, we can infer that

$$f(x) \in f\big[\sup\big(\Gamma_f(y)\big] \subset \sup\big(f[\Gamma_f(y)]\big) \subset \mathrm{lb}\big(\mathrm{ub}\big(f[\Gamma_f(y)]\big)\big).$$

Moreover, by Definition 5.1, we have $f(u) \leq y$ for all $u \in \Gamma_f(y)$, and thus $y \in \mathrm{ub}\big(f[\Gamma_f(y)]\big)$. Hence, it is clear that $f(x) \leq y$, and thus $x \in \Gamma_f(y)$. Therefore,

$$x \in \Gamma_f(y) \cap \sup\big(\Gamma_f(y)\big) = \max\big(\Gamma_f(y)\big) = g_f(y),$$

and thus $g_f(y) \neq \emptyset$ is also true. □

Now, as an immediate consequence of Theorems 7.1 and 7.3, we can also state the following extension of an observation of Pickert [8] and the first part of [5, Proposition 7.34].

Corollary 7.4. *If f is a structure on a sup-complete proset X to Y, then the following assertions are equivalent:*

(1) *f is increasingly normal;*
(2) *$f[\sup(A)] \subset \sup(f[A])$ for all $A \subset X$.*

8. Supremum properties of increasingly regular structures

From Theorem 7.1, by using Theorem 6.7, we can immediately derive

Theorem 8.1. *If f is an increasingly regular structure on X onto Y, then for any $A \subset X$ we have*

$$f[\sup(A)] \subset \sup\big(f[A]\big).$$

Proof: In this case, by Theorem 6.7, the structure f is increasingly normal. Therefore, Theorem 7.1 can be applied to get the required inclusion. □

From the above theorem, it is clear that in particular we also have

Corollary 8.2. *If f is an increasingly regular structure on a sup-complete proset X onto a poset Y, then $f[\sup(A)] = \sup(f[A])$ for all $A \subset X$.*

Moreover, analogously to Theorem 7.3, we can also prove the following

Theorem 8.3. *If f is a structure on a quasi-sup-complete proset X to Y such that*

$$f[\sup(A)] \subset \sup\big(f[A]\big)$$

for all $A \subset X$ with $A \neq \emptyset$, then f is increasingly regular.

Proof: By Theorem 6.5, it is enough to show only that f is increasing and $\varphi_f(x) \neq \emptyset$ for all $x \in X$.

From the proof of Theorem 7.3, it is clear that f is increasing. Moreover, if $x \in X$, then by Definition 6.1 we have $x \in \Lambda_f(x)$, and thus $\Lambda_f(x) \neq \emptyset$. Therefore, by the assumed quasi-sup-completeness of X, there exists $\alpha \in X$ such that $\alpha \in \sup\big(\Lambda_f(x)\big)$. Hence, by using the assumed sup-preservingness of f, we can infer that

$$f(\alpha) \in f\big[\sup\big(\Lambda_f(x)\big] \subset \sup\big(f[\Lambda_f(x)]\big) \subset \mathrm{lb}\big(\mathrm{ub}\big(f[\Lambda_f(x)]\big)\big).$$

Moreover, by Definition 6.1, we also have $f(u) \leq f(x)$ for all $u \in \Lambda_f(x)$, and thus $f(x) \in \mathrm{ub}\big(f[\Lambda_f(x)]\big)$. Hence, it is clear that $f(\alpha) \leq f(x)$, and thus $\alpha \in \Lambda_f(x)$. Therefore,

$$\alpha \in \Lambda_f(x) \cap \sup\big(\Lambda_f(x)\big) = \max\big(\Lambda_f(x)\big) = \varphi_f(x),$$

and thus $\varphi_f(x) \neq \emptyset$ is also true. □

Now, as an immediate consequence of Theorems 8.1 and 8.3, we can also state

Corollary 8.4. *If f is a structure on a quasi-sup-complete proset X onto Y, then the following assertions are equivalent:*

(1) *f is increasingly regular;*
(2) *$f[\sup(A)] \subset \sup(f[A])$ for all $A \subset X$.*

Remark 8.5. Note that if f is as an increasingly regular structure on X onto Y, or f is an increasingly normal structure on X to Y, then by Theorems 8.1 and 7.1 we also have

$$f[\min(X)] = f[\sup(\emptyset)] \subset \sup\big(f[\emptyset]\big) = \min(Y).$$

9. Further supremum properties of increasingly regular structures

In addition to Theorem 8.1, we can also prove the following

Theorem 9.1. *If f is a increasingly regular structure on X onto a poset Y, then for any $A \subset X$ we have*

$$\sup\big(f[A]\big) \subset f\big[\min\big(\mathrm{ub}(A) \cap \varphi_f[X]\big)\big].$$

Proof: If $y \in \sup(f[A])$, then by the corresponding definitions we have

$$y \in \mathrm{ub}\big(f[A]\big) \qquad \text{and} \qquad y \in \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big).$$

Thus, in particular, for any $a \in A$ we have $f(a) \leq y$. Moreover, since $Y = f[X]$, there exists $x \in X$ such that $y = f(x)$. Therefore, we also have $f(a) \leq f(x)$. Hence, by taking a $\varphi \in \mathcal{O}_f$, we can infer that $a \leq \varphi(x)$. Therefore,

$$\varphi(x) \in \mathrm{ub}(A) \cap \varphi[X] \subset \mathrm{ub}(A) \cap \varphi_f[X].$$

Namely, by Theorem 6.3, we have $\varphi \subset \varphi_f$, and thus $\varphi[X] \subset \varphi_f[X]$.

On the other hand, if $v \in \mathrm{ub}(A) \cap \varphi_f[X]$, then for any $a \in A$ we have $a \leq v$, and moreover there exists $u \in X$ such that $v \in \varphi_f(u)$. Hence, by Theorem 6.6, we can see that there exists $\psi \in \mathcal{O}_f$ such that $v = \psi(u)$. Therefore, we also have $a \leq \psi(u)$, and thus $f(a) \leq f(u)$. Hence, it is clear that $f(u) \in \mathrm{ub}(f[A])$. Moreover, since

$$f(x) = y \in \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big),$$

we can also see that $f(x) \leq f(u)$. Hence, by using Corollary 4.4, we can infer that $\psi(x) \leq \psi(u) = v$. Moreover, by Theorem 6.3, we also have

$$\varphi(x), \psi(x) \in \varphi_f(x) = \max\big(\Lambda_f(x)\big) = \Lambda_f(x) \cap \mathrm{ub}\big(\Lambda_f(x)\big).$$

Hence, in particular, we can see that $\varphi(x) \leq \psi(x)$, and thus $\varphi(x) \leq v$ also holds. Consequently, $\varphi(x) \in \mathrm{lb}\big(\mathrm{ub}(A) \cap \varphi_f[X]\big)$, and thus

$$\varphi(x) \in \big(\mathrm{ub}(A) \cap \varphi_f[X]\big) \cap \mathrm{lb}\big(\mathrm{ub}(A) \cap \varphi_f[X]\big) = \min\big(\mathrm{ub}(A) \cap \varphi_f[X]\big)$$

is also true. Now, by Corollary 4.2, it is clear that

$$y = f(x) = f\big(\varphi(x)\big) \in f\big[\min\big(\mathrm{ub}(A) \cap \varphi_f[X]\big)\big].$$

Therefore, the required inclusion is true. □

Remark 9.2. From the above proof, we can also see that if f is an increasingly φ-regular structure on X onto a poset Y, then for any $A \subset X$ we have

$$\sup\big(f[A]\big) \subset f\big[\min\big(\mathrm{ub}(A) \cap \varphi[X]\big)\big].$$

Moreover, as an immediate consequence of Theorems 8.1 and 9.1, we can also state

Theorem 9.3. *If f is an increasingly regular structure on X onto a poset Y, then*

$$f[\sup(A)] = \sup\big(f[A]\big)$$

for all $A \subset X$ with $\mathrm{ub}(A) \subset \varphi_f[X]$.

Proof: Namely, if $A \subset X$, then by Theorem 8.1 we have

$$f[\sup(A)] \subset \sup\big(f[A]\big)$$

even if Y only a proset.

Moreover, if $\mathrm{ub}(A) \subset \varphi_f[X]$, then by Theorem 9.1 we also have

$$\sup\big(f[A]\big) \subset f\big[\min\big(\mathrm{ub}(A) \cap \varphi_f[X]\big)\big] = f\big[\min\big(\mathrm{ub}(A)\big)\big] = f[\sup(A)].$$

Therefore, the required equality is also true. □

From the latter theorem, by Theorems 6.8 and 4.7, it is clear that in particular we also have

Theorem 9.4. *If f is an injective increasingly regular structure on one poset X onto another Y, then for any $A \subset X$ we have*

$$f[\sup(A)] = \sup\big(f[A]\big).$$

Proof: Namely, by Theorem 6.8, the structure f is increasingly φ_f-regular. Hence, by Theorem 4.7, we can see that φ_f is the identity function of X. Therefore, $\varphi_f[X] = X$. Now, by Theorem 9.3, it is clear that the required assertion is true. $\square$

10. Further supremum properties of increasingly normal structures

From Theorem 9.1, by using Theorem 4.5 and Remark 6.2, we can also get the following

Theorem 10.1. *If f is a increasingly normal structure on X onto a poset Y, then for any $A \subset X$ we have*

$$\sup\big(f[A]\big) \subset f\big[\min\big(\mathrm{ub}(A) \cap g_f[Y]\big)\big].$$

Proof: Now, by Theorem 4.5, the structure f is increasingly regular. Moreover, by Remark 6.2, we have $\varphi_f = g_f \circ f$. Thus, in particular

$$\varphi_f[X] = (g_f \circ f)[X] = g_f\big[f[X]\big] = g_f[Y].$$

Hence, by Theorem 9.1, it is clear that the required inclusion is also true. $\square$

Now, as an immediate consequence of Theorems 7.1 and 10.1, we can also state

Theorem 10.2. *If f is an increasingly normal structure on X onto a poset Y, then*

$$f[\sup(A)] = \sup\big(f[A]\big)$$

for all $A \subset X$ with $\mathrm{ub}(A) \subset g_f[Y]$.

Hence, by Theorems 5.7 and 4.8, it is clear that in particular we also have the following

Theorem 10.3. *If f is an injective increasingly normal structure on one poset X onto another Y, then for any $A \subset X$ we have*

$$f[\sup(A)] = \sup\big(f[A]\big).$$

Proof: Now, by Theorem 5.7, the structure f is g_f-normal. Hence, by Theorem 4.8, we can see that $X = g_f[Y]$. Therefore, by Theorem 10.2, the required assertion is true. $\square$

However, it now more interesting that, analogously to Theorem 9.1, we can also prove the following

Theorem 10.4. *If f is an increasingly normal structure on X to Y, then for any $A \subset X$ we have*

$$g_f\left[\sup\bigl(f[A]\bigr)\right] \subset \min\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr).$$

Proof: If $y \in \sup(f[A])$, then by the corresponding definitions we have

$$y \in \mathrm{ub}\bigl(f[A]\bigr) \qquad \text{and} \qquad y \in \mathrm{lb}\bigl(\mathrm{ub}\bigl(f[A]\bigr)\bigr).$$

Thus, in particular, for any $a \in A$ we have $f(a) \leq y$. Hence, by taking any $g \in \mathcal{Q}_f$, we can infer that $a \leq g(y)$. Therefore,

$$g(y) \in \mathrm{ub}(A) \cap g[Y] \subset \mathrm{ub}(A) \cap g_f[Y].$$

Namely, by Theorem 5.3, we have $g \subset g_f$, and thus $g[Y] \subset g_f[Y]$.

On the other hand, if $u \in \mathrm{ub}(A) \cap g_f[Y]$, then for any $a \in A$ we have $a \leq u$, and moreover there exists $v \in Y$ such that $u \in g_f(v)$. Hence, by Theorem 5.6, we can see that there exists $h \in \mathcal{Q}_f$ such that $u = h(v)$. Therefore, $a \leq h(v)$, and thus $f(a) \leq v$. This shows that $v \in \mathrm{ub}(f[A])$. Hence, by using that $y \in \mathrm{lb}(\mathrm{ub}(f[A]))$, we can infer that $y \leq v$. Now, by Theorem 4.6, it is clear that $h(y) \leq h(v) = u$. Moreover, by Theorem 5.3, we also have

$$g(y),\ h(y) \in g_f(y) = \max\bigl(\Gamma_f(y)\bigr) = \Gamma_f(y) \cap \mathrm{ub}\bigl(\Gamma_f(y)\bigr).$$

Hence, in particular, we can see that $g(y) \leq h(y)$, and thus $g(y) \leq u$ also holds. Consequently, $g(y) \in \mathrm{lb}\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr)$, and thus

$$g(y) \in \mathrm{ub}(A) \cap g_f[Y] \cap \mathrm{lb}\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr) = \min\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr)$$

is also true. Now, by Theorem 5.6, it is clear that

$$g_f(y) = \bigl\{g(y) : g \in \mathcal{Q}_f\bigr\} \subset \min\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr).$$

Therefore,

$$g_f\left[\sup\bigl(f[A]\bigr)\right] = \bigcup\bigl\{g_f(y) : y \in \sup\bigl(f[A]\bigr)\bigr\} \subset \min\bigl(\mathrm{ub}(A) \cap g_f[Y]\bigr)$$

is also true. □

Remark 10.5. From the above proof, we can also see that if f is an increasingly g-normal structure on X to Y, then for any $A \subset X$ we have

$$g\bigl[\sup\bigl(f[A]\bigr)\bigr] \subset \min\bigl(\mathrm{ub}(A) \cap g[Y]\bigr).$$

Moreover, as an immediate consequence of Theorem 10.4, we can also state

Corollary 10.6. *If f is an increasingly normal structure on X to Y, then*

$$g_f\big[\sup\big(f[A]\big)\big] \subset \sup(A)$$

for all $A \subset X$ with $\operatorname{ub}(A) \subset g_f[Y]$.

Hence, it is clear that in particular we also have

Corollary 10.7. *If f is an increasingly normal structure on a poset X to a sup-complete proset Y, then*

$$\sup(A) = g_f\big[\sup\big(f[A]\big)\big]$$

for all $A \subset X$ with $\operatorname{ub}(A) \subset g_f[Y]$.

Moreover, from Corollary 10.6, by using Theorems 5.7 and 4.8, we can also immediately get the following

Theorem 10.8. *If f is an injective increasingly normal structure on a poset X to another Y, then for any $A \subset X$ we have*

$$g_f\big[\sup\big(f[A]\big)\big] \subset \sup(A).$$

Proof: Now, by Theorem 5.7, the structure f is g_f-normal. Hence, by Theorem 4.8, we can see that $X = g_f[Y]$. Therefore, by Corollary 10.6, the required assertion is true. □

From the above theorem, it is clear that in particular we also have

Corollary 10.9. *If f is an injective increasingly normal structure on a poset X to a sup-complete poset Y, then for any $A \subset X$ we have*

$$\sup(A) = g_f\big[\sup\big(f[A]\big)\big].$$

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Institute of Mathematics, University of Debrecen, H-4010 Debrecen, Pf. 12, Hungary

E-mail: szaz@math.klte.hu

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