Retral spaces and continua with the fixed point property

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Abstract. We show that every retral continuum with the fixed point property is locally connected. It follows that an indecomposable continuum with the fixed point property is not a retract of a topological group.

Keywords: retraction, homogeneous space, topological groups, coset space

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1. Introduction

A space X is called a retral space if it is a retract of a topological group. If G is a topological group, then the function μ defined by $\mu(x, y, z) = xy^{-1}z$ has the property that $\mu(x, y, y) = x = \mu(y, y, x)$ for all $x, y \in G$. A continuous function with this property is called a Mal'tsev function and if there is a Mal'tsev function on a space X then X is called a Mal'tsev space. The class of Mal'tsev spaces is closed under retractions and we have just shown that every topological group is a Mal'tsev space. So every retral space is also a Mal'tsev space.

Mal'tsev functions were introduced by Mal'tsev in [7] and Uspenskiĭ has shown that much of the behaviour of topological groups generalizes to Mal'tsev spaces (see for example [15], [16] and [11]). Since every retral space is a Mal'tsev space, it is natural to ask whether the converse is also true. It was shown by Sipacheva in [12] that every compact Mal'tsev space is retral (see also [6, Corollary 6] and [11, Theorem 1.6] for generalizations). In [6], Gartside, Reznichenko and Sipacheva provide an example of a Mal'tsev space which is not retral and Cauty [4] has provided an example of a compact Mal'tsev space which is not a retract of a compact topological group.

In this note we prove that every retral continuum with the fixed point property is locally connected. This leads to interesting examples. The fixed point property is crucial as the dyadic solenoid demonstrates.

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2. Preliminaries

We assume that all spaces are regular. A continuum is a compact and connected set. A space X is locally connected if for all $x \in X$ and every open neighbourhood

U of x, there is a connected set $C \subseteq U$ such that $x \in \operatorname{Int} C$. If $x, y \in X$ and $A \subseteq X$, then we say that x and y are connected in A provided there is a connected set C containing x and y such that $C \subseteq A$. In particular x and y are connected in X if and only if they belong to the same component of X. By convention, the empty set is a connected set.

A space X is said to have the fixed point property provided that for every continuous function $f : X \to X$, there is some $x \in X$ such that $f(x) = x$.

Definition 2.1. Let X be a topological space. If $x \in X$, then we say that the components of X are *regularly locally connected at* x if for every neighbourhood U of x there is a neighbourhood V of x such that for every component C of X, there is a connected set C' in X such that

$$V \cap C \subseteq C' \subseteq U.$$

We say that the components of X are *regularly locally connected* if for every $x \in X$, the components of X are regularly locally connected at x.

The following straightforward lemma explains our terminology. The easy proof is left to the reader.

Lemma 2.2. *Consider the following statements regarding a space X.*

(1) *X is locally connected.*
(2) *The components of X are regularly locally connected.*
(3) *The components of X are locally connected.*

For every space X we have (1) $\to$ (2) *and* (2) $\to$ (3). *If X is connected then* (2) $\to$ (1).

We now provide examples to show that in the previous lemma (3) $\not\to$ (2) and (2) $\not\to$ (1).

Example 2.3. By $\mathbb{I}$ we denote the usual unit interval $[0, 1]$. The space Z is given by $\{0\} \cup \{1/n : n \in \mathbb{N}\}$. Let X be the subspace of the plane $\mathbb{R}^2$ given by:

$$X = \big(Z \times \mathbb{I}\big) \cup \bigcup \big\{[1/2n, 1/(2n-1)] \times \{1\} : n \in \mathbb{N}\big\}.$$

The space X is a compact metric space. The components of X are locally connected since every component of X is homeomorphic to $\mathbb{I}$. Using Proposition 2.5 below one verifies easily that the components of X are not regularly locally connected. This shows that (3) $\not\to$ (2) in Lemma 2.2.

Example 2.4. Let X be the subspace of $\mathbb{R}$ given by:

$$X = \{0\} \cup \bigcup \big\{[1/2n, 1/(2n-1)] : n \in \mathbb{N}\big\}.$$

One verifies easily that the components of X are regularly locally connected. Since X is not locally connected, this example shows that (2) $\not\to$ (1) in Lemma 2.2.

The following proposition provides a reformulation of Definition 2.1. We omit the easy proof.

Proposition 2.5. *Suppose X is a space and $x \in X$. The following are equivalent.*

(1) *The components of X are regularly locally connected at x.*
(2) *For every neighbourhood U of x there is a neighbourhood V of x such that for all $y, z \in V$ the following holds: If y and z are connected in X, then y and z are connected in U.*

3. Retral continua with the fixed point property

Let $\mathcal{U}$ be a cover of the space X. We say that another cover $\mathcal{V}$ of X refines $\mathcal{U}$ if for all $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subseteq U$.

If $A \subseteq X$ and $f : A \to X$ then we say that f is limited by $\mathcal{U}$ provided that for every $z \in A$ there is an element $U \in \mathcal{U}$ containing both z and $f(z)$. For a set A, by $\Delta(A)$ we denote the diagonal in A^2 which is given by $\{(a, a) : a \in A\}$.

Lemma 3.1. *Let X be a Mal'tsev space. For every open cover $\mathcal{U}$ of X and compact subset $K \subseteq X$, there is an open cover $\mathcal{V}$ of X such that whenever $x, y \in V \in \mathcal{V}$, there is a continuous map $f : X \to X$ such that $f(x) = y$ and $f \restriction K$ is limited by $\mathcal{U}$.*

Proof: Fix a Mal'tsev function μ on X. By regularity we may refine the open cover $\mathcal{U}$ by an open cover $\mathcal{W}$ of X, such that the cover $\{\overline{W} : W \in \mathcal{W}\}$ is also a refinement of $\mathcal{U}$. For every $W \in \mathcal{W}$ we choose $U_W \in \mathcal{U}$ such that $\overline{W} \subseteq U_W$. We have that

$$\Delta(X) \times \overline{W} \subseteq \mu^{-1}[U_W].$$

Claim. For every $W \in \mathcal{W}$ there is an open cover $\mathcal{V}_W$ of X such that

$$\bigcup\{V \times V \times \big(K \cap \overline{W}\big) : V \in \mathcal{V}_W\} \subseteq \mu^{-1}[U_W].$$

Proof of Claim: Let $W \in \mathcal{W}$ be given. If $K \cap \overline{W} = \emptyset$ then there is nothing to prove. So assume $K \cap \overline{W} \neq \emptyset$. We fix $x \in X$. For every $w \in K \cap \overline{W}$, there are open sets E_w and G_w such that

$$(x, x, w) \in E_w \times E_w \times G_w \subseteq \mu^{-1}[U_W].$$

Since $K \cap \overline{W}$ is compact, there is a finite set $F \subseteq K \cap \overline{W}$ such that the collection $\{G_w : w \in F\}$ covers $K \cap \overline{W}$. Let V_x be given by $\bigcap\{E_w : w \in F\}$. Then V_x is an open neighbourhood of x and clearly

$$V_x \times V_x \times \big(K \cap \overline{W}\big) \subseteq \mu^{-1}[U_W].$$

Therefore, the open cover $\mathcal{V}_W$ may be given by $\{V_x : x \in X\}$. □

Since K is compact, we may fix a finite subcollection $\mathcal{W}'$ of $\mathcal{W}$, such that $\mathcal{W}'$ is an open cover of K. Since $\mathcal{W}'$ is finite we may find an open cover $\mathcal{V}$ of X, such that $\mathcal{V}$ is a refinement of $\mathcal{V}_W$ for every $W \in \mathcal{W}'$.

We claim that the cover $\mathcal{V}$ is as required. To show this, suppose that $x, y \in V$ for some $V \in \mathcal{V}$. We define $f : X \to X$ by the formula $f(z) = \mu(y, x, z)$. Then f is continuous and $f(x) = \mu(y, x, x) = y$. To show that $f \restriction K$ is limited by $\mathcal{U}$, suppose $z \in K$. Then $z \in W$ for some $W \in \mathcal{W}'$. Since $x, y \in V \in \mathcal{V}$ and $\mathcal{V}$ refines $\mathcal{V}_W$, we have

$$(y, x, z) \in V \times V \times \left(K \cap \overline{W}\right) \subseteq \mu^{-1}[U_W].$$

It follows that $f(z) = \mu(y, x, z) \in U_W$. Recall that $z \in W \subseteq U_W$ and therefore it follows that $\{z, f(z)\} \subseteq U_W$. Since z was an arbitrary element of K, this shows that $f \restriction K$ is limited by $\mathcal{U}$. □

For the remainder of this section, we are mainly concerned with compact spaces. Recall from [12] that every compact Mal'tsev space is retral.

Theorem 3.2. *Let X be a compact Mal'tsev space in which all components have the fixed point property. Then the components of X are regularly locally connected.*

Proof: We use the formulation provided by Proposition 2.5. So fix $x \in X$ and let U be some arbitrary neighbourhood of x. Fix a Mal'tsev function μ on X. Let G be a neighbourhood of x such that $\overline{G} \subseteq U$. We have

$$\overline{G} \times \Delta(X) \subseteq \mu^{-1}[U].$$

So we may find an open cover $\mathcal{W}$ of X such that

$$\bigcup\{\overline{G} \times W \times W : W \in \mathcal{W}\} \subseteq \mu^{-1}[U].$$

We apply Lemma 3.1 to the open cover $\mathcal{W}$ to obtain an open cover $\mathcal{V}$ of X such that whenever $y, z \in V \in \mathcal{V}$, there is a continuous function $f : X \to X$ such that $f(y) = z$ and f is limited by $\mathcal{W}$.

Now choose $V \in \mathcal{V}$ such that $x \in V$. We claim that $V \cap G$ is as required. To see this, let $y, z \in V \cap G$ and suppose that y and z are connected in X. Let C be the component of X containing y and z. By construction there is a continuous function $f : X \to X$ such that $f(y) = z$ and f is limited by $\mathcal{W}$. We define a function $g : X \to X$ by $g(w) = \mu(y, w, f(w))$. Then g is continuous. Since $f(y) = z$, we have $f[C] \subseteq C$ and therefore there is some $v \in C$ such that $f(v) = v$. It follows that $g(v) = \mu(y, v, f(v)) = \mu(y, v, v) = y$. Since we also have that $g(y) = \mu(y, y, f(y)) = f(y) = z$, the set $C' = g[C]$ is a connected set containing both y and z. We will show that $C' \subseteq U$.

So suppose that $w \in C$. Then $\{w, f(w)\} \subseteq W$ for some $W \in \mathcal{W}$. Since $y \in G$, it follows that $(y, w, f(w)) \in \mu^{-1}[U]$ and therefore

$$g(w) = \mu(y, w, f(w)) \in U.$$

Since w was an arbitrary element of C, we have shown that $C' = g[C] \subseteq U$ and this completes the proof. $\square$

It follows in particular that the space X given in Example 2.3 is not a Mal'tsev space and by compactness it follows that this space is not retral. We now obtain our main result:

Corollary 3.3. *Every Mal'tsev continuum with the fixed point property is locally connected.*

PROOF: This follows from Theorem 3.2 and Lemma 2.2 $\square$

The following result is a generalization of the previous corollary, but it is in fact equivalent to it.

Corollary 3.4. *Suppose X is a Mal'tsev space and let $C \subseteq X$ be a compact component of X with the fixed point property. Then C is locally connected.*

PROOF: Let μ be a Mal'tsev function on X. Since C is a component of X it follows that $\mu[C^3]$ is a connected subset of X. If $x \in C$, then $\mu(x,x,x) = x$ and therefore $\mu[C^3] = C$. It follows that $\mu \restriction C^3 : C^3 \to C$ is a Mal'tsev function on C. Now apply Corollary 3.3 to the Mal'tsev continuum C. $\square$

A continuum X is called decomposable if it is the union of two proper subcontinua. A continuum is indecomposable if it is not decomposable. It is well-known that indecomposable continua are not locally connected, see for example [8, Corollary 1.10.14]. We obtain the following corollary:

Corollary 3.5. *Suppose X is an indecomposable continuum with the fixed point property. Then X is not a Mal'tsev space and hence not a retract of a topological group.*

Example 3.6. Let S be the dyadic solenoid, see [8, p. 85] for a description. Then S is indecomposable but S admits a group structure which makes S into a topological group. Thus S is a metric continuum which is both retral and indecomposable. This example shows that the fixed point property is essential in Corollary 3.3.

Let $\mathbb{S}^n$ be the n-dimensional unit sphere in $\mathbb{R}^{n+1}$ and B^n the closed n-dimensional unit ball in $\mathbb{R}^n$. A metric space X is *connected in dimension m*, abbreviated C^n, if for every $0 \leq m \leq n$, every continuous function $f : \mathbb{S}^m \to X$ can be extended to a continuous function $\overline{f} : B^{m+1} \to X$. A metric space X is called *locally connected in dimension n*, abbreviated LC^n, if for every $x \in X$ and for every neighbourhood U of x and for every $0 \leq m \leq n$, there exists a neighbourhood V of x such that every continuous function $f : \mathbb{S}^m \to V$ can be extended to a continuous function $\overline{f} : B^{m+1} \to U$. Note that a metric space X is path-connected if and only if it is C^0 and locally path-connected if and only if it is LC^0.

Let X be a metric continuum with the fixed point property and assume that X is a Mal'tsev space. Then X is locally connected by Corollary 3.3. It follows from the Mazurkiewicz Theorem (cf. [8, Theorem 1.5.22] and [8, Corollary 1.5.24]) that X is path-connected and locally path-connected and thus X is C^0 and LC^0. We do not know whether this result generalizes to higher dimensional forms of connectivity. For example, let X be a metric continuum with the fixed point property and $n > 0$. Suppose X is a Mal'tsev space which is C^n, is then X also LC^n?

4. Retracts of coset spaces and Mal'tsev spaces

In this section we study the connection between Mal'tsev spaces and retracts of coset spaces. A space X is called a coset space provided that there is a topological group G with closed subgroup H such that X and G/H are homeomorphic. A topological space X is called homogeneous provided that for every $x, y \in X$, there exists a homeomorphism h of X such that $h(x) = y$. Every topological group is homogeneous. Every coset space is also homogeneous but the converse is not true in general, see Ford [5], van Mill [9] and van Mill and Ridderbos [10] for examples. However, Ungar [13] has shown that every locally compact homogeneous metric space is a coset space.

Given a certain class $\mathcal{A}$ of homogeneous spaces, it is natural to ask which spaces are retracts of members of $\mathcal{A}$. Uspenskiĭ has shown in [14], that every space is a retract of a homogeneous space. By a result of Motorov (cf. [1]), not every compact space is a retract of a compact homogeneous space, an example is the well-known $\sin(1/x)$-curve. Other results in this spirit have already been mentioned in the introduction. In addition it follows from Corollary 3.3 that the $\sin(1/x)$-curve is not a Mal'tsev space and therefore it is not a retract of a topological group. This observation also follows from the following simple argument which was brought to our attention by Uspenskiĭ. Every Mal'tsev space X has the following property: for every $x, y \in X$ there is a continuous function $f : X \to X$ such that $f(x) = y$ and $f(y) = x$. The $\sin(1/x)$-curve does not have this property, hence it is not a Mal'tsev space.

Mal'tsev spaces and retracts of coset spaces both arise naturally as retracts of certain classes of homogeneous spaces. In the first case this is the class of all topological groups and in the second case this is the class of all coset spaces. Since every topological group is a coset space, every Mal'tsev space which is retral is a retract of a coset space. Not every Mal'tsev space is retral by [6] and this raises the following question:

Question 4.1. *Is every Mal'tsev space a retract of a coset space?*

The previous question is motivated by the fact that the conclusion of Lemma 3.1 also holds for retracts of coset spaces, this was proved in [10, Corollary 2.3]. Since the $\sin(1/x)$-curve does not satisfy the conclusion of Lemma 3.1, it follows that this space is not a retract of a coset space. Further examples of spaces which are not retracts of coset spaces can be found in [10, Section 4].

We may also ask the converse question: *Is every retract of a coset space a Mal'tsev space?* Uspenskiĭ ([16, Proposition 16]) has noted that if $n \notin \{0, 1, 3, 7\}$, then $\mathbb{S}^n$ is not a Mal'tsev space. For example, the 2-dimensional sphere $\mathbb{S}^2$ is a coset space of a compact group which is not a Mal'tsev space and this answers the previous question negatively. As an application of Corollary 3.5 we provide yet another example of a compact coset space which is not a Mal'tsev space.

Recall that the pseudo-arc is the unique non-degenerate metric continuum which is chainable and hereditarily indecomposable (see Bing [3]). It was shown by Bing in [2] that the pseudo-arc is homogeneous. By Ungar's result in [13] it follows that the pseudo-arc is a compact coset space.

Example 4.2. Let M be the pseudo-arc. Then M is an indecomposable metric continuum with the fixed point property. It follows from Corollary 3.5 that M is not a Mal'tsev space and hence not a retract of a topological group. So the pseudo-arc is an example of a coset space which is not a Mal'tsev space.

Remark 4.3. As an application of [10, Corollary 2.3], some results are proved in [10] for retracts of coset spaces. By Lemma 3.1 the conclusion of Corollary 2.3 in [10] also holds for Mal'tsev spaces and therefore the results in [10, Section 3] are also valid for Mal'tsev spaces. This applies in particular to Theorem 3.1, Theorem 3.3, Theorem 3.4 and Corollary 3.5 in [10]. Not all of these results are new.

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