Comment.Math.Univ.Carolin. 48,1 (2007)107–117

Locally realcompact and HN-complete spaces

David Buhagiar, Emmanuel Chetcuti

Abstract. Two classes of spaces are studied, namely locally realcompact spaces and HN-complete spaces, where the latter class is introduced in the paper. Both of these classes are superclasses of the class of realcompact spaces. Invariance with respect to subspaces and products of these spaces are investigated. It is shown that these two classes can be characterized by demanding that certain equivalences hold between certain classes of Baire measures or by demanding that certain classes of Baire measures have non empty support. It is known that a space is locally realcompact if and only if it is open in its Hewitt-Nachbin realcompactification; we give an external characterization of HN-completeness with respect to the Hewitt-Nachbin realcompactification. In addition, a complete characterization of products of these classes is given.

Keywords: Baire measure, realcompactness, local realcompactness, HN-completeness

Classification: Primary 28C15, 54D60; Secondary 54D45, 54D99

1. Introduction

Realcompact spaces (originally called Q-spaces) were introduced by Hewitt in 1948 [13]. One can define realcompact spaces as those spaces which are homeomorphic to a closed subspace of a product of real lines and therefore, it is evident that realcompactness is a generalization of compactness. One can note that the above definition requires realcompact spaces to be at least Tychonoff (a T_1 space on which every point x and every closed set F disjoint from x are functionally separated).

Many generalizations of realcompact spaces have been studied, see for example [2], [6], [9], [10], [17], [18]. This paper is devoted to the study of two properties weaker than realcompactness, namely *local realcompactness* and *HN-completeness*. It will be shown that these two properties are measurable. The notion of locally realcompact space was studied in [15], [14]. It is known that a space is locally realcompact if and only if it is open in its Hewitt-Nachbin realcompactification. Here we give an external characterization of HN-completeness relative to the Hewitt-Nachbin realcompactification.

Throughout the paper, all (topological) spaces are assumed to be at least Tychonoff. For well-known characterizations and properties concerning realcompact spaces one can consult [11], [19].

2. Definitions, notation and basic results

For sake of completeness, we now give some definitions and well-known results that are needed below. Let $\mathcal{A}(X)$ be the *algebra* generated by the collection $\mathcal{Z}(X)$ of all *zero sets* of a space X. By a measure μ on $\mathcal{A}(X)$ we mean a finitely additive non-negative real-valued function on $\mathcal{A}(X)$. A measure μ is called *regular* if $\mu(B) = \inf\{\mu(U) : B \subset U \in \mathcal{C}(X)\}$ for each $B \in \mathcal{A}(X)$, where $\mathcal{C}(X)$ denotes the collection of all *cozero sets* of X. Equivalently, μ is *regular* if $\mu(B) = \sup\{\mu(Z) : B \supset Z \in \mathcal{Z}(X)\}$ for each $B \in \mathcal{A}(X)$. From now on by a measure we mean a *regular measure*.

Definition 2.1. Let μ be a measure on $\mathcal{A}(X)$.

(I) μ is called σ-additive if

$$\mu\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mu(B_i)$$

whenever $\{B_i : i = 1, 2, \dots\}$ is a disjoint countable subcollection of $\mathcal{A}(X)$ with $\bigcup_{i=1}^{\infty} B_i \in \mathcal{A}(X)$.

(II) μ is called τ-additive if for every open cover $\mathcal{U}$ of X by cozero sets and for every $\epsilon > 0$ there is a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mu(\bigcup \mathcal{V}) > \mu(X) - \epsilon$.

A measure μ on X is called a *two-valued* measure if $\mu(\mathcal{A}(X)) = \{0, 1\}$. Let x be a fixed point of X. Then, a *Dirac measure* δ_x is defined by

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B \in \mathcal{A}(X), \\ 0 & \text{if } x \notin B \in \mathcal{A}(X). \end{cases}$$

We denote by $\mathcal{T}(X), \mathcal{T}_\sigma(X), \mathcal{T}_\tau(X)$ and $\mathcal{D}(X)$ the set of all two-valued measures, two-valued σ-additive measures, two-valued τ-additive measures and Dirac measures on X respectively. It is not difficult to see that for any space X we have:

$$\mathcal{T}_\tau(X) = \mathcal{D}(X) \subset \mathcal{T}_\sigma(X) \subset \mathcal{T}(X).$$

Let μ be a measure on X. Then by the *support* of μ we mean the set

$$S(\mu) = \bigcap\{Z \in \mathcal{Z}(X) : \mu(Z) = \mu(X)\} = X \setminus \bigcup\{U \in \mathcal{C}(X) : \mu(U) = 0\}.$$

Two-valued measures are known to be in one-to-one correspondence with maximal zero filters. Indeed, if $\mathcal{F}$ is a maximal zero filter of a space X, then the map $\mu : \mathcal{A}(X) \to \{0, 1\}$ defined by $\mu(B) = 1$ if and only if there exists $Z \in \mathcal{Z}(X)$ with $Z \subset B$, is an element of $\mathcal{T}(X)$. Moreover, if $\mathcal{F}$ has the *countable intersection property* (*c.i.p.*) then μ is an element of $\mathcal{T}_\sigma(X)$. Conversely, if $\mu : \mathcal{A}(X) \to \{0, 1\}$ is an element of $\mathcal{T}(X)$, then the collection $\mathcal{F} = \{Z \in \mathcal{Z}(X) : \mu(Z) = 1\}$ defines a maximal zero filter of the space X. Moreover, if μ is an element of $\mathcal{T}_\sigma(X)$, then $\mathcal{F}$ has c.i.p. One can consult [5] or [16] for a proof.

3. Locally realcompact spaces and HN-complete spaces

We begin this section by giving the definition of HN-complete spaces.

Definition 3.1. A space X is said to be *Hewitt-Nachbin complete* (HN-complete for short) if there is a sequence $\mathfrak{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ of open (cozero) covers of X such that every maximal zero $\mathfrak{U}$-Cauchy filter $\mathcal{F}$ with c.i.p. on X converges, where $\mathcal{F}$ is said to be $\mathfrak{U}$-Cauchy if for every $\mathcal{U} \in \mathfrak{U}$ there exists some $U \in \mathcal{U}$ such that $F \subset U$ for some $F \in \mathcal{F}$.

The notion of $\mathfrak{U}$-positive measure for some collection of cozero covers $\mathfrak{U}$ is given in [3].

Definition 3.2. Let $\mathcal{U}$ be a cozero cover of a space X and μ a measure on X. Then μ is said to be *$\mathcal{U}$-positive* if there exists a $U \in \mathcal{U}$ such that $\mu(U) > 0$.

If $\mathfrak{U}$ is a collection of cozero covers, then μ is said to be *$\mathfrak{U}$-positive* if μ is $\mathcal{U}$-positive for every $\mathcal{U} \in \mathfrak{U}$.

Remark 3.1. One can easily see that any τ-additive measure is $\mathcal{U}$-positive for any cozero cover $\mathcal{U}$ of X. Consequently, any Dirac measure is $\mathfrak{U}$-positive for any $\mathfrak{U}$.

Definition 3.3. For a collection of cozero covers $\mathfrak{U}$ of X we denote by $\mathcal{T}(X, \mathfrak{U})$ ($\mathcal{T}_\sigma(X, \mathfrak{U})$) the set of $\mathfrak{U}$-positive measures in $\mathcal{T}(X)$ ($\mathcal{T}_\sigma(X)$).

We can now show that HN-completeness is a measurable property.

Theorem 3.1. *The following conditions are equivalent for a space X.*

(i) *X is HN-complete.*
(ii) *There exists a sequence of cozero covers $\mathfrak{U}$ of X such that every $\mathfrak{U}$-positive two-valued σ-additive measure on X has a non empty support.*
(iii) *There exists a sequence of cozero covers $\mathfrak{U}$ of X such that*

$$\mathcal{T}_\sigma(X, \mathfrak{U}) = \mathcal{T}_\tau(X, \mathfrak{U}) = \mathcal{D}(X).$$

Proof: (i) $\Longrightarrow$ (ii). Let X be HN-complete and let $\mathfrak{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ be a sequence of cozero covers of X such that every maximal zero $\mathfrak{U}$-Cauchy filter with c.i.p. converges. Let $\mu \in \mathcal{T}_\sigma(X, \mathfrak{U})$ and $\mathcal{F} = \{Z \in \mathcal{Z}(X) : \mu(Z) = 1\}$. Then $\mathcal{F}$ is a maximal zero filter of the space X with c.i.p. and it is not difficult to see that it is $\mathfrak{U}$-Cauchy and so converges to some point $x \in X$. Then, for every $U \in \mathcal{C}(X)$ with $x \in U$, we have $\mu(U) = 1$ so that μ has a non empty support.

(ii) $\Longrightarrow$ (iii). Let there exist a sequence of cozero covers $\mathfrak{U}$ of X such that the trivial measure 0 is the only $\mathfrak{U}$-positive two-valued σ-additive measure on X with an empty support. Assume that there is a measure $\mu \in \mathcal{T}_\sigma(X, \mathfrak{U})$ which is not in $\mathcal{T}_\tau(X, \mathfrak{U})$. Then, there exists a cozero cover $\mathcal{V}$ of X such that $\mu(\bigcup_{i=1}^n V_i) = 0$ for every finite subcollection $V_1, \dots, V_n$ of $\mathcal{V}$. In particular, we have that $\mu(V) = 0$ for every $V \in \mathcal{V}$ and therefore μ has an empty support, so that $\mu = 0$.

(iii) $\Longrightarrow$ (i). Let $\mathfrak{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ be a sequence of cozero covers of X such that $\mathcal{T}_\sigma(X, \mathfrak{U}) = \mathcal{D}(X)$. We show that $\mathfrak{U}$ satisfies the conditions in Definition 3.1. Consider a maximal zero $\mathfrak{U}$-Cauchy filter $\mathcal{F}$ with c.i.p. Construct $\mu \in \mathcal{T}(X)$ by $\mu(Y) = 1$ if and only if there exists $F \in \mathcal{F}$ with $F \subset Y$. Then μ is σ-additive and since $\mathcal{F}$ is $\mathfrak{U}$-Cauchy, μ is also $\mathfrak{U}$-positive. By (iii), $\mu \in \mathcal{D}(X)$ and therefore there exists some $x \in X$ such that $\mu = \delta_x$. It is not difficult to see that this implies that $\mathcal{F}$ converges to x as required. □

The definition of HN-complete spaces can be restated in terms of ideals. An ideal $\mathcal{I}$ is said to have the *countable union property* (*c.u.p.*) if $\bigcup_{n\in\mathbb{N}} U_n \neq X$ for every countable collection $U_n \in \mathcal{I}$. We will consider ideals of cozero sets, i.e. cz-ideals. A cozero set C is said to be a co-neighborhood of a point x in X if $X \setminus C$ is a zero neighborhood of x. A cz-ideal $\mathcal{I}$ is said to co-converge to a point x if it contains all the cozero co-neighborhoods of x. A maximal cz-ideal co-converges if $\bigcup \mathcal{I} \neq X$. In this case $\bigcup \mathcal{I} = X \setminus \{x\}$, where $\mathcal{I}$ co-converges to x. One can see from Definition 3.1 that a space X is HN-complete if there is a sequence $\mathfrak{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ of cozero covers of X such that every maximal cozero $\mathfrak{U}$-Cauchy ideal $\mathcal{I}$ with c.u.p. on X co-converges, where $\mathcal{I}$ is said to be $\mathfrak{U}$-Cauchy if for every $\mathcal{U} \in \mathfrak{U}$ there exists some $U \in \mathcal{U}$ such that $X \setminus U \subset C$ for some $C \in \mathcal{I}$.

We now give an external characterization of HN-complete spaces. One can compare this to the external and internal characterizations of Čech-complete spaces [1], [4], [8]. In the proof we use the well-known fact (see for example [11]), that for any countable family of cozero sets C_n in X,

$$(*) \qquad \bigcup_{n\in\mathbb{N}} \operatorname{int}_{\nu X}(C_n \cup (\nu X \setminus X)) = \operatorname{int}_{\nu X}\Big(\bigcup_{n\in\mathbb{N}} C_n \cup (\nu X \setminus X)\Big).$$

Theorem 3.2. *A space X is HN-complete if and only if it is a G_δ set in its Hewitt-Nachbin realcompactification νX.*

Proof: Let X be a HN-complete space and let $\mathfrak{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence of cozero covers of X having the property in Definition 3.1. For every $n \in \mathbb{N}$ and every $U \in \mathcal{U}_n$ take a cozero set $V(U)$ in νX such that $V(U) \cap X = U$. Evidently we have the following subset inclusion

$$X \subset \bigcap_{n\in\mathbb{N}} \bigcup_{U\in\mathcal{U}_n} V(U)$$

and we need to show that equality holds.

Consider any $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{U\in\mathcal{U}_n} V(U)$ and denote by $\mathcal{I}_x$ the ideal of cozero co-neighborhoods of x in νX. Evidently, $\mathcal{I}_x$ has c.u.p. Now $\mathcal{I}_x \cap X$ is a cz-ideal in X and by complete regularity, it is $\mathfrak{U}$-Cauchy. Using equation $(*)$ one can show that it has c.u.p. Indeed, say there exists a countable collection C_n in $\mathcal{I}_x \cap X$ such that $\bigcup_{n\in\mathbb{N}} C_n = X$. Then $\bigcup_{n\in\mathbb{N}} \operatorname{int}_{\nu X}(C_n \cup (\nu X \setminus X)) = \nu X$. In particular,

$x \in \operatorname{int}_{\nu X}(C_n \cup (\nu X \setminus X))$ for some C_n. But $C_n = C_n' \cap X$, where C_n' is a cozero co-neighborhood of x in νX, which evidently cannot be. Now let $\mathcal{J}_x$ be a maximal cz-ideal in X containing the ideal $\mathcal{I}_x \cap X$. Then $\mathcal{J}_x$ is $\mathfrak{U}$-Cauchy and we only need to show that it has c.u.p. Again this is done by using $(*)$. Indeed, say there exists some countable collection $C_n \in \mathcal{J}_x$ such that $\bigcup C_n = X$. Then, there exists some $n \in \mathbb{N}$ such that $x \in \operatorname{int}_{\nu X}(C_n \cup (\nu X \setminus X))$ and therefore, there exists some zero neighborhood Z of x satisfying $Z \subset C_n \cup (\nu X \setminus X)$. Consequently, $C = \nu X \setminus Z \in \mathcal{I}_x$ and $C \cup (C_n \cup (\nu X \setminus X)) = \nu X$. Thus, $D = (C \cap X) \in \mathcal{I}_x \cap X$ and $D \cup C_n = X$, a contradiction. We thus proved that $\mathcal{J}_x$ has c.u.p. and since it is maximal and $\mathfrak{U}$-Cauchy it converges in X. Say $\bigcup \mathcal{J}_x = X \setminus \{z\}$. It is now not difficult to see that $x = z \in X$ as required to proof.

Conversely, say $X = \bigcap_{n\in\mathbb{N}} G_n$, where G_n are open in νX. For every $x \in X$ and every $n \in \mathbb{N}$ choose cozero sets $U_n(x)$ and $Z_n(x)$ in νX such that $x \in U_n(x) \subset Z_n(x) \subset G_n$. Let $\mathcal{U}_n = \{X \cap U_n(x) : x \in X\}$, for every $n \in \mathbb{N}$, and we show that the sequence of cozero covers $\mathfrak{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ has the required property. Take any maximal $\mathfrak{U}$-Cauchy z-filter $\mathcal{F}$ with c.i.p. in X. For every $F \in \mathcal{F}$ there exists some zero set F' in νX such that $F' \cap X = F$. Let $\mathcal{F}'$ be the collection of all zero sets in νX such that if $Z \in \mathcal{F}'$, then $Z \cap X = F$ for some $F \in \mathcal{F}$. Then $\mathcal{F}'$ is a filter with c.i.p. We now show that it is prime. Let Z_1 and Z_2 be two zero sets in νX such that $Z_1 \cup Z_2 \in \mathcal{F}'$. If Z_1 or Z_2 does not intersect X then there is nothing to prove, indeed if say $Z_2 \cap X = \emptyset$ then $(Z_1 \cup Z_2) \cap X = Z_1 \cap X \in \mathcal{F}$ and therefore, $Z_1 \in \mathcal{F}'$. If both Z_1 and Z_2 intersect X then $(Z_1 \cap X) \cup (Z_2 \cap X) \in \mathcal{F}$. But $\mathcal{F}$ is maximal and therefore prime. Hence, either $(Z_1 \cap X)$ or $(Z_2 \cap X)$ must be in $\mathcal{F}$ and consequently, either Z_1 or Z_2 must be in $\mathcal{F}'$. We have just shown that $\mathcal{F}'$ is a prime z-filter with c.i.p. in νX and so converges to some point $x \in \nu X$. Thus $x \in Z_n(x_n)$ for some x_n for every $n \in \mathbb{N}$, so that

$$x \in \bigcap_{n\in\mathbb{N}} Z_n(x_n) \subset \bigcap_{n\in\mathbb{N}} G_n = X,$$

showing that $x \in X$. Consequently, $\mathcal{F}$ converges as required to prove. □

We next give the definition of locally realcompact spaces and show that local realcompactness is also a measurable property.

Definition 3.4. A space X is said to be *locally realcompact* if every $x \in X$ has a neighborhood U_x such that $\overline{U_x}$ is realcompact. Equivalently, X is said to be *locally realcompact* if every $x \in X$ has a cozero realcompact neighborhood U_x.

Theorem 3.3. *The following conditions are equivalent for a space X.*

(i) *X is locally realcompact.*
(ii) *There exists a cozero cover $\mathcal{U}$ of X such that*

$$\mathcal{T}_\sigma(X,\mathcal{U}) = \mathcal{T}_\tau(X,\mathcal{U}) = \mathcal{D}(X).$$

(iii) *There exists a cozero cover $\mathcal{U}$ of X such that every $\mathcal{U}$-positive two-valued σ-additive measure on X has a non empty support.*

Proof: (i) $\Longrightarrow$ (ii). Let X be locally realcompact. For every $x \in X$ there exists a cozero realcompact neighborhood U_x of x. Since U_x is cozero, $\mathcal{A}(X) \cap U_x = \mathcal{A}(U_x)$. Let $\mathcal{U} = \{U_x : x \in X\}$ and let $\mu \in \mathcal{T}_\sigma(X, \mathcal{U})$. There exists some $y \in X$ such that $\mu(U_y) = 1$ and therefore, $\mu(A) = \mu(A \cap U_y)$ for every $A \in \mathcal{A}(X)$. Define $\mu_y \in \mathcal{T}(U_y)$ by $\mu_y(A) = \mu(B)$ for every $A \in \mathcal{A}(U_y)$, where B is any element in $\mathcal{A}(X)$ such that $A = B \cap U_y$. Then μ_y is well defined and since μ is σ-additive, so is μ_y, that is $\mu_y \in \mathcal{T}_\sigma(U_y)$. Since U_y is realcompact, $\mu_y \in \mathcal{D}(U_y)$ and there exists an $x \in U_y$ such that $\mu_y = \delta_x(U_y)$. Thus $\mu = \delta_x(X) = \delta_x$ and consequently, $\mu \in \mathcal{D}(X)$ as required.

(ii) $\Longrightarrow$ (iii). Let there exist a cozero cover $\mathcal{U}$ of X such that $\mathcal{T}_\sigma(X, \mathcal{U}) = \mathcal{D}(X)$ and let $\mu \in \mathcal{T}_\sigma(X, \mathcal{U})$. There exists some $x \in X$ such that $\mu = \delta_x$ and therefore, $\mu(V) = 1$ for every cozero V containing x showing that μ has a non empty support.

(iii) $\Longrightarrow$ (i). Let $\mathcal{U}$ be a cozero cover of X such that every $\mathcal{U}$-positive two-valued σ-additive measure on X has a non empty support. For every $x \in X$ there exists a $U \in \mathcal{U}$ such that $x \in U$ and also a cozero set V_x and a zero set Z_x such that $x \in V_x \subset \overline{V_x} \subset Z_x \subset U$. Let $\mathcal{V} = \{V_x : x \in X\}$ and we prove that V_x is realcompact for all $x \in X$ by showing that Z_x is realcompact for all $x \in X$. Indeed, say there is an $x \in X$ such that Z_x is not realcompact. Then there is a measure $\mu \in \mathcal{T}_\sigma(Z_x)$ which is not in $\mathcal{D}(Z_x)$. Consider the extension $\tilde{\mu}$ of μ defined by $\tilde{\mu}(A) = \mu(A \cap Z_x)$ for $A \in \mathcal{A}(X)$. Then $\tilde{\mu} \in \mathcal{T}_\sigma(X)$ and since $Z_x \in \mathcal{A}(X)$ we have $\tilde{\mu}(Z_x) = 1$ which shows that $\tilde{\mu} \in \mathcal{T}_\sigma(X, \mathcal{U})$. By (iii), $\tilde{\mu}$ has a non empty support and so there exists some $y \in X$ such that $\tilde{\mu}(V) = 1$ for every cozero V containing y. Since $\tilde{\mu}(Z_x) = 1$ we have that $y \in Z_x$. We finally show that $\mu = \delta_y(Z_x)$ leading to a contradiction. Indeed, take any $W \in \mathcal{C}(Z_x)$ with $y \in W$. There exists some open $G \subset X$ such that $W = G \cap Z_x$ and some cozero in X set O_y such that $y \in O_y \subset G$. Then $O_y \cap Z_x$ is a cozero in Z_x set containing y. Now $\tilde{\mu}(O_y) = 1$ and therefore, $\mu(O_y \cap Z_x) = 1$ which gives $\mu(W) = 1$. Consequently, since μ is regular, we have just proved that $\mu(B) = 1$ for every $B \in \mathcal{A}(Z_x)$ with $x \in B$. Also, if $B \in \mathcal{A}(Z_x)$ with $x \notin B$ then $\mu(B) = 0$ and thus, $\mu = \delta_y(Z_x)$. $\square$

The following external characterization of locally realcompact spaces is proved in [15].

Theorem 3.4. *A space X is locally realcompact if and only if it is open in its Hewitt-Nachbin realcompactification νX.*

It is clear from the definitions that we have the following implications,

$$\text{realcompact} \longrightarrow \text{locally realcompact} \longrightarrow \text{HN-complete}.$$

We have examples to show that none of the above implications is reversible. Indeed, the space $[0, \omega_1[$ is a locally realcompact space (in fact it is locally compact) but is not realcompact. From Theorems 4.2 and 4.4 below we get that the space $[0,\omega_1[^{\omega_0}$ is HN-complete but is not locally realcompact.

It is also evident that every locally compact space is locally realcompact and that every Čech-complete space is HN-complete. On the other hand both the Sorgenfrey line S and the set of rationals $\mathbb{Q}$ (as a subspace of $\mathbb{R}$) are realcompact but S is not locally compact while $\mathbb{Q}$ is not Čech-complete.

4. Subspaces and products of locally realcompact and HN-complete spaces

The next result is that both HN-completeness and local realcompactness are invariant with respect to both closed and Baire subsets and also to finite products. The proof is given for HN-complete spaces, the proof for locally realcompact spaces is simpler.

Theorem 4.1.

(I) *A closed subset of a locally realcompact* (*HN-complete*) *space is locally realcompact* (*HN-complete*).

(II) *A Baire subset of a locally realcompact* (*HN-complete*) *space is locally realcompact* (*HN-complete*).

(III) *A finite product of locally realcompact* (*HN-complete*) *spaces is locally realcompact* (*HN-complete*).

Proof: (I) and (II). Let X be a subspace of a HN-complete space Y and let $\mathfrak{U} = \{\mathcal{U}_i : i \in \mathbb{N}\}$ be a sequence of cozero covers of Y such that $\mathcal{T}_\sigma(Y,\mathfrak{U}) = \mathcal{D}(Y)$. Let $\mathcal{V}_i = \{U \cap X : U \in \mathcal{U}_i\}$ and let $\mathfrak{V} = \{\mathcal{V}_i : i \in \mathbb{N}\}$. Let $\mu \in \mathcal{T}_\sigma(X,\mathfrak{V})$. Then $\tilde{\mu} \in \mathcal{T}_\sigma(Y)$, where $\tilde{\mu}(B) = \mu(B \cap X)$ for every $B \in \mathcal{A}(Y)$. For every $i \in \mathbb{N}$ there exists a $V = U \cap X \in \mathcal{V}_i$ such that $\mu(V) = 1$ so that $\tilde{\mu}(U) = 1$ and $\tilde{\mu} \in \mathcal{T}_\sigma(Y,\mathfrak{U})$. Since Y is HN-complete (with respect to $\mathfrak{U}$) there exists some $y \in Y$ such that $\tilde{\mu} = \delta_y$.

To prove (I), let X be closed in Y and let $U \in \mathcal{C}(Y)$ contain y. Then $\mu(U \cap X) = \tilde{\mu}(U) = 1$ and hence, $U \cap X \neq \emptyset$. This implies that y is an element of X. Next, let $U \in \mathcal{C}(X)$ contain y. Since Y is Tychonoff, there exists a $U' \in \mathcal{C}(Y)$ such that $y \in U' \cap X \subset U$. Since $\tilde{\mu} = \delta_y$ we have that $\tilde{\mu}(U') = 1$ and hence, $\mu(U) = 1$. This implies that $\mu^*(\{y\}) = 1$ and consequently we have that $\mu \in \mathcal{T}(X) \cap \mathcal{M}_t(X) = \mathcal{D}(X)$.

To prove (II), let X be a Baire subset of Y. Since $\tilde{\mu}$ is σ-additive we may assume that $\tilde{\mu}$ is defined on $\mathcal{B}(Y)$. X is in $\mathcal{B}(Y)$ and therefore $\tilde{\mu}(X) = 1$. Since $\tilde{\mu} = \delta_y$ this implies that $y \in X$. From this we may conclude that $\mu \in \mathcal{D}(X)$.

(III). Let $\{X_i : i = 1,\dots,n\}$ be a finite collection of HN-complete spaces. Let $\mathfrak{U}^i = \{\mathcal{U}^i_k : k \in \mathbb{N}\}$ be a sequence of cozero covers of X_i such that $\mathcal{T}_\sigma(X_i,\mathfrak{U}^i) = \mathcal{D}(X_i)$ for $i = 1,\dots,n$. Consider the sequence of cozero covers $\mathfrak{U} = \{\mathcal{U}_k : k \in \mathbb{N}\}$ of $\prod_{i=1}^n X_i$, where $\mathcal{U}_k = \{U_1 \times \dots \times U_n : U_i \in \mathcal{U}^i_k \text{ for } i = 1,\dots,n\}$.

Let μ be an element of $\mathcal{T}_\sigma(\prod_{i=1}^n X_i, \mathfrak{U})$. If π_i denotes the projection from $\prod_{j=1}^n X_j$ onto X_i and μ_i is defined by

$$\mu_i(B) = \mu(\pi_i^{-1}[B]) \text{ for every } B \in \mathcal{A}(X_i),$$

then μ_i is an element of $\mathcal{T}_\sigma(X_i)$.

For every $k \in \mathbb{N}$ there exists a $U \in \mathcal{U}_k$ such that $\mu(U) = 1$. Let $U = U_1 \times \cdots \times U_n$, then

$$\mu_i(U_i) = \mu(\pi_i^{-1}[U_i]) = \mu(X_1 \times \cdots \times U_i \times \cdots \times X_n) = 1,$$

so that $\mu_i \in \mathcal{T}_\sigma(X_i, \mathfrak{U}^i)$.

Hence there exists an $x_i \in X_i$ satisfying $\mu_i = \delta_{x_i}$. Define $x = (x_1, \dots, x_n) \in \prod_{i=1}^n X_i$ and let U be an arbitrary element of $\mathcal{C}(\prod_{i=1}^n X_i)$ that contains x. Then there exist sets $U_i \in \mathcal{C}(X_i)$, such that $x_i \in U_i$, for every $i = 1, \dots, n$, and $U_1 \times \cdots \times U_n \subset U$. Since $\mu_i = \delta_{x_i}$ we have that $\mu(\pi_i^{-1}[U_i]) = \mu_i(U_i) = 1$ for every $i = 1, \dots, n$, and hence $\mu(U) \geqslant \mu(\bigcap_{i=1}^n \pi_i^{-1}[U_i]) = 1$. This implies that $\mu^*(\{x\}) = 1$, where μ^* is the outer measure defined by μ, and therefore, $\mu \in \mathcal{D}(\prod_{i=1}^n X_i)$. $\square$

Our next aim is to improve on the results of Theorem 4.1(III).

Theorem 4.2. *The product $\prod_{\alpha\in\kappa} X_\alpha$, where $X_\alpha \neq \emptyset$ for all $\alpha \in \kappa$, is locally realcompact if and only if all spaces X_α are locally realcompact and there exists a finite subset $\kappa_0 \subset \kappa$ such that X_α is realcompact for all $\alpha \in \kappa \setminus \kappa_0$.*

Proof: Since any product of realcompact spaces is realcompact, sufficiency follows from Theorem 4.1(III).

To prove necessity, let $\prod_{\alpha\in\kappa} X_\alpha$ be a non-empty locally realcompact space. Take any $\beta \in \kappa$ and a point $x \in X_\beta$; we show that x has a cozero realcompact neighborhood W in X_β. Let x_α be an arbitrary point of X_α for $\alpha \neq \beta$ and let $x_\beta = x$. The point $x = (x_\alpha)_{\alpha\in\kappa} \in \prod_{\alpha\in\kappa} X_\alpha$ has a cozero realcompact neighborhood U. There exists a member $\prod_{\alpha\in\kappa} W_\alpha$ of the canonical base for $\prod_{\alpha\in\kappa} X_\alpha$ such that $(x_\alpha)_{\alpha\in\kappa} \in \prod_{\alpha\in\kappa} W_\alpha \subset U$ and $W_\alpha = X_\alpha$ for $\alpha \in \kappa \setminus \kappa_0$, where $|\kappa_0| < \aleph_0$. One can assume that each W_α is cozero in X_α and therefore, $\prod_{\alpha\in\kappa} W_\alpha$ is also cozero (being a finite intersection of cozero sets) and hence is realcompact. Consequently, W_α is realcompact for every $\alpha \in \kappa$. $\square$

Theorem 4.3. *The product of countably many HN-complete spaces is HN-complete.*

Proof: Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of HN-complete spaces. Let $\mathfrak{U}^i = \{\mathcal{U}_k^i : k \in \mathbb{N}\}$ be a sequence of cozero covers of X_i such that $\mathcal{T}_\sigma(X_i, \mathfrak{U}^i) = \mathcal{D}(X_i)$ for every $i \in \mathbb{N}$. Consider the sequence of cozero covers $\mathfrak{U} = \{\mathcal{U}_{kn} : k, n \in \mathbb{N}\}$ of $\prod_{i\in\mathbb{N}} X_i$, where

$$\mathcal{U}_{kn} = \left\{ \prod_{i\in\mathbb{N}} W_i : W_i = X_i \ \text{ for } \ i \neq k \ \text{ and } \ W_k \in \mathcal{U}_n^k \right\}.$$

Let μ be an element of $\mathcal{T}_\sigma(\prod_{i\in\mathbb{N}} X_i, \mathfrak{U})$. If π_i denotes the projection from $\prod_{j\in\mathbb{N}} X_j$ onto X_i and μ_i is defined by

$$\mu_i(B) = \mu(\pi_i^{-1}[B]) \ \text{ for every } \ B \in \mathcal{A}(X_i),$$

then μ_i is an element of $\mathcal{T}_\sigma(X_i)$.

For every $n \in \mathbb{N}$ there exists a $U \in \mathcal{U}_{in}$ such that $\mu(U) = 1$. Let $U = \prod_{i\in\mathbb{N}} W_i$, where $W_j = X_j$ for $j \neq i$ and $W_i \in \mathcal{U}_n^i$; then

$$\mu_i(W_i) = \mu(\pi_i^{-1}[W_i]) = \mu\left(\prod_{i\in\mathbb{N}} W_i \right) = \mu(U) = 1,$$

so that $\mu_i \in \mathcal{T}_\sigma(X_i, \mathfrak{U}^i)$.

Hence there exists an $x_i \in X_i$ satisfying $\mu_i = \delta_{x_i}$. Define $x = (x_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} X_i$ and let U be an arbitrary element of $\mathcal{C}(\prod_{i\in\mathbb{N}} X_i)$ that contains x. Then there exist a finite subset $\mathbb{N}_0$ of $\mathbb{N}$ and sets $U_i \in \mathcal{C}(X_i)$ for every $i \in \mathbb{N}_0$, such that $x_i \in U_i$, for every $i \in \mathbb{N}_0$, and $\prod_{i\in\mathbb{N}} V_i \subset U$, where $V_i = U_i$ for every $i \in \mathbb{N}_0$ and $V_i = X_i$ for every $i \in \mathbb{N} \setminus \mathbb{N}_0$. Since $\mu_i = \delta_{x_i}$ we have that $\mu(\pi_i^{-1}[V_i]) = \mu_i(V_i) = 1$ for every $i \in \mathbb{N}$, and hence $\mu(U) \geqslant \mu(\bigcap_{i\in\mathbb{N}} \pi_i^{-1}[V_i]) = 1$. This implies that $\mu^*(\{x\}) = 1$, where μ^* is the outer measure defined by μ, and therefore, $\mu \in \mathcal{D}(\prod_{i\in\mathbb{N}} X_i)$. □

Theorem 4.4. *The product $\prod_{\alpha\in\kappa} X_\alpha$, where $X_\alpha \neq \emptyset$ for $\alpha \in \kappa$, is HN-complete if and only if all spaces X_α are HN-complete and there exists a countable set $\kappa_0 \subset \kappa$ such that X_α is realcompact for $\alpha \in \kappa \setminus \kappa_0$.*

Proof: Since any product of realcompact spaces is realcompact, sufficiency follows from Theorem 4.3.

To prove necessity, let $\prod_{\alpha\in\kappa} X_\alpha$ be a non-empty HN-complete space. Let $\mathfrak{U} = \{\mathcal{U}_n : n \in \mathbb{N}\}$ be a sequence cozero covers of $\prod_{\alpha\in\kappa} X_\alpha$ such that every prime zero $\mathfrak{U}$-Cauchy filter $\mathcal{F}$ with c.i.p. on $\prod_{\alpha\in\kappa} X_\alpha$ is fixed.

Fix a point $x = (x_\alpha)_{\alpha\in\kappa} \in \prod_{\alpha\in\kappa} X_\alpha$ and any $U_i \in \mathcal{U}_i$ such that $x \in U_i$ for all $i \in \mathbb{N}$. There exists a member $\prod_{\alpha\in\kappa} W_\alpha^i$ of the canonical base for $\prod_{\alpha\in\kappa} X_\alpha$ such that $x \in \prod_{\alpha\in\kappa} W_\alpha^i \subset U_i$ and $W_\alpha^i = X_\alpha$ for $\alpha \in \kappa \setminus \kappa_i$, where $|\kappa_i| < \aleph_0$. One can assume that each W_α^i is cozero in X_α. Let $\kappa_0 = \bigcup_{i\in\mathbb{N}} \kappa_i$, so that κ_0 is countable.

Take an $\beta \in \kappa \setminus \kappa_0$ and let $\mathcal{F}_\beta$ be a prime zero filter with c.i.p.; we show that $\mathcal{F}_\beta$ is fixed in X_β. Let

$$\mathcal{N}_\alpha = \{A \in \mathcal{Z}(X_\alpha) : x_\alpha \in A\}$$

be the maximal zero filter fixed at x_α in X_α and consider the filter base $\mathcal{F}$ in $\prod_{\alpha\in\kappa} X_\alpha$ given by

$$\left\{ \prod_{\alpha\in\kappa} F_\alpha : F_\alpha = \{x_\alpha\} \text{ for every } \alpha \neq \beta \text{ and } F_\beta \in \mathcal{F}_\beta \right\}$$

and let $\mathcal{G}$ be the zero filter in $\prod_{\alpha\in\kappa} X_\alpha$ given by

$$\left\{ G : G \in \mathcal{Z}\left(\prod_{\alpha\in\kappa} X_\alpha \right), F \subset G \text{ for some } F \in \mathcal{F} \right\}.$$

Evidently, $\mathcal{G}$ has c.i.p. and sets of the form

$$\left\{ \prod_{\alpha\in\kappa} F_\alpha : F_\alpha \in \mathcal{N}_\alpha \text{ for every } \alpha \in \kappa_0,\ F_\alpha = X_\alpha \text{ for every } \alpha \in \kappa \setminus (\kappa_0 \cup \{\beta\}) \text{ and } F_\beta \in \mathcal{F}_\beta \right\}$$

are in $\mathcal{G}$ and therefore $\mathcal{G}$ is $\mathfrak{U}$-Cauchy. We now show that $\mathcal{G}$ is prime.

Let $G_1, G_2 \in \mathcal{Z}(\prod_{\alpha\in\kappa} X_\alpha)$ such that $G_1 \cup G_2 \in \mathcal{G}$. By definition, there exists $F \in \mathcal{F}$ such that $F \subset G_1 \cup G_2$. Let $F = \prod_{\alpha\in\kappa} F_\alpha$ where $F_\alpha = \{x_\alpha\}$ for every $\alpha \neq \beta$ and $F_\beta \in \mathcal{F}_\beta$. Let us denote by Z^β the subspace $\prod_{\alpha\in\kappa} A_\alpha$ of $\prod_{\alpha\in\kappa} X_\alpha$, where $A_\alpha = \{x_\alpha\}$ for every $\alpha \neq \beta$ and $A_\beta = X_\beta$. Then Z^β is homeomorphic to X_β where as a homeomorphism $f : Z^\beta \to X_\beta$ one can take the map $f[(x_\alpha)_{\alpha\in\kappa}] = x_\beta$, that is the restriction of the projection π_β on Z^β. Since $F \subset G_1 \cup G_2$ we have that $H = (G_1 \cup G_2) \cap Z^\beta \neq \emptyset$ and H is a zero set in Z^β. Thus $f(H) \in \mathcal{Z}(X_\beta)$ and $F_\beta \subset f(H)$. This shows that $f(H) \in \mathcal{F}_\beta$ and therefore, either $f(G_1 \cap Z^\beta)$ or $f(G_2 \cap Z^\beta)$ is in $\mathcal{F}_\beta$. Consequently we have that either G_1 or G_2 is in $\mathcal{G}$ as required.

By assumption we get that $\mathcal{G}$ is fixed, that is $\bigcap \mathcal{G} \neq \emptyset$. Say $y = (y_\alpha)_{\alpha\in\kappa} \in \bigcap \mathcal{G}$, then $y_\beta \in \bigcap \mathcal{F}_\beta$. Indeed, if there exists some $H \in \mathcal{F}_\beta$ not containing y_β, then $G = \prod_{\alpha\in\kappa} F_\alpha$, where $F_\alpha = X_\alpha$ for every $\alpha \neq \beta$ and $F_\beta = H$, is in $\mathcal{G}$ but $y \notin G$. Thus we proved that $\mathcal{F}_\beta$ is fixed as required.

Finally, using the same notation as above, if a countable product $\prod_{i\in\mathbb{N}} X_i$ is HN-complete then X_i is HN-complete for all $i \in \mathbb{N}$ since it is homeomorphic to the closed subspace Z^i. $\square$

References

[1] Arhangelskiĭ A.V., *On topological spaces which are complete in the sense of Čech*, Vestnik Moskov. Univ. Ser. I Mat. Meh. (1961), no. 2, 37–40 (Russian).

[2] Blair R.L., van Douwen E., *Nearly realcompact spaces*, Topology Appl. **47** (1992), no. 3, 209–221.

[3] Buhagiar D., Chetcuti E., Dvurečenskij A., *Measure-theoretic characterizations of certain topological properties*, Bull. Pol. Acad. Sci. **53** (2005), no. 1, 99–109.

[4] Čech E., *On bicompact spaces*, Ann. Math. **38** (1937), 823–844.

[5] Dijkstra J.J., *Measures in topology*, Master Thesis, Univ. of Amsterdam, 1977.

[6] Dykes N., *Generalizations of realcompact spaces*, Pacific J. Math. **33** (1970), 571–581.

[7] Engelking R., *General Topology*, Heldermann, Berlin, 1989 (revised edition).

[8] Frolík Z., *Generalizations of the G_δ-property of complete metric spaces*, Czechoslovak Math. J. **10** (1960), 359–379.

[9] Frolík Z., *A generalization of realcompact spaces*, Czechoslovak Math. J. **13** (1963), 127–138.

[10] Gardner R.J., Pfeffer W.F., *Borel Measures*, in: Handbook of Set-Theoretic Topology, Elsevier, 1984, pp. 961–1043.

[11] Gillman L., Jerison M., *Rings of Continuous Functions*, Springer, New York, 1976.

[12] Hart K.P., Nagata J., Vaughan J.E. (Eds.), *Encyclopedia of General Topology*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 2004.

[13] Hewitt E., *Linear functionals on spaces of continuous functions*, Fund. Math. **37** (1950), 161–189.

[14] Isiwata T., *On locally Q-complete spaces,* I, II*, and* III, Proc. Japan Acad. **35** (1959), 232–236, 263–267, 431–434.

[15] Mack J., Rayburn M., Woods G., *Lattices of topological extensions*, Trans. Amer. Math. Soc. **189** (1972), 163–174.

[16] Nagata J., *Modern General Topology*, Elsevier Science Publishers B.V., Amsterdam, The Netherlands, 1985, Second revised edition.

[17] Rice M.D., Reynolds G.D., *Weakly Borel-complete topological spaces*, Fund. Math. **105** (1980), 179–185.

[18] Sakai M., *A new class of isocompact spaces and related results*, Pacific J. Math. **122** (1986), no. 1, 211–221.

[19] Weir M., *Hewitt-Nachbin Spaces*, North Holland Math. Studies, American Elsevier, New York, 1975.

Department of Mathematics, Faculty of Science, University of Malta,
Msida MSD.06, Malta

E-mail: david.buhagiar@um.edu.mt

Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49,
SK-814 73 Bratislava, Slovakia

E-mail: chetcuti@mat.savba.sk

(*Received* July 29, 2005, *revised* May 30, 2006)