Baire-one mappings contained in a usco map

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Abstract. We investigate Baire-one functions whose graph is contained in the graph of a usco mapping. We prove in particular that such a function defined on a metric space with values in $\mathbb{R}^d$ is the pointwise limit of a sequence of continuous functions with graphs contained in the graph of a common usco map.

Keywords: Baire-one function, usco map, usco-bounded sequence of continuous functions

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1. Introduction

We study the following question:

Let X be a metric space, Y a convex subset of a normed linear space and $f : X \to Y$ a Baire-one function whose graph is contained in the graph of a usco mapping. Is there a sequence of continuous functions $f_n : X \to Y$ pointwise converging to f such that the graphs of all the f_n's are contained in a usco map $\varphi : X \to Y$?

This question appeared in the joint research of R. Anguelov and the author — see [1]. In the quoted paper a natural convergence structure on the space of minimal usco maps is introduced and studied. The question we address here is closely related to the possibility to approximate minimal usco maps by continuous functions. Namely, if X is a Baire space and the answer to our question is positive, then continuous functions form a dense subset in the convergence space of minimal usco maps from X to Y.

Moreover, this question is also natural and interesting in itself as there are several theorems on the existence of Baire-one selections of multivalued (in particular usco) maps — see e.g. [4], [5], [7]. In particular, every usco map from a metric space into a normed linear space admits a Baire-one selection; therefore Baire-one functions whose graph is contained in the graph of a usco mapping are quite common.

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We do not know the full answer to our question but we prove some partial results. One of them is that the answer is positive if Y is a closed convex subset of a finite-dimensional space. This is used in [1] to show that if X is a Baire metric space then continuous functions from X to $\mathbb{R}^d$ form a dense subset of the convergence space of minimal usco maps.

Let us start by recalling and introducing some notions.

A nonempty-valued mapping $\varphi : X \to Y$ is called *upper semi-continuous compact valued* (shortly *usco*) if $\varphi(x)$ is a (nonempty) compact subset of Y for each $x \in X$ and $\{x \in X : \varphi(x) \subset U\}$ is open in X for each $U \subset Y$ open.

A function $f : X \to Y$ is called *Baire-one* if it is the pointwise limit of a sequence of continuous functions.

We say that a family of functions (defined on X with values in Y) is *usco-bounded* if there is a usco map $\varphi : X \to Y$ whose graph, i.e. the set

$$\{(x, y) \in X \times Y : y \in \varphi(x)\}$$

contains the graphs of all the functions from the family. We will use this terminology for single functions and for sequences of functions.

2. Basic facts and examples

An important role is played by the following characterization of usco maps and maps whose graph is contained in the graph of a usco map.

Lemma 2.1. *Let X and Y be metric spaces and $\varphi : X \to Y$ a nonempty-valued set-valued mapping.*

(i) *The mapping φ is usco if and only if whenever x_n is a sequence in X converging to some $x \in X$ and $y_n \in \varphi(x_n)$ for each $n \in \mathbb{N}$, there is a subsequence of y_n converging to an element of $\varphi(x)$.*

(ii) *There is a usco map $\psi : X \to Y$ with $\varphi \subset \psi$ (in the sense of inclusion of graphs) if and only if whenever x_n is a sequence in X converging to some $x \in X$ and $y_n \in \varphi(x_n)$ for each $n \in \mathbb{N}$, there is a convergent subsequence of y_n.*

Proof: The point (i) is an analogue of [2, Lemma 3.1.1]. The assertion of the quoted lemma is the same — only X and Y are arbitrary topological spaces and nets are used instead of sequences. As we are now dealing with metric spaces, sequences are enough and the same proof works.

Let us show the point (ii). The 'only if' part follows immediately from (i). It remains to prove the 'if' part. Let ψ be the multivalued mapping whose graph is the closure in $X \times Y$ of the graph of φ. We will show that ψ is usco using (i). Denote by d and ρ the metrics of X and Y, respectively.

Let x_n be a sequence in X converging to $x \in X$ and $y_n \in \psi(x_n)$ for each $n \in \mathbb{N}$. Then each pair (x_n, y_n) belongs to the graph of ψ. As the graph of φ

is dense in the graph of ψ, there are pairs (x'_n, y'_n) in the graph of φ such that $d(x'_n, x_n) < \frac{1}{n}$ and $\rho(y'_n, y_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$. Then $x'_n \to x$, and hence there is a subsequence y'_{n_k} converging to some $y \in Y$. Then y_{n_k} converge to y as well. Hence (x_{n_k}, y_{n_k}) converges to (x, y). As the graph of ψ is closed, we get $y \in \psi(x)$. This completes the proof. $\square$

An important subclass of Baire-one functions consists of so-called simple functions. We recall the definition.

Let X and Y be metric spaces and $f : X \to Y$ be a function. The function f is called *simple* if there is a σ-discrete partition of X into F_σ sets such that f is constant on each element of the partition.

Recall that a family of subsets of X is *discrete* if each point of X has a neighborhood meeting at most one element of the family; and a family is *σ-discrete* if it is a countable union of discrete families.

It is easy to check that f is simple if and only if there is a partition F_γ, $\gamma \in \Gamma$, of X such that

- f is constant on each F_γ;
- each F_γ can be expressed as an increasing union of closed sets F_γ^n such that the family F_γ^n, $\gamma \in \Gamma$, is discrete for each $n \in \mathbb{N}$.

If Y is a convex subset of a normed linear space (or more generally if Y is an arcwise connected metric space), then any simple function with values in Y is Baire-one (see e.g. [6, Lemma 2.13]).

Moreover, we need the following well-known approximation result.

Lemma 2.2. *Let X and Y be metric spaces. Then any Baire-one function $f : X \to Y$ is the uniform limit of a sequence of simple functions.*

Proof: It follows from [3, Lemmata 1.1 and 1.4] that f has a σ-discrete function base consisting of closed sets (i.e., a σ-discrete family $\mathcal{B}$ of closed subsets of X such that $f^{-1}(U)$ is the union of a subfamily of $\mathcal{B}$ for each $U \subset Y$ open). The assertion then follows by using Lemma 2.7 of [6]. $\square$

Now we are going to give some examples of Baire-one mappings which are usco-bounded and some which are not. The first one is trivial. It follows from the fact that the closed bounded sets in $\mathbb{R}^n$ are compact.

Example 2.3. *Any bounded (Baire-one) function $f : X \to \mathbb{R}^n$ is usco-bounded.*

Also the second example is trivial:

Example 2.4. (i) *The function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is a Baire-one function which is not usco-bounded.

(ii) *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is usco-bounded (it is continuous) although it is not bounded.*

We continue by two more general statements.

Example 2.5. *Let Y be an infinite dimensional normed space. Then there is a bounded Baire-one function $f : \mathbb{R} \to Y$ which is not usco-bounded.*

Proof: Let y_n be a sequence in the unit ball of Y which has no convergent subsequence (as Y is infinite-dimensional, the unit ball is not compact and hence such a sequence exists). Define the function f by the formula

$$f(x) = \begin{cases} 0, & x \in (-\infty, 0] \cup (1, +\infty), \\ y_n, & x \in (\frac{1}{n+1}, \frac{1}{n}],\ n \in \mathbb{N}. \end{cases}$$

Then f is bounded, it is easily seen to be simple, and hence Baire-one. However, it is not usco-bounded due to Lemma 2.1. Indeed, $\frac{1}{n} \to 0$ and the sequence $f(\frac{1}{n}) = y_n$ has no convergent subsequence. □

Example 2.6. *Let Y be a non-complete metric space. Then there is a sequence of simple functions $f_n : \mathbb{R} \to Y$ which uniformly converges to a simple function $f : \mathbb{R} \to Y$ such that each f_n is usco-bounded but f is not.*

Proof: Let y_n be a Cauchy sequence in Y which is not convergent. Define the functions f_n by the formula:

$$f_n(x) = \begin{cases} 0, & x \in (-\infty, 0] \cup (1, +\infty), \\ y_n, & x \in (0, \frac{1}{n}], \\ y_k, & x \in (\frac{1}{k+1}, \frac{1}{k}],\ 1 \le k < n. \end{cases}$$

Then each f_n is easily seen to be a usco-bounded simple function. Moreover, as the sequence y_n is Cauchy, the sequence f_n uniformly converges to the function f defined by the formula

$$f(x) = \begin{cases} 0, & x \in (-\infty, 0] \cup (1, +\infty), \\ y_n, & x \in (\frac{1}{n+1}, \frac{1}{n}],\ n \in \mathbb{N}. \end{cases}$$

This function is simple and is not usco-bounded. This can be proved by the argument used in the previous example (note that y_n is a non-converging Cauchy sequence and hence has no convergent subsequence). □

3. Main results

In this section we show some partial positive answers to the question from the introduction. The first result shows that the answer is positive if f is a simple function.

Theorem 3.1. *Let X be a metric space, Y a convex subset of a normed linear space and $f : X \to Y$ be a usco-bounded simple function. Then there is a usco-bounded sequence of continuous functions $f_n : X \to Y$ which pointwise converges to f.*

Proof: We will imitate the proof of the fact that simple functions are Baire-one in [6, Lemmata 2.12 and 2.13]. Without loss of generality suppose that $0 \in Y$. Fix a partition F_γ, $\gamma \in \Gamma$, of X such that

- f is constant on each F_γ;
- each F_γ can be expressed as an increasing union of closed sets F_γ^n such that the family F_γ^n, $\gamma \in \Gamma$, is discrete for each $n \in \mathbb{N}$.

For $n \in \mathbb{N}$ and $\gamma \in \Gamma$ we define the following functions:

$$\begin{aligned} d^n(x) &= \operatorname{dist}(x, \bigcup_{\gamma\in\Gamma} F_\gamma^n), \\ d_\gamma^n(x) &= \operatorname{dist}(x, F_\gamma^n), \\ e_\gamma^n(x) &= \operatorname{dist}(x, \bigcup_{\delta\in\Gamma\setminus\{\gamma\}} F_\delta^n). \end{aligned}$$

All these functions are continuous and hence the set

$$G_\gamma^n = \{x \in X : d_\gamma^n(x) < \tfrac{1}{3} e_\gamma^n(x)\}$$

is open for each n and γ. Moreover, as the family F_γ^n, $\gamma \in \Gamma$, is discrete, we get $F_\gamma^n \subset G_\gamma^n$. It is proved in [6, pp. 34–35] that the family G_γ^n, $\gamma \in \Gamma$, is discrete for each $n \in \mathbb{N}$, too.

Denote by y_γ the value of f on F_γ and define the functions f_n as follows:

$$f_n(x) = \begin{cases} 0, & x \in X \setminus \bigcup_{\gamma\in\Gamma} G_\gamma^n, \\ \left(1 - \frac{4d_\gamma^n(x)}{d_\gamma^n(x)+e_\gamma^n(x)}\right) y_\gamma, & x \in G_\gamma^n. \end{cases}$$

The function f_n is continuous (see [6, pp. 35–36]) and $f_n(x) = y_\gamma$ for $x \in F_\gamma^n$. Hence clearly f_n pointwise converges to f.

It remains to show that the sequence f_n is usco-bounded. To see this we will use Lemma 2.1. Fix a sequence x_n converging to $x \in X$ and a sequence k_n of natural numbers. We will be done if we show that the sequence $f_{k_n}(x_n)$ has a converging subsequence.

Up to passing to a subsequence we can suppose that the sequence k_n is either constant or increasing. If $k_n = k$ for each n, then $f_{k_n}(x_n) = f_k(x_n)$ converges to $f_k(x)$ due to the continuity of f_k.

Hence suppose that k_n is increasing. If $x_n \in X \setminus \bigcup_{\gamma\in\Gamma} G_\gamma^{k_n}$ for infinitely many n's, then $f_{k_n}(x_n) = 0$ for infinitely many n's and hence we have converging subsequence.

Thus suppose that for each $n \in \mathbb{N}$ there is some $\gamma_n \in \Gamma$ with $x_n \in G_{\gamma_n}^{k_n}$. By the definition of the set $G_{\gamma_n}^{k_n}$ there is some $z_n \in F_{\gamma_n}^{k_n}$ with $\rho(x_n, z_n) < \frac{1}{3} e_{\gamma_n}^{k_n}(x_n)$. We have $f_{k_n}(x_n) = c_n y_{\gamma_n}$ for some $c_n \in [0,1]$. Without loss of generality we may suppose that the sequence c_n converges to some $c \in [0,1]$.

Let $\alpha \in \Gamma$ be such that $x \in F_\alpha$. If $\gamma_n = \alpha$ for infinitely many n's, then there is a subsequence of $f_{k_n}(x_n)$ converging to cy_α.

Finally, suppose that $\gamma_n \neq \alpha$ for each n. Then we have

$$\rho(x_n, z_n) < \frac{1}{3} e_{\gamma_n}^{k_n}(x_n) \leq \frac{1}{3} e_{\gamma_n}^{k_n}(x) + \frac{1}{3}\rho(x_n, x)$$

for $n \in \mathbb{N}$. As there is some n_0 such that $x \in F_\alpha^{n_0}$, $e_{\gamma_n}^{k_n}(x) = 0$ for n sufficiently large. Therefore $\rho(x_n, z_n)$ converges to 0 and thus z_n converges to x. As f is usco-bounded we get (by Lemma 2.1) a converging subsequence of $f(z_n) = y_{\gamma_n}$. Then $f_{k_n}(x_n) = c_n y_{\gamma_n}$ has a converging subsequence as well. □

The next result is an analogue of the standard fact that Baire-one functions are preserved by the uniform limits. Note, that the assumption that the range is complete is necessary, while in the standard setting completeness is not needed.

Theorem 3.2. *Let X be a metric space and Y be a convex subset of a normed linear space which is complete in the norm metric. Let $f_n : X \to Y$ be a sequence of mappings which uniformly converges to a mapping $f : X \to Y$. If each f_n is the pointwise limit of a usco-bounded sequence of continuous functions, then f has the same property.*

The completeness assumption on Y cannot be omitted.

Proof: We will imitate the proof of the fact that Baire-one functions are preserved by uniform limits given in [6, Lemma 2.14]. As the quoted proof contains a large number of misprints, we give a complete proof.

Denote by Z the normed space which contains Y. Without loss of generality suppose that $\|f_m(x) - f(x)\| < 2^{-m}$ for each $x \in X$ and $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ let $(f_{m,n} : n \in \mathbb{N})$ be a usco-bounded sequence pointwise converging to f_m.

We define continuous functions $g_{m,n} : X \to Y$ as follows:

$$g_{1,n} = f_{1,n} \quad \text{for } \ n \in \mathbb{N},$$

$$g_{m+1,n}(x) = \begin{cases} f_{m+1,n}(x), & \|f_{m+1,n}(x) - g_{m,n}(x)\| \leq 2^{-m+1}, \\ g_{m,n}(x) + 2^{-m+1} \frac{f_{m+1,n}(x) - g_{m,n}(x)}{\|f_{m+1,n}(x) - g_{m,n}(x)\|}, & \\ & \|f_{m+1,n}(x) - g_{m,n}(x)\| > 2^{-m+1}. \end{cases}$$

We have

$$\forall\, x \in X\, \forall\, m \in \mathbb{N}\, \exists\, n_0 \in \mathbb{N}\, \forall\, n \geq n_0 : g_{m,n}(x) = f_{m,n}(x).$$

Indeed, let $x \in X$ be arbitrary. As $g_{1,n} = f_{1,n}$ for all n, the assertion is true for $m = 1$. Suppose it is true for some m. As

$$\begin{aligned}\|f_m(x) - f_{m+1}(x)\| &\leq \|f_m(x) - f(x)\| + \|f(x) - f_{m+1}(x)\| \\ &< 2^{-m} + 2^{-m-1} < 2^{-m+1},\end{aligned}$$

we have $\|f_{m,n}(x)-f_{m+1,n}(x)\| < 2^{-m+1}$ for n large enough. Now using the induction hypothesis and the definition of $g_{m+1,n}(x)$, we get $g_{m+1,n}(x) = f_{m+1,n}(x)$ for n large enough.

Set $h_n = g_{n,n}$ for $n \in \mathbb{N}$. Then the sequence h_n pointwise converges to f and, moreover, is usco-bounded.

Let us show the first assertion. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Choose $m \in \mathbb{N}$ such that $2^{-m+4} < \varepsilon$. Fix $n_0 \in \mathbb{N}$ such that $g_{m,n}(x) = f_{m,n}(x)$ and $\|f_{m,n}(x) - g_m(x)\| < \frac{\varepsilon}{2}$ for $n \geq n_0$. Then for $n \geq n_0$ we have

$$\begin{aligned}\|h_n(x) - f(x)\| &\leq \|g_{n,n}(x) - g_{m,n}(x)\| + \|g_{m,n}(x) - f_m(x)\| + \|f_m(x) - f(x)\| \\ &\leq 2^{-m+1} + \cdots + 2^{-n+1} + \|f_{m,n}(x) - f_m(x)\| + 2^{-m} \\ &< 2^{-m+3} + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

This shows that $h_n(x)$ converges to $f(x)$.

Now we are going to prove that the sequence h_n is usco-bounded. So take an arbitrary sequence $x_n \in X$ converging to some $x \in X$ and a sequence k_n of natural numbers. We need to show that the sequence $h_{k_n}(x_n)$ has a converging subsequence.

If the sequence k_n has a constant subsequence, we are done (as $x_n \to x$ and each f_k is continuous). Otherwise we can without loss of generality suppose that the sequence k_n is increasing.

Now, as $(f_{m,n} : n \in \mathbb{N})$ is usco-bounded for each $m \in \mathbb{N}$, the sequence $f_{m,k_n}(x_n)$ has a converging subsequence for each $m \in \mathbb{N}$. Thus, we can suppose without loss of generality that, for each $m \in \mathbb{N}$ the sequence $f_{m,k_n}(x_n)$ converges to some $y_m \in Y$.

Further, for each $n \in \mathbb{N}$ we have

$$h_{k_n}(x_n) = f_{1,k_n}(x_n) + \sum_{j=1}^{k_n-1} c_j^n \frac{f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)}{\|f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)\|},$$

where $c_j^n \in [0, 2^{-j+1}]$ for $n \in \mathbb{N}$ and $j = 1, \ldots, k_n-1$. (If $f_{j+1,k_n}(x_n) = g_{j,k_n}(x_n)$, we set $c_j^n = 0$ and suppose the fraction equals some unit vector.)

We can consider sequences $(c_j^n : j = 1, \dots, k_n - 1)$ as elements of the set

$$C = \{(t_j : j \in \mathbb{N}) : \forall\, j \in \mathbb{N} : t_j \in [0, 2^{-j+1}]\}.$$

The embedding is done by completing the finite sequence by zeros since the k_n-th place. The set C is a compact subset of the Banach space ℓ_1, hence we can suppose without loss of generality that the sequences (c_j^n) converge (for $n \to \infty$) in the ℓ_1-norm to a sequence $(c_j : j \in \mathbb{N}) \in C$.

Observe that for each $j \in \mathbb{N}$ the sequence $g_{j,k_n}(x_n)$ converges in Y. Indeed, for $j = 1$ we have $g_{1,k_n}(x_n) = f_{1,k_n}(x_n)$ which converges to y_1. Suppose now that $j \in \mathbb{N}$ is such that $g_{j,k_n}(x_n)$ converges in Y. Then

$$g_{j+1,k_n}(x_n) = g_{j,k_n}(x_n) + c_j^n \frac{f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)}{\|f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)\|},$$

and hence it converges in Z due to the assumption that $f_{j+1,k_n}(x_n)$ converges to y_{j+1} and that c_j^n converges to c_j. (If $f_{j+1,k_n}(x_n) = g_{j,k_n}(x_n)$ only for a finite number of n's, the conclusion is clear. If the equality holds for infinitely many n's then by the above convention $c_j^n = 0$ for infinitely many n's, and hence $c_j = 0$. So $c_j^n \to 0$ and hence the limit of $g_{j+1,k_n}(x_n)$ is the same as that of $g_{j,k_n}(x_n)$.) Moreover, as Y is complete and the values of each $g_{m,n}$ are in Y, we get that the limit belongs to Y.

Finally, we have that for each $j \in \mathbb{N}$ the sequence $c_j^n \frac{f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)}{\|f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)\|}$ converges for $n \to \infty$ to some $z_j \in Z$. As $\|z_j\| \le 2^{-j+1}$, the sequence $(y_1 + \sum_{j=1}^{N-1} z_j : N \in \mathbb{N})$ is Cauchy. Moreover, $y_1 + \sum_{j=1}^{N-1} z_j \in Y$ for each $N \in \mathbb{N}$, as it is equal to

$$\lim_{n\to\infty} \left(f_{1,k_n}(x_n) + \sum_{j=1}^{N-1} c_j^n \frac{f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)}{\|f_{j+1,k_n}(x_n) - g_{j,k_n}(x_n)\|} \right) = \lim_{n\to\infty} g_{N,k_n}(x_n),$$

which belongs to Y as Y is complete and $g_{N,k_n}(x_n) \in Y$ for each N and n. Hence the above sequence converges in Y to $y_1 + \sum_{j=1}^{\infty} z_j$. Now it is clear that $h_{k_n}(x_n)$ converges to $y_1 + \sum_{j=1}^{\infty} z_j$, which completes the proof of the positive part.

If Y is not complete, Example 2.6 shows that the assertion is not true. Indeed, there are usco-bounded simple functions $f_n : \mathbb{R} \to Y$ uniformly converging to a function f which is not usco-bounded. Each f_n is the pointwise limit of a usco-bounded sequence of continuous function by Theorem 3.1. On the other hand, f cannot be expressed as such a limit as in such a case it would be usco-bounded. $\square$

Finally, we give the result for the case when Y is finite-dimensional.

Theorem 3.3. *Let X be a metric space, Y a closed convex subset of $\mathbb{R}^d$ and $f : X \to Y$ be a usco-bounded Baire-one function. Then f is the pointwise limit of a usco-bounded sequence of continuous functions.*

In the proof we will need two simple lemmata.

Lemma 3.1. *Let X be a metric space, f and g two functions defined on X with values in $\mathbb{R}^d$ such that $f - g$ is bounded. If f is usco-bounded, then so is g.*

Proof: Let x_n be a sequence in X converging to some $x \in X$. As f is usco-bounded, we can suppose that the sequence $f(x_n)$ converges. Further, the sequence $g(x_n) - f(x_n)$ is bounded, and hence has a convergent subsequence. It follows that $g(x_n)$ has a convergent subsequence. This shows that g is usco-bounded. □

Lemma 3.2. *Let X be a metric space, Y be a closed convex subset of $\mathbb{R}^d$ and $f : X \to Y$ be a usco-bounded Baire-one function. Then f is a uniform limit of a sequence of usco-bounded simple functions.*

Proof: Fix $\varepsilon > 0$. By Lemma 2.2 there is a simple function g with $\|f(x) - g(x)\| < \varepsilon$ for all $x \in X$. By Lemma 3.1 the function g is usco-bounded. This completes the proof. □

Proof of Theorem 3.3: Let $f : X \to Y$ be a usco-bounded Baire-one function. By Lemma 3.2 it is the uniform limit of a usco-bounded sequence of simple functions. Now the result follows from Theorems 3.1 and 3.2. □

4. Final remarks and open questions

Of course, the main problem is whether the answer to the question from the introduction is positive in general. However, let us formulate some more questions.

Question 4.1. *Let X and Y be metric spaces and $f : X \to Y$ a usco-bounded Baire-one function. Is there a sequence of usco-bounded simple functions uniformly converging to f?*

The positive answer to this question would imply (using Theorem 3.2) the positive answer to our main problem under the assumption that Y is complete. Due to Example 2.6 it would not help to solve the general case.

By Lemma 3.2 the answer to the above question is positive if Y is a closed convex subset of $\mathbb{R}^d$. Moreover, the answer is positive if f is continuous (and X, Y are general metric spaces). Let us sketch the proof of this (although it yields nothing new with respect to our main problem).

Let $f : X \to Y$ be continuous and $\varepsilon > 0$. Then for each $x \in X$ there is an open neighborhood U_x of x with $\operatorname{diam} f(U_x) < \varepsilon$. Let $\mathcal{V}$ be a locally finite open refinement of the open cover $\{U_x : x \in X\}$ of X. Let $\{V_\alpha : \alpha < \kappa\}$ be an enumeration of $\mathcal{V}$ by ordinal numbers. Set

$$W_\alpha = V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$$

for each $\alpha < \kappa$. If $W_\alpha \neq \emptyset$ choose $y_\alpha \in f(W_\alpha)$. Define the function $g : X \to Y$ by setting $g(x) = y_\alpha$ for $x \in W_\alpha$. It is clear that the distance of $f(x)$ and $g(x)$ is less than ε for each $x \in X$. Further, it is easy to see that g is a simple function. Finally, as the partition $\{W_\alpha : \alpha < \kappa\}$ is locally finite, the function g is easily seen to be usco-bounded (using Lemma 2.1).

Another question concerns possible modification of Theorem 3.2.

Question 4.2. *Let X be a metric space and Y a convex subset of a normed linear space. Suppose that the sequence of functions $f_n : X \to Y$ is usco-bounded and uniformly converges to a function f. Suppose, moreover, that each f_n is the pointwise limit of a usco-bounded sequence of continuous functions. Is the same true for f?*

If Y is complete, the answer is positive (even without the assumption of the usco-boundedness of the sequence) by Theorem 3.2. Note that the sequence from Example 2.6 is not usco-bounded.

The positive answer to this question would help to solve the problem if the following strengthening of the first question has positive answer.

Question 4.3. *Let X and Y be metric spaces and $f : X \to Y$ a usco-bounded Baire-one function. Is there a usco-bounded sequence of simple functions uniformly converging to f?*

Note, that the answer is positive if Y is a closed convex subset of $\mathbb{R}^d$. This follows easily from the proofs of Lemmata 3.1 and 3.2. (Note that if f is usco-bounded and Y is a closed subset of $\mathbb{R}^n$, then the graph of the set-valued map $x \mapsto B(f(x), \varepsilon)$ is contained in the graph of a usco map — we can use Lemma 2.1.)

Added in proof.

Question 4.1 was recently answered in the positive by J. Spurný. Therefore the answer to our main question is positive provided the range space Y is complete.

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