# Central limit theorem for Hölder processes on $\mathbb{R}^m$-unit cube

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*Abstract.* We consider a sequence of stochastic processes $(X_n(\mathbf{t}), \mathbf{t} \in [0,1]^m)$ with continuous trajectories and we show conditions for the tightness of the sequence in the Hölder space with a parameter $\gamma$.

*Keywords:* Hölder space, tightness, weak convergence
*Classification:* 60F17, 60G50

## 1. Introduction

The invariance principle in the Hölder space for i.i.d. case and the index set $[0,1]$ was established by Lamperti, see Lamperti (1962). Račkauskas and Suquet (2001) found necessary and sufficient conditions for the weak convergence of normalized sums in the Hölder space. However, they studied also only i.i.d. case with the index set $[0,1]$. According to Hamadouche (2000), we can find in Kerkyacharian and Roynette (1991) the condition for the tightness in the Hölder space, but this condition is only for the case when $t \in [0,1]$.

The invariance principle for processes indexed by arbitrary sets can be found for example in Ledoux and Talagrand (1991), where the invariance principle is presented solely in the space of continuous functions.

We present the tightness condition in the Hölder subspace of the space of continuous functions ($C^{0,\gamma}$, see Definition 1) with supreme metric, in case of stochastic processes indexed by $\mathbf{t} \in [0,1]^m$.

In Račkauskas and Suquet (1998) and Račkauskas and Suquet (2005), we can find conditions for the tightness of sequences in Hölder spaces — $H^0_\rho$. The space $H^0_\rho$ is the space of real valued continuous functions $f$ on $[0,1]^m$ such that $w_\rho(f,1) < \infty$ and $\lim_{\delta \to 0} w_\rho(f,\delta) = 0$. This space is equipped with the norm

$$\|f\|_\rho := |f(\mathbf{0})| + w_\rho(f,1),$$

where $\rho$ is a modulus of smoothness which satisfies any special conditions and $w_\rho(\cdot,\cdot)$ is defined as follows

$$w_\rho = \sup_{\mathbf{s},\mathbf{t},\|\mathbf{s}-\mathbf{t}\|_s<\delta} \frac{|f(\mathbf{t}) - f(\mathbf{s})|}{\rho(\|\mathbf{s}-\mathbf{t}\|_s)},$$

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where $\|\mathbf{s}\|_s := \max_{1\le i\le m}|s_i|$. In fact, they used more general norms than we do. Some of their norms should be equivalent to ours. For special choice of $\rho$, it could be shown that $C^{0,\gamma} \subset H^0_\rho$. But their theorems consider processes in the Hölder space and show the tightness. We consider only continuous processes and we prove the tightness in the Hölder subspace under our conditions.

From now on $\|\cdot\|$ denotes the Euclidean norm.

**Definition 1.** *The space $C^{0,\beta}([0,1]^m)$, where $0<\beta\le 1$, is the space of all continuous functions which satisfy for all $\mathbf{s},\mathbf{t}\in[0,1]^m$ and a finite constant $K$:*

$$|f(\mathbf{t})-f(\mathbf{s})| \le K\|\mathbf{t}-\mathbf{s}\|^\beta. \tag{1}$$

## 2. Main result

**Theorem 1.** *Let $(X_n(\mathbf{t}),\mathbf{t}\in[0,1]^m)$ be a sequence of stochastic processes with continuous trajectories. Let the finite dimensional distributions of $(X_n)$ converge weakly. Moreover, let there exist constants $\alpha$, $\beta$ and $K$ such that*

$$\Pr(|X_n(\mathbf{t})-X_n(\mathbf{s})|\ge\varepsilon) \le \frac{K}{\varepsilon^\alpha}\|\mathbf{s}-\mathbf{t}\|^{\alpha\beta},\ \forall\,\mathbf{s},\mathbf{t}\in[0,1]^m, \forall\,\varepsilon>0, \forall\,n\in\mathbb{N}, \tag{2}$$

*where*

$$0<\beta\le 1 \ \text{ and } \ \alpha\beta > \frac{2}{\log_2\frac{4m}{4m-3}}\,. \tag{3}$$

*Then the sequence of processes converges weakly to some process in the space $C^{0,\gamma}([0,1]^m)$ where $\gamma$ is as follows*

$$0<\gamma<\beta-\frac{m}{\alpha}\,. \tag{4}$$

## 3. Proof

The weak convergence in the Hölder space of a sequence of processes $(X_n)$ is equivalent to the tightness in the Hölder space of distributions of $(X_n)$ and the convergence of finite dimensional distributions of $(X_n)$. Our aim is to prove the tightness of distributions of sequence $(X_n)$.

First we formulate one useful lemma.

**Remark.** We will denote by $\mathbf{1}$, $\mathbf{L}$, the elements of $\mathbb{R}^m$ with all coordinates equal to 1, $L$, respectively.

The interval $I_{\langle\mathbf{j},\mathbf{k}\rangle}$ in the space $\mathbb{R}^m$ will denote the set of all $\mathbf{l}$ such that $\mathbf{l}$ lies between $\mathbf{j}$ and $\mathbf{k}$ or is equal to $\mathbf{j}$ or $\mathbf{k}$. So in case of negative coordinates it will be used in a nonstandard way. We will use it only if for each $i: 1\le i\le m$ the coordinates $j_i, l_i, k_i$ are all either nonnegative or nonpositive and it will mean that for all coordinates $|j_i|\le|l_i|\le|k_i|$ is satisfied. In a similar way, the symbol $I_{(\mathbf{j},\mathbf{k})}$ stands for all $\mathbf{l}$ which lie between $\mathbf{j}$ and $\mathbf{k}$ and are equal neither to $\mathbf{j}$ nor to $\mathbf{k}$.

**Lemma 1.** *Let us have* $\mathbf{s}, \mathbf{t} \in [0,1]^m$ *and*

$$J = \{(\mathbf{k} - \mathbf{1}/2)\delta, \quad \mathbf{k} \in \mathbb{N}^m \cap [\mathbf{1}, \mathbf{1}/\delta]\}, \tag{5}$$

*for some* $\delta > 0$ *such that* $\delta^{-1} \in \mathbb{N}$. *Let us consider close balls* $B(\mathbf{j}, \frac{\delta}{2})$ *in* $\mathbb{R}^m$ *with centers in* $\mathbf{j} \in J$ *and with radii* $\frac{\delta}{2}$ *in supreme metric. If* $\|\mathbf{s} - \mathbf{t}\| < \delta$ *(Euclidean norm) then*

$$|X(\mathbf{s}) - X(\mathbf{t})| \le 4 \max_{\mathbf{j} \in J} \sup_{\mathbf{z} \in B(\mathbf{j}, \frac{\delta}{2})} |X(\mathbf{z}) - X(\mathbf{j})|.$$

Proof: It is clear that in such a case of $\mathbf{s}, \mathbf{t}$, the points $\mathbf{s}, \mathbf{t}$ belong to coinciding or abuting balls $B(\mathbf{j}, \frac{\delta}{2})$. The absolute value of each coordinate of $(\mathbf{s} - \mathbf{t})$ is smaller or equal to $\delta$ (since the Euclidean norm of $(\mathbf{s} - \mathbf{t})$ is smaller or equal to $\delta$). So the distance of $\mathbf{s}$ and $\mathbf{t}$ in supreme norm is smaller or equal to $\delta$ (the diameter of balls).

So let us suppose that $\mathbf{s} \in B(\mathbf{j}, \frac{\delta}{2})$ and $\mathbf{t} \in B(\mathbf{i}, \frac{\delta}{2})$. The balls $B(\mathbf{j}, \frac{\delta}{2})$ and $B(\mathbf{i}, \frac{\delta}{2})$ coincide or abut and the radius is in the supreme norm, so the point $\frac{\mathbf{j}+\mathbf{i}}{2}$ is equal to $\mathbf{i}$ or belongs to the both balls (the balls are close and their radii are in supreme metric). We get

$$|X(\mathbf{s}) - X(\mathbf{t})| \le |X(\mathbf{s}) - X(\mathbf{j})| + |X(\mathbf{j}) - X(\frac{\mathbf{j}+\mathbf{i}}{2})| + |X(\frac{\mathbf{j}+\mathbf{i}}{2}) - X(\mathbf{i})| + |X(\mathbf{i}) - X(\mathbf{t})|,$$

which finishes the proof. □

The following Proposition 1 will prove Theorem 1.

**Proposition 1.** *The sequence of stochastic processes* $(X_n(\mathbf{t}), \mathbf{t} \in [0,1]^m)$ *with continuous trajectories is tight in* $C^{0,\gamma}([0,1]^m)$, *if the following conditions are satisfied:*

- (i) *the sequence* $(X_n(\mathbf{0}))$ *is tight;*
- (ii) *there exist constants* $K, \alpha, \beta > 0$ *such that* (2), (3) *and* (4) *are satisfied.*

The proof of Proposition 1 is based on the proofs of theorems Billingsley (1968, Theorems 12.1–12.3) and uses the following proposition.

**Proposition 2.** *Let us consider a net* $(S_{\mathbf{i}}, \mathbf{i} \in I)$, *where*

$$I = \{\mathbf{k}\delta/(2L), \ \mathbf{k} \in \mathbb{Z}^m \cap [-L, L]^m\} \tag{6}$$

*for any* $L \in \mathbb{N}$. *If the net satisfies for every* $\mathbf{i}^1, \mathbf{i}^2 \in I$, *every* $\varepsilon > 0$ *and any constants* $K, \alpha, \beta > 0$ *the following inequality*

$$\Pr(|S_{\mathbf{i}^1} - S_{\mathbf{i}^2}| \ge \varepsilon) \le \frac{K}{\varepsilon^\alpha} \|\mathbf{i}^1 - \mathbf{i}^2\|^{\alpha\beta}, \tag{7}$$

*condition* (3) *and* $S_{\mathbf{0}} = 0$ *then there exists a constant* $M$ *depending only on* $\alpha, \beta, K, m$ *such that*

$$\Pr(\max_{\mathbf{i} \in I: \mathbf{0} \leq \mathbf{i} \leq (\frac{\delta}{2})\mathbf{v}} |S_{\mathbf{i}}| \geq \varepsilon) \leq \frac{M}{\varepsilon^{\alpha}} \left(\frac{\delta}{2}\right)^{\alpha\beta} \tag{8}$$

*for every* $m$*-dimensional vector* $\mathbf{v}$ *containing only* $1, -1$.

Proof of Proposition 1: First we prove that the sequence is tight in the space of continuous functions $C([0,1]^m)$ and then we show that it is tight in the subspace $C^{0,\gamma}([0,1]^m)$.

For the tightness in $C([0,1]^m)$ we need only to show that for every $\varepsilon, \eta > 0$ there exists $\delta > 0$ such that

$$\Pr(w(X_n, \delta) \geq 4\varepsilon) \leq \eta \quad \forall n \in \mathbb{N}, \tag{9}$$

where

$$w(X_n, \delta) = \sup_{\mathbf{s}, \mathbf{t}: \|\mathbf{s} - \mathbf{t}\| < \delta} |X_n(\mathbf{s}) - X_n(\mathbf{t})|. \tag{10}$$

Let us recall Lemma 1 and use the balls with centers $\mathbf{j} \in J$ (see (5)) and with radii in supreme metric. For $\delta$ such that $\delta^{-1} \in \mathbb{N}$ the inequality

$$\sum_{\mathbf{j} \in J} \Pr(\sup_{\mathbf{s} \in B(\mathbf{j}, \frac{\delta}{2})} |X_n(\mathbf{s}) - X_n(\mathbf{j})| \geq \varepsilon) \leq \eta \tag{11}$$

would assert the inequality (9). Let us prove the inequality (11).

We need to bound each term of (11). Let us fix $n, L, \delta > 0$ and $\mathbf{j} \in J$ and define new variables $S_{\mathbf{i}}$:

$$S_{\mathbf{i}} := X_n(\mathbf{j} + \mathbf{i}) - X_n(\mathbf{j}), \tag{12}$$

where $\mathbf{i} \in I$, see (6).

According to (2), we obtain for each $\mathbf{i}^1, \mathbf{i}^2 \in I$:

$$\Pr(|S_{\mathbf{i}^1} - S_{\mathbf{i}^2}| \geq \varepsilon) \leq \frac{K}{\varepsilon^{\alpha}} \|\mathbf{i}^1 - \mathbf{i}^2\|^{\alpha\beta}. \tag{13}$$

Now we apply Proposition 2. The constant $M$ (in inequality (8)) does not depend on $L$, and so due to the continuity of the process we get:

$$\Pr(\sup_{\mathbf{j} \leq \mathbf{s} \leq \mathbf{j} + \mathbf{v}\frac{\delta}{2}} |X_n(\mathbf{s}) - X_n(\mathbf{j})| \geq \varepsilon) \leq \frac{M}{\varepsilon^{\alpha}} \left(\frac{\delta}{2}\right)^{\alpha\beta}. \tag{14}$$

Since it is possible to cover the ball with radius $\frac{\delta}{2}$ in supreme metric by at most $2^m$ parts in the form $[\mathbf{0}, \frac{\delta}{2} \cdot \mathbf{v}]$ ($\mathbf{v} \in \mathbb{Z}^m$ and consists only on $+1, -1$) we get

$$\Pr\Big( \sup_{\mathbf{s}\in B(\mathbf{j},\frac{\delta}{2})} |X_n(\mathbf{s}) - X_n(\mathbf{j})| \geq \varepsilon\Big) \leq \frac{M_1}{\varepsilon^\alpha} \left(\frac{\delta}{2}\right)^{\alpha\beta}, \tag{15}$$

where $M_1$ depends only on $K, \alpha, \beta, m$.

The unit cube $[0,1]^m$ can be covered by $\delta^{-m}$ balls with centers $\mathbf{j} \in J$ and radii $\frac{\delta}{2}$ in supreme metric, so we can write

$$\sum_{\mathbf{j}\in J} \Pr\Big( \sup_{\mathbf{s}\in B(\mathbf{j},\frac{\delta}{2})} |X_n(\mathbf{s}) - X_n(\mathbf{j})| \geq \varepsilon\Big) \leq \frac{M_2}{\varepsilon^\alpha} \left(\frac{\delta}{2}\right)^{\alpha\beta - m}, \tag{16}$$

where $M_2$ depends again only on the constants $K, \alpha, \beta, m$.

Especially for $\delta = 2^{1-k}$ and $\varepsilon = K2^{-k\gamma}$:

$$\sum_{\mathbf{j}\in J} \Pr\Big( \sup_{\mathbf{s}\in B(\mathbf{j},1/2^k)} |X_n(\mathbf{s}) - X_n(\mathbf{j})| \geq K2^{-k\gamma}\Big) \leq \frac{M_2}{K^\alpha} \left(\frac{1}{2^k}\right)^{\alpha\beta - m - \alpha\gamma}. \tag{17}$$

The following sum is bounded for $\alpha, \beta, \gamma$ with $\beta - m/\alpha > \gamma$:

$$\sum_{k=1}^{+\infty} \frac{M_2}{K^\alpha} \left(\frac{1}{2^k}\right)^{\alpha\beta - m - \alpha\gamma} < +\infty.$$

So, for each fixed $\varepsilon, \eta > 0$ we can find $k \in \mathbb{N}$ such that $\varepsilon > K2^{-k\gamma}$ and the right-hand side of (17) is less than $\eta$. If we put $\delta = 2^{1-k}$ we obtain inequality (11) from (17).

Condition (9) is satisfied so the sequence is tight in the space of continuous functions $C([0,1]^m)$. Now we need to show the tightness in the subspace $C^{0,\gamma}([0,1]^m)$. So our aim is to show that for every $\varepsilon > 0$ there exists a constant $\bar{K}_\varepsilon$ such that the trajectories of the process $X$ satisfy condition (1) with constants $\gamma$ and $\bar{K}_\varepsilon$ with probability at least $1 - \varepsilon$.

We see at once that for every $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

$$\Pr\Big(\exists\, k \in \mathbb{N} : \max_{\mathbf{j}\in J} \sup_{\mathbf{s}\in B(\mathbf{j},1/2^k)} |X_n(\mathbf{s}) - X_n(\mathbf{j})| > K_\varepsilon 2^{-k\gamma}\Big) \leq \varepsilon. \tag{18}$$

We put

$$\Omega_\varepsilon^n = \{\omega :\ \forall\, k \in \mathbb{N} :\ \max_{j\in J} \sup_{\mathbf{s}\in B(\mathbf{j},1/2^k)} |X_n(\omega,\mathbf{s}) - X_n(\omega,\mathbf{j})| \leq K_\varepsilon 2^{-k\gamma}\},$$

then $\Pr(\Omega_\varepsilon^n) \geq 1-\varepsilon$. Now we consider only $\omega \in \Omega_\varepsilon^n$. And we show that there exists a constant $\bar{K}_\varepsilon > 0$ such that

$$\forall\, \mathbf{s}, \mathbf{t} \in [0,1]^m : |X_n(\omega, \mathbf{s}) - X_n(\omega, \mathbf{t})| \leq \bar{K}_\varepsilon \|\mathbf{s}-\mathbf{t}\|^\gamma.$$

We can find for each $\mathbf{s}, \mathbf{t} \in [0,1]^m$ a constant $l$ such that $\frac{1}{2^l} \leq \|\mathbf{s}-\mathbf{t}\| < \frac{2}{2^l}$. Using Lemma 1 with $\delta := 2^{1-l}$ we get for $\omega \in \Omega_\varepsilon^n$

$$\begin{aligned} |X_n(\mathbf{s}) - X_n(\mathbf{t})| &\leq 4 \cdot \max_{\mathbf{j} \in J} \sup_{\mathbf{u} \in B(\mathbf{j}, 1/2^l)} |X_n(\mathbf{u}) - X_n(\mathbf{j})| \\ &\leq 4 \cdot K_\varepsilon 2^{-l\gamma} \leq 4 \cdot K_\varepsilon \|\mathbf{s}-\mathbf{t}\|^\gamma. \end{aligned}$$

Let us put $\bar{K}_\varepsilon := 4K_\varepsilon$. We have derived that for $\gamma < \beta - \frac{m}{\alpha}$ we can find for every $\varepsilon > 0$ a constant $\bar{K}_\varepsilon > 0$ such that the process $X_n$ is Hölder with constants $\gamma, \bar{K}_\varepsilon$ with probability at least $1-\varepsilon$.

Further, we can find for every $\varepsilon > 0$ a compact subset $\mathsf{C}_\varepsilon$ of $C^{0,\gamma}([0,1]^m)$ such that $\Pr(X_n \in \mathsf{C}_\varepsilon) \geq 1-\varepsilon$ for all $n \in \mathbb{N}$. (For every $\varepsilon$ and every process $X_n$ we find a subset $\mathsf{C}^n_{\varepsilon/2^n}$ such that $\Pr(X_n \in \mathsf{C}^n_{\varepsilon/2^n}) \geq 1 - \varepsilon/2^n$ and put $\mathsf{C}_\varepsilon := \bigcap_{i=1}^{+\infty} \mathsf{C}^n_{\varepsilon/2^n}$.)

It remains to prove Proposition 2. □

Proof of Proposition 2: We begin with observing that $\Pr(E_1 \cap E_2) \leq P(E_1) \wedge P(E_2)$. Then the following inequality follows for $\mathbf{i}, \mathbf{l}_1, \mathbf{l}_2, \mathbf{k}$ from condition (7):

$$\begin{aligned} \Pr(|S_{\mathbf{i}} - S_{\mathbf{l}_1}| \geq \varepsilon, |S_{\mathbf{l}_2} - S_{\mathbf{k}}| \geq \varepsilon) &\leq \left(\frac{K}{\varepsilon^\alpha} (\|\mathbf{i}-\mathbf{l}_1\|)^{\alpha\beta}\right) \wedge \left(\frac{K}{\varepsilon^\alpha} (\|\mathbf{l}_2-\mathbf{k}\|)^{\alpha\beta}\right) \\ &\leq \frac{K}{\varepsilon^\alpha} (\max(\|\mathbf{i}-\mathbf{l}_1\|, \|\mathbf{l}_2-\mathbf{k}\|))^{\alpha\beta}. \end{aligned} \tag{19}$$

Set for $\mathbf{n} \in I$ (recall that $\mathbf{l} \in I_{\langle \mathbf{0},\mathbf{n}\rangle}$ denotes that $\mathbf{l}$ lies between $\mathbf{0}$ and $\mathbf{n}$)

$$\begin{aligned} M'_{\mathbf{n}} &:= \max_{\mathbf{l} \in I_{\langle \mathbf{0},\mathbf{n}\rangle}} \min\{|S_{\mathbf{l}}|, |S_{\mathbf{n}} - S_{\mathbf{l}}|\}, \\ M_{\mathbf{n}} &:= \max_{\mathbf{i} \in I_{\langle \mathbf{0},\mathbf{n}\rangle}} |S_{\mathbf{i}}|. \end{aligned}$$

Analogously to Billingsley (1968, Theorem 12.1), we show

$$\Pr(M'_{\mathbf{n}} \geq \varepsilon) \leq \frac{\hat{K}}{\varepsilon^\alpha} \|\mathbf{n}\|^{\alpha\beta} \tag{20}$$

where the constant $\hat{K}$ depends only on $\alpha, \beta, K, m$. More precisely, $\hat{K}$ is such that $\hat{K} \geq 3^m K$ and it is large enough so that inequality (23) is satisfied.

In order to prove (20), we use induction. First, we need to show that inequality (20) holds for $\mathbf{n} : |\mathbf{n}| \leq \mathbf{1}\delta/(2L)$ (by $|\mathbf{n}|$ we denote the number $(|n_1|, |n_2|, \cdots, |n_m|)$). According to (19), we can write for $\mathbf{i} \in I$ such that $\mathbf{i} \in I_{\langle \mathbf{0},\mathbf{n}\rangle}$:

$$\Pr(|S_{\mathbf{i}}| \geq \varepsilon, |S_{\mathbf{n}} - S_{\mathbf{i}}| \geq \varepsilon) \leq \frac{K}{\varepsilon^{\alpha}}(\max(\|\mathbf{i}\|, \|\mathbf{n} - \mathbf{i}\|))^{\alpha\beta} \leq \frac{K}{\varepsilon^{\alpha}}\|\mathbf{n}\|^{\alpha\beta}. \tag{21}$$

Since $\mathbf{i} \in I$ and $\mathbf{n}$ are such that $|\mathbf{n}| \leq \mathbf{1}\delta/(2L)$ it is obvious that there are no more than $2^m$ of $\mathbf{i}$'s the maximum is taken over.

So according to (21), condition (20) is satisfied with the constant $K2^m$.

And similarly we are able to prove this condition for all $\mathbf{n}$ such that $|\mathbf{n}| \leq \delta/L$ with a constant $K3^m$.

Now we assume that there exists $\mathbf{h} \in I$ such that condition (20) is satisfied for all $\mathbf{n} \in I$ such that $\mathbf{n} \in I_{\langle \mathbf{0},\mathbf{h})}$. We need to show that the condition is satisfied also for $\mathbf{h}$. We use the same idea as Billingsley (1968).

We divide the space between the points $\mathbf{j}$ and $\mathbf{j}+\mathbf{h}$ into two parts, we take two reference points. Let $h_k$ be the biggest (in absolute value) of $\mathbf{h}$-coordinates. Then one of the reference points, we denote it by $\mathbf{r}_1$, will be the point with the same coordinates as $\mathbf{h}$ except the $k^{\text{th}}$ coordinate which will be equal to $\operatorname{sign}(h_k)\lfloor \frac{|h_k|}{2} \rfloor_I$, where $\lfloor x \rfloor_I = \lfloor x \cdot \frac{2L}{\delta} \rfloor \cdot \frac{\delta}{2L}$ and $\lfloor x \rfloor$ denotes the biggest integer smaller or equal to $x$, and the second point, $\mathbf{r}_2$, will have the same coordinates as $\mathbf{0}$ but the $k^{\text{th}}$ coordinate will be equal to $\operatorname{sign}(h_k)\lfloor \frac{|h_k|}{2} \rfloor_I + \operatorname{sign}(h_k)\delta/(2L)$. Since we suppose $\mathbf{i} \in I$ and $\mathbf{i} \in I_{\langle \mathbf{0},\mathbf{h})}$, either $\mathbf{i} \in I_{\langle \mathbf{0},\mathbf{r}_1\rangle}$ or $\mathbf{i} \in I_{\langle \mathbf{r}_2,\mathbf{h})}$. Moreover

$$\|\mathbf{r}_1\|, \|\mathbf{h} - \mathbf{r}_2\| \leq \|\mathbf{h}\|\sqrt{1 - \frac{3}{4m}} = \|\mathbf{h}\| \left(\frac{1}{2}\right)^{1-\frac{1}{2}\log_2(4-3/m)}.$$

Now we put:

$$\begin{aligned}
U_1 &:= \max_{\mathbf{i}\in I_{\langle \mathbf{0},\mathbf{r}_1\rangle}} \min\{|S_{\mathbf{i}}|, |S_{\mathbf{r}_1} - S_{\mathbf{i}}|\} \\
U_2 &:= \max_{\mathbf{i}\in I_{\langle \mathbf{r}_2,\mathbf{h})}} \min\{|S_{\mathbf{i}} - S_{\mathbf{r}_2}|, |(S_{\mathbf{h}} - S_{\mathbf{r}_2}) - (S_{\mathbf{i}} - S_{\mathbf{r}_2})|\} \\
&= \max_{\mathbf{i}\in I_{\langle \mathbf{r}_2,\mathbf{h})}} \min\{|S_{\mathbf{i}} - S_{\mathbf{r}_2}|, |S_{\mathbf{h}} - S_{\mathbf{i}}|\},
\end{aligned}$$

where the maximum is taken over $\mathbf{i} \in I$ in the corresponding part. Note that $U_2$ is almost the same term as $U_1$ if we replace $\mathbf{i}$ by $\mathbf{i} + \mathbf{r}_2$. Further we put

$$D_l := \min\{|S_{\mathbf{r}_l}|, |S_{\mathbf{h}} - S_{\mathbf{r}_l}|\}, \quad l = 1, 2.$$

Taking into consideration the induction assumption and condition (19), we get:

$$\Pr(U_l \geq \varepsilon) \leq \frac{\hat{K}}{\varepsilon^\alpha}\|\mathbf{h}\|^{\alpha\beta}\left(\frac{1}{2}\right)^{(1-\frac{1}{2}\log_2(4-3/m))\alpha\beta},$$

$$\Pr(D_l \geq \varepsilon) \leq \frac{K}{\varepsilon^\alpha}\|\mathbf{h}\|^{\alpha\beta}.$$

We defined $U_l, D_l$ so that there is satisfied:

$$\min\{|S_{\mathbf{i}}|, |S_{\mathbf{h}} - S_{\mathbf{i}}|\} \leq U_l + D_l \quad \forall\, \mathbf{i} \text{ in the corresponding part.}$$

So for every $\mathbf{h}$

$$M'_{\mathbf{h}} \leq \max\{U_1 + D_1, U_2 + D_2\},$$

and then

$$\Pr(M'_{\mathbf{h}} \geq \varepsilon) \leq \Pr(U_1 + D_1 \geq \varepsilon) + \Pr(U_2 + D_2 \geq \varepsilon).$$

For all $\varepsilon_1, \varepsilon_2 \geq 0 : \varepsilon = \varepsilon_1 + \varepsilon_2$ we have

$$\begin{aligned}\Pr(U_l + D_l \geq \varepsilon) &\leq \min_{\varepsilon_1,\varepsilon_2\geq 0:\varepsilon_1+\varepsilon_2=\varepsilon}(\Pr(U_l \geq \varepsilon_1) + \Pr(D_l \geq \varepsilon_2)) \\ &\leq \min_{\varepsilon_1,\varepsilon_2\geq 0:\varepsilon_1+\varepsilon_2=\varepsilon}\left(\frac{\hat{K}}{\varepsilon_1^\alpha}\|\mathbf{h}\|^{\alpha\beta}\left(\frac{1}{2}\right)^{(1-\frac{1}{2}\log_2(4-3/m))\alpha\beta} + \frac{K}{\varepsilon_2^\alpha}\,||\mathbf{h}||^{\alpha\beta}\right) \\ &\leq \hat{K}\,||\mathbf{h}||^{\alpha\beta}\min_{\varepsilon_1,\varepsilon_2\geq 0:\varepsilon_1+\varepsilon_2=\varepsilon}\left(\frac{1}{\varepsilon_1^\alpha}\left(\frac{1}{2}\right)^{\alpha\beta(1-\frac{1}{2}\log_2(4-3/m))} + \frac{K}{\varepsilon_2^\alpha\hat{K}}\right).\end{aligned}$$

Looking for the minimum of

$$\min_{\varepsilon_1,\varepsilon_2\geq 0:\varepsilon_1+\varepsilon_2=\varepsilon}\left(\frac{1}{\varepsilon_1^\alpha}\left(\frac{1}{2}\right)^{\alpha\beta(1-\frac{1}{2}\log_2(4-3/m))} + \frac{1}{\varepsilon_2^\alpha}\frac{K}{\hat{K}}\right),$$

we need to find the smallest value of

$$\min_{0\leq\varepsilon_1\leq\varepsilon}\left(\frac{C_1}{\varepsilon_1^\alpha} + \frac{C_2}{(\varepsilon-\varepsilon_1)^\alpha}\right),$$

which is

$$\frac{1}{\varepsilon^\alpha}\left(C_1^{\frac{1}{1+\alpha}} + C_2^{\frac{1}{1+\alpha}}\right)^{1+\alpha}.$$

Using our constants, we get:

$$\frac{1}{\varepsilon^\alpha}\left(\left(\frac{K}{\hat{K}}\right)^{\frac{1}{1+\alpha}} + \left(\frac{1}{2}\right)^{\frac{\alpha\beta(1-\frac{1}{2}\log_2(4-3/m))}{1+\alpha}}\right)^{1+\alpha}. \tag{22}$$

Now we look for such constants that the expression (22) will be smaller or equal to $\varepsilon^{-\alpha}2^{-1}$. Especially, we require:

$$(23)\qquad 2\left(\left(\frac{K}{\hat{K}}\right)^{\frac{1}{1+\alpha}} + \left(\frac{1}{2}\right)^{\frac{\alpha\beta(1-\frac{1}{2}\log_2(4-3/m))}{1+\alpha}}\right)^{1+\alpha} \le 1.$$

Recalling our condition (3) on $\alpha, \beta$, we can choose a constant $\hat{K}$, such that $\hat{K} \ge 3^m K$ and is large enough so that inequality (23) is satisfied. So, relation (20) is proved by induction.

It remains to realize that

$$M_{\mathbf{n}} \le M'_{\mathbf{n}} + |S_{\mathbf{n}}|$$

and observe that

$$(24)\qquad \Pr(M_{\mathbf{n}} \ge \varepsilon) \le \Pr(M'_{\mathbf{n}} \ge \varepsilon/2) + \Pr(|S_{\mathbf{n}}| \ge \varepsilon/2).$$

The first term of the right-hand side was already bounded so we need to bound the second term of right-hand side. For the second term, we can use relation (7) and we get:

$$\Pr(|S_{\mathbf{n}}| \ge \varepsilon) \le \frac{K}{\varepsilon^{\alpha}} \|\mathbf{n}\|^{\alpha\beta}.$$

□

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