# Locally soluble-by-finite groups with small deviation for non-subnormal subgroups

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*Abstract.* A group $G$ has subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > \dots$ of non-subnormal subgroups of $G$, for all but finitely many $i$ there is no infinite descending chain of non-subnormal subgroups of $G$ that contain $H_{i+1}$ and are contained in $H_i$. This property $\mathfrak{P}$, say, was investigated in a previous paper by the authors, where soluble groups with $\mathfrak{P}$ and locally nilpotent groups with $\mathfrak{P}$ were effectively classified. The present article affirms a conjecture from that article by showing that locally soluble-by-finite groups with $\mathfrak{P}$ are soluble-by-finite and are therefore classified.

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This article continues the study of groups with "small deviation for non-subnormal subgroups"; more precisely, of groups $G$ for which this deviation is at most 1, where we write $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. The discussion of groups with this property was begun in [6], following some general results on groups $G$ for which the non-subnormal deviation exists at all. We refer the reader to [6] for the more general case and for precisely what it means to say that the non-subnormal deviation of a group $G$ *exists*, but we recall here that a group $G$ has non-subnormal deviation at most 1 if, for every descending chain $H_0 > H_1 > H_2 > \dots$ of non-subnormal subgroups of $G$, almost all of the "intervals" $I[H_{i+1}, H_i]$ satisfy the minimal condition for non-subnormal subgroups of $G$, that is, for all but finitely many $i$ there is no infinite descending chain of non-subnormal subgroups of $G$ all of which contain $H_{i+1}$ and are contained in $H_i$. It was observed in [6] that a group with the weak minimal condition for non-subnormal subgroups has non-subnormal deviation at most 1 but that the converse is false — groups with this weak minimal condition were the topic of discussion in [4] and [5]. The main results of [6] were concerned with soluble groups $G$ satisfying $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ and with locally nilpotent groups $G$ with this property. We summarize the results on soluble groups as follows.

**Theorem 1.** *Let $G$ be a soluble group such that* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$, *and let $T$ be the torsion subgroup of the Baer radical $B$ of $G$.*

(i) *If $B$ is minimax then $G$ too is minimax.*

(ii) *If $B/T$ is not minimax then $G = B$ and every subgroup of $G$ is subnormal.*
(iii) *If $B/T$ is minimax but $T$ is not minimax then $T$ contains all elements of $G$ of finite order, $G/T$ is minimax and $T$ satisfies* $\min-\langle g\rangle$ *for each element $g \in G\backslash B$. Furthermore, every subgroup of $B$ is subnormal.*

We were also able to show that a locally nilpotent group $G$ with $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ is soluble and of a certain restricted structure. It is clear from the above theorem that the structure of a soluble-by-finite group with non-subnormal deviation at most 1 has also been determined, and we conjectured in [6] that a locally soluble-by-finite group with deviation at most 1 is soluble-by-finite. Verification of this conjecture is the main purpose of the present article.

**Theorem 2.** *Let $G$ be a locally soluble-by-finite group such that* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. *Then $G$ is soluble-by-finite.*

We approach the proof of this theorem by means of a sequence of lemmas.

**Lemma 1.** *Let $G$ be a locally finite group satisfying* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. *Then $G$ is soluble-by-finite.*

Proof: If $G$ is Chernikov then the result is clear, while if $G$ is not Chernikov then it is a Baer group, by Lemma 2.8 of [6], hence locally nilpotent and hence soluble, as pointed out above. □

Recall next that a group $G$ is *radical* if it is the union of an ascending series of normal subgroups with successive factor groups locally nilpotent.

**Lemma 2.** *Let $G$ be a radical group satisfying* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. *Then $G$ is soluble.*

Proof: By Theorem 5.7 of [6] every locally nilpotent section of $G$ is soluble. Let $H_1$ be the locally nilpotent radical of $G, H_2/H_1$ the locally nilpotent radical of $G/H_1$, and let $T$ denote the maximal normal torsion subgroup of $G$. By Lemma 1 $T$ is soluble and, factoring, we may assume that $T = 1$. Let $B = B(G)$ denote the Baer radical of $G$; then $B = B(H_2)$. If $B$ is not minimax then $H_2 = B$, by Theorem 1, and so $G = H_1$, which is soluble. But if $B$ is minimax then every abelian subgroup of $G$ is minimax, by Corollary 2.3 of [6], and then $G$ is minimax, by Theorem 10.35 of [8], and hence soluble. This establishes the result. □

**Lemma 3.** *Let $G$ be a finitely generated soluble-by-finite group satisfying* $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$. *Then $G$ is minimax.*

Proof: We may suppose that $G$ is soluble. If $G$ is not minimax then it has infinite (Prüfer) rank, by Theorem 10.38 of [8], and hence a section $K$ isomorphic to the wreath product of a cyclic group of prime order by an infinite cyclic group, by [3]. But the Baer radical of $K$ is abelian of prime exponent and clearly does not satisfy $\min-K$, and Theorem 1(iii) gives a contradiction. □

Our next result reduces the proof of Theorem 2 to that of the locally soluble case. This is followed by two straightforward lemmas, the first of which is well-known. These will assist in establishing a Proposition that, in turn, provides a key step in the final stage of the proof of the theorem. In the statement of Lemma 6 (and elsewhere) we employ for convenience the term "periodic index" to indicate that a subgroup $H$ of a group $G$ has the property that every element of $G$ has a non-zero power in $H$.

**Lemma 4.** *Suppose that every locally soluble group with non-subnormal deviation at most* 1 *is soluble. Then every locally soluble-by-finite group with non-subnormal deviation at most* 1 *is soluble-by-finite.*

Proof: Let $G$ be a locally soluble-by-finite group with $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ and suppose that $G$ is not soluble-by-finite. If there are integers $d$ and $k$ such that every finitely generated subgroup of $G$ has its soluble radical of derived length at most $d$ and index at most $k$ then $G$ too has this property, by Proposition 1.K.2 of [2]. Thus we may assume that $G$ is countable and, by Lemma 2, we may further assume (by factoring if necessary) that $G$ has trivial Hirsch-Plotkin radical. $G$ is an ascending union of finitely generated subgroups $F_1 < F_2 < \dots$; for each $i$ the soluble radical $S_i$ of $F_i$ has finite index, and it is easy to see that the subgroup $S$ generated by all the $S_i$ is locally soluble and hence, by hypothesis, soluble. Since $G$ has trivial Baer radical we deduce from Corollary 2.3 of [6] that every abelian subgroup of $G$ is minimax and hence that $S$ is minimax, by Theorem 10.35 of [8]. It follows that there is a finite upper bound $r$ for the torsion-free ranks of the subgroups $F_i$. By Lemma 3 each $F_i$ is minimax and hence nilpotent-by-abelian-by-finite (by Theorem 3.25 of [8]), and we may apply Proposition 1 of [1] to deduce that $G$ has a finite ascending normal series whose factors are soluble or locally finite. But each locally finite factor of this series is soluble-by-finite, by Lemma 1, and since each finite $G$-invariant factor of $G$ has its centralizer of finite index in $G$ we easily obtain the contradiction that $G$ is soluble-by-finite. □

**Lemma 5.** *Let $X$ be a finitely generated nilpotent-by-finite group, $Y$ a subgroup of $X$, and let $U$ be a normal subgroup of finite index in $Y$. Then there is a normal subgroup $N$ of finite index in $X$ such that $N \cap Y \leq U$.*

Proof: There is a normal nilpotent subgroup $V$ of finite index in $X$, and if $M \cap (Y \cap V) \leq U \cap V$ for some normal subgroup $M$ of finite index in $V$ then, with $N = \mathrm{Core}_X(M)$, we have $X/N$ finite and $N \cap Y = N \cap (Y \cap V) \leq (U \cap V) \leq U$, and the result follows. Thus we may suppose that $X = V$ and in particular that $Y$ is subnormal in $X$. If $Y$ is normal in $X$ then $Y/\mathrm{Core}_X(U)$ is finite, and since $X/\mathrm{Core}_X(U)$ is residually finite the result follows in this case. By induction on the subnormal defect of $Y$ in $X$ we may assume that there is a normal subgroup $W$ of $X$ containing $Y$, and a normal subgroup $K$ of finite index in $W$, such that $K \cap Y \leq U$; replacing $K$ by its core if necessary we may assume that $K$ is normal in $X$. Since $X/K$ is residually finite there is a normal subgroup $N$ of finite index in $X$ such that $N \cap W = K$, and then $N \cap Y = N \cap W \cap Y = K \cap Y \leq U$, and the lemma is proved. $\square$

**Lemma 6.** *Let $G$ be a locally soluble group with $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ and suppose that $H$ is a subnormal subgroup of $G$ that has periodic index in $G$. Then there is a positive integer $d$ such that $G^{(d)} \leq H$ (where $G^{(d)}$ denotes the $d$th term of the derived series of $G$).*

Proof: There is a finite subnormal series from $H$ to $G$, and each factor of this series is locally finite and hence soluble, by Lemma 1. $\square$

The following Proposition, with its rather technical hypotheses, will allow us to deal, in particular, with the case where $G$ is locally nilpotent-by-finite, and its generality will also allow us to reduce to that case. Much of the proof of the Proposition is similar to that of Theorem 4 of [1], though there are significant differences.

**Proposition.** *Let $G$ be a locally soluble group that is the ascending union of a chain of finitely generated subgroups $F_1 < F_2 < \dots$ and, for each $i$, let $K_i$ be a normal subgroup of $F_i$ such that $F_i/K_i$ is nilpotent-by-finite, $F_1/K_1$ and each index $[F_{i+1} : F_iK_{i+1}]$ is infinite, and $K_{i+1} \cap F_i \leq K_i$. Suppose also that $F_i$ has derived length exactly $d_i$ and that $F_{i+1}/K_{i+1}$ has derived length at least $2d_i$ (for each $i$). Then there are subgroups $M$ and $L$ of $G$ such that $L \leq M, L$ has periodic index in $G$ and $G^{(d)} \not\leq M$ for any positive integer $d$, and there is an infinite descending chain of subgroups of $M$ that contain $L$.*

Proof: There is a nilpotent normal subgroup $M_1/K_1$ of finite index in $F_1/K_1$ and a proper $F_1$-invariant subgroup $L_1/K_1$ of finite index in $M_1/K_1$. By Lemma 5, applied to the group $F_2/K_2$, there is a normal subgroup $M_2/K_2$ of finite index in $F_2/K_2$ such that $M_2 \cap F_1 \leq K_2L_1$ and so $M_2 \cap F_1 \leq K_2L_1 \cap F_1 = L_1(K_2 \cap F_1) = L_1$. Clearly we may choose $M_2$ such that $M_2/K_2$ is nilpotent and $F_2/M_2$ has derived length equal to that of $F_2/K_2$ (using the residual finiteness of $F_2/K_2$). Since $F_2/K_2$ is polycyclic, the intersection of all subgroups $U$ of finite index in $F_2$ that contain $K_2F_1$ is $K_2F_1$, which has infinite index in $F_2$. So we may choose such a $U$ that does not contain $M_2$, and then $L_2 := \mathrm{Core}_{F_2}(U \cap M_2)$ contains $K_2$, has finite index in $M_2$, and is such that $M_2$ is not contained in $F_1L_2$. We continue in this manner: for each $i$ there is a normal nilpotent subgroup $M_{i+1}/K_{i+1}$ of finite index in $F_{i+1}/K_{i+1}$ such that $M_{i+1} \cap F_i \leq L_i$ and $F_{i+1}/M_{i+1}$ has derived length equal to that of $F_{i+1}/K_{i+1}$, and an $F_{i+1}$-invariant subgroup $L_{i+1}/K_{i+1}$ of finite index in $M_{i+1}/K_{i+1}$ such that $M_{i+1} \not\leq F_iL_{i+1}$. Let $L = \langle L_i : i \geq 1\rangle, M = \langle M_i : i \geq 1\rangle$; clearly $L$ has periodic index in $G$ and $L \leq M$.

Suppose next that $G^{(d)} \leq M$ for some integer $d$, and choose $i > 1$ such that $d_{i-1} > d$. We have $F_i^{(d)} \leq M \cap F_i$, and since $M_{j+1} \cap F_j \leq M_j$ for all $j$ it follows that $M \cap F_i = M_1 \dots M_i$ (see Lemma 2 of [1]) and hence that $F_i^{(d)} \leq F_{i-1}M_i$.

But this gives $F_i^{(d+d_{i-1})} \leq M_i$, so $d_i \leq (d + d_{i-1}) < 2d_{i-1}$, a contradiction.

Now let $H_0 = M$ and, for $i \geq 1$, let $H_i = (L_1 \dots L_i)(M_{i+1}M_{i+2} \dots)$; thus $L \leq H_i \leq M$ for all $i$. Assume for a contradiction that the chain $H_0 \geq H_1 \geq H_2 \geq \dots$ terminates after finitely many steps, so that $M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots)$ for some $t$. Now if $x \in (M_{t+1} \dots M_{t+k}) \cap F_t$ for some $k > 1$ then

$$x \in (M_{t+1} \dots M_{t+k-1})(M_{t+k} \cap F_{t+k-1}) \cap F_t = (M_{t+1} \dots M_{t+k-1}) \cap F_t,$$

and an easy induction on $k$ shows that $x \in M_{t+1} \cap F_t \leq L_t$. Thus

$$\begin{aligned} M_t \leq (L_1 \dots L_t)(M_{t+1}M_{t+2} \dots) \cap F_t &= (L_1 \dots L_t)((M_{t+1}M_{t+2} \dots) \cap F_t) \\ &= (L_1 \dots L_t) \leq F_{t-1}L_t, \end{aligned}$$

contradicting the choice of $L_t$. Thus the $H_i$ form an infinite descending chain, and the Proposition is proved. □

Proof of Theorem 2: By Lemma 4 we may assume that $G$ is locally soluble, and our aim is to show that $G$ is soluble, so we may also assume that $G$ is countable. Suppose the result false. The iterated Hirsch-Plotkin radical of $G$ is soluble, by Lemma 2, so we may factor and assume that $G$ has no nontrivial subnormal soluble subgroups; as in the proof of Lemma 4, every soluble subgroup of $G$ is therefore minimax. If every insoluble subgroup of $G$ is subnormal then $G$ satisfies the weak minimal condition for non-subnormal subgroups (since soluble minimax groups satisfy the weak minimal condition for *all* subgroups), and Theorem B of [5] implies that either $G$ is soluble or $G$ has all subgroups subnormal, in which case $G$ is again soluble [7], a contradiction. Thus we see that every insoluble subgroup of $G$ contains a non-subnormal insoluble subgroup, and since $\mathrm{dev}_{\text{non-sn}}(G) \leq 1$ there exists an insoluble subgroup $G_0$ of $G$ such that, if $X < Y$ are any non-subnormal insoluble subgroups of $G_0$ then the interval $I[X, Y]$ satisfies the minimal condition for non-subnormal subgroups of $G$. We may assume that $G = G_0$ and, again factoring if necessary, we may retain the condition that the Hirsch-Plotkin radical of $G$ is trivial and hence that every soluble subgroup is minimax.

By Lemma 3, $G$ is the ascending union of finitely generated subgroups $F_1 < F_2 < \dots$, each of which is minimax. For each $i$, let $K_i$ denote the locally nilpotent radical of $F_i$; then $K_i$ is nilpotent and $F_i/K_i$ is abelian-by-finite, by Theorem 3.25 of [8]. Certainly $F_i \cap K_{i+1} \leq K_i$ for each $i$. If there is an upper bound $b$ for the derived lengths of the factors $F_i/K_i$ then $G^{(b)}$ is locally nilpotent and therefore trivial, and we have the contradiction that $G$ is soluble. Passing to an appropriate subsequence of the $F_i$ if necessary, we may therefore assume that the derived length of $F_{i+1}/K_{i+1}$ is at least twice that of $F_i$, for each $i$.

If there is no upper bound for the torsion-free ranks (that is, Hirsch lengths) of the $F_i/K_i$ then we may again re-label the $F_i$ and assume that the torsion-free rank of $F_{i+1}/K_{i+1}$ is greater than that of $F_i$ for each $i$ and that $F_1/K_1$ is infinite. But now $G$ satisfies the hypotheses and hence the conclusion of the Proposition and, with the notation of the Proposition, we may apply Lemma 6 to deduce that neither $L$ nor any of the subgroups $H_i$ is subnormal in $G$, so the interval $I[L,M]$ does not satisfy min for non-subnormal subgroups of $G$, so that $L$ must be soluble and hence minimax, which gives, since $L$ has periodic index in $G$, the contradiction that the torsion-free ranks of the $F_i$ are bounded. It follows that the torsion-free rank of each $F_{i+1}/K_{i+1}$ is at most $r$, for some fixed positive integer $r$.

For each $i$ there is a normal torsion-free abelian subgroup $U_i/K_i$ of finite index in $F_i/K_i$, and the rank of each $U_i/K_i$ is at most $r$. By Theorem 3.23 of [8], there is an integer $d = d(r)$ such that (for all $i$), if $C_i$ denotes the centralizer in $F_i$ of $U_i/K_i$, then $F_i/C_i$ has derived length at most $d$. As $C_i/K_i$ is centre-by-finite it has finite derived group, and so $F_i^{(d+1)}$ is nilpotent-by-finite, which gives $G^{(d+1)}$ locally nilpotent-by-finite, so there is no loss in assuming that $G$ is locally nilpotent-by-finite, that is, $F_i/K_i$ is finite for each $i$. Observe that, by factoring by the iterated Hirsch-Plotkin radical if necessary, we may again retain the condition that all soluble subgroups of $G$ are minimax.

If the torsion-free ranks of the $F_i$ are bounded above then, as in the proof of Lemma 4, we may apply Proposition 1 of [1] to deduce that $G$ has a finite ascending normal series whose factors are soluble or locally finite and hence to obtain the contradiction that $G$ is soluble. Thus we may assume that the torsion-free ranks of the $F_i$ increase with $i$. But now $G$ satisfies the hypotheses of the Proposition, with each "$K_i$" of those hypotheses trivial, and as in the earlier part of the present proof we have subgroups $L$ and $M$ and an infinite descending chain of subgroups $H_i$ of $M$, each containing $L$, such that no $H_i$ is subnormal in $G$. Thus $L$ is soluble and hence minimax and, since $L$ has periodic index in $G$, we obtain the contradiction that the torsion-free ranks of the $F_i$ are bounded. This completes the proof of Theorem 2. □

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