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Regularity for entropy solutions of a class of parabolic equations with irregular data

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Abstract. Using as a main tool the time-regularizing convolution operator introduced by R. Landes, we obtain regularity results for entropy solutions of a class of parabolic equations with irregular data. The results are obtained in a very general setting and include known previous results.

Keywords: regularity, entropy solutions, parabolic equations, irregular data

Classification: 35D10, 35K55

1. Introduction and statement of the results

In this paper, we study the following class of nonlinear parabolic equations

$$\text{(P)} \qquad \begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}(a(x,t,u,Du)) = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $R^N (N \geq 2)$ and $T > 0$, $Q = \Omega \times (0,T)$, Σ denotes the lateral surface of Q, $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$. The function $a(x,t,s,\xi) : Q \times R \times R^N \to R^N$ is a *Carathéodory* function satisfying for almost every $(x,t) \in Q$ and every $(s,\xi) \in R^{N+1}$, $\xi \in R^N$, $\xi' \in R^N$, $\xi \neq \xi'$,

$$a(x,t,s,\xi)\xi \geq b(|s|)|\xi|^p, \tag{1.1}$$

$$|a(x,t,s,\xi)| \leq \beta(\eta(x,t) + b(|s|)|\xi|^{p-1}), \tag{1.2}$$

$$[a(x,t,s,\xi)) - a(x,t,s,\xi')][\xi - \xi'] > 0, \tag{1.3}$$

where β is a positive constant, $p > 1$, η is a nonnegative function and belongs to $L^{p'}(Q)$, $p' = \frac{p}{p-1}$, $b : [0,+\infty) \to (0,+\infty)$ is a continuous function such that

$$b(|s|) \geq \alpha > 0, \tag{1.4}$$

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where α is a positive constant.

The simplest model, in the case $p = 2$, of $a(x,t,s,\xi)$ is $a(x,t,s,\xi) = (1+|s|)^m\xi$ with $m \geq 0$.

Recently the concept of entropy solutions to elliptic equations and parabolic equations was introduced in [1] and [2], respectively. The existence of entropy solutions to problem (P) was obtained in [3].

Let $T_k(s) = \min\{k, \max\{-k, s\}\}$, $S_k(s) = \int_0^s T_k(\tau)\,d\tau$ denote its primitive function for every $s \in R$ and $k > 0$.

Definition 1.1. A measurable function $u \in L^\infty(0,T;L^1(\Omega))$ will be called an entropy solution of problem (P) if $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, $S_k(u(\cdot,t)) \in L^1(\Omega)$, $\forall\, k > 0$, $\forall\, t \in [0,T]$, and u satisfies

$$\begin{aligned}
&\int_\Omega S_k(u(T)-\phi(T))\,dx + \int_0^T \langle \phi_t, T_k(u-\phi)\rangle\,dt \\
&\quad + \int_Q a(x,t,u,Du)DT_k(u-\phi)\,dx\,dt \\
&\leq \int_\Omega S_k(u_0-\phi(0))\,dx + \int_Q fT_k(u-\phi)\,dx\,dt,
\end{aligned} \tag{1.5}$$

$\forall\, k > 0$, $\forall\, \phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\phi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$.

Definition 1.2 (see [5], [10], [14])**.** For $0 < q < +\infty$, the set of all measurable functions $u : Q \to R$ such that the functional $[u]_q = \sup_{k>0} k\,\mathrm{meas}\{(x,t) \in Q : |u(x,t)| > k\}^{\frac{1}{q}}$ is finite, is called the Marcinkiewicz space and is denoted by $\mathcal{M}^q(Q)$.

One can deduce that $\mathcal{M}^q(Q) \subset \mathcal{M}^r(Q)$ for $r < q$. The connection between Marcinkiewicz and Lebesgue spaces is as follows: $L^q(Q) \subset \mathcal{M}^q(Q) \subset L^r(Q)$ for $r < q$ (see [5], [14]). The Marcinkiewicz spaces are also known as weak-Lebesgue spaces. When $q > 1$, the Marcinkiewicz space $\mathcal{M}^q(Q)$ is a Banach space with the norm defined by $\|u\|_q = \sup_{t>0} t^{\frac{1-q}{q}} \int_0^t u^*(\tau)d\tau$, where $u^*(\tau) = \inf\{k > 0 : \mathrm{meas}\{|u| > k\} \leq \tau\}$ defines the non-increasing rearrangement of u (see [14]).

Considering the growth of $a(x,t,s,\xi)$ with respect to s, not only it can be proved the existence of entropy solution u, but also that a fast growth of $a(x,t,s,\xi)$ as s goes to infinity improves the regularity of u. What is most remarkable is that the growth of $b(|s|)$ at infinity affects also the summability of Du. Regularity results in a similar context to elliptic equations can be found in [4].

Now we state the main results of this paper.

Theorem 1.1. *Assume* (1.1) *and* (1.4), *and let* $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$. *Assume moreover that there exist positive constants* γ *and* $s_0, m \geq 0$ *such that*

$$b(|s|) \geq \gamma |s|^m, \ \forall\, s : |s| \geq s_0. \tag{1.6}$$

Let u be an entropy solution to problem (P). *Then we have*

(i) *if $m > 1$, then $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q)$, $r = \frac{N+1}{N}p$;*
(ii) *if $0 \le m < 1$, then $u \in \mathcal{M}^r(Q)$, $r = \frac{(N+1)p-N}{N} + m$, $|Du| \in \mathcal{M}^q(Q)$, $q = p - \frac{N(1-m)}{N+1}$;*
(iii) *if $m = 1$, then $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^r(Q)$, $1 \le q < p$, $r < \frac{(N+1)p}{N}$;*

where $\mathcal{M}^r(Q), \mathcal{M}^q(Q)$ are the Marcinkiewicz spaces.

Theorem 1.2. *Assume* (1.1), (1.4) *and* (1.6), *and let $f \in L^d(Q)$, $1 < d < \frac{(N+2)p}{(N+2)p-N}$, $u_0 = 0$. Let u be an entropy solution to problem* (P) *and $u \in L^\infty(0,T;L^{\frac{(2-p-d+dp)N}{N+p-pd}}(\Omega))$. Then we have*

(i) *if $m \ge 1 - \frac{(N+2-d)p}{(N+p-pd)d'}$, $d' = \frac{d}{d-1}$, then $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q)$, $r = \frac{(N+2-d)p}{N+p-pd}$;*
(ii) *if $0 \le m < 1 - \frac{(N+2-d)p}{(N+p-pd)d'}$ and one of the following conditions is satisfied:*
(1) *$p \ge 2 - \frac{1}{N+1}$,*
(2) *$1 < p < 2 - \frac{1}{N+1}$ but $\frac{N+2}{(N+1)p-(N-1)} \le d$,*
then $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^r(Q)$ with $q = d[p - \frac{N+p-pd}{N+2-d}(1-m)]$ and $r = d[\frac{p(N+2-d)}{N+p-pd} - 1 + m]$.

Theorem 1.3. *Assume* (1.1), (1.4) *and* (1.6), *and let $u_0 \in L^d(\Omega)$, $1 < d < 2$, $f = 0$. Let u be an entropy solution to problem* (P) *and $u \in L^\infty(0,T;L^d(\Omega))$. Then we have*

(i) *if $m \ge 2 - d$, then $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^r(Q)$, $r = \frac{N+d}{N}p$;*
(ii) *if $0 \le m < 2 - d$ and one of the following conditions is satisfied:*
(1) *$p \ge 2 - \frac{1}{N+1}$,*
(2) *$1 < p < 2 - \frac{1}{N+1}$ but $\frac{N(3-p)}{N+p-1} \le d$,*
then $u \in L^q(0,T;W_0^{1,q}(\Omega)) \cap L^r(Q)$ with $q = p - \frac{N}{N+d}(2-m-d)$ and $r = \frac{N+d}{N}p - 2 + m + d$.

Remark 1.1. Theorems 1.1–1.3 show that not only the right term f and initial value u_0 can affect the regularity of entropy solution u, but also the growth of $b(|s|)$ at infinity affects the regularity.

Remark 1.2. The exponents q, r of Theorem 1.1 in the case of $m = 0$ are the same as that of [7]. This theorem extends Theorem 3.6 in [7] to the general setting. Moreover two cases of $0 \le m < 1$ and $m = 1$ are studied in the framework of Marcinkiewicz and Sobolev space in this paper.

Remark 1.3. In Theorem 1.2, if d tends to 1 then $\frac{(2-p-d+dp)N}{N+p-pd}$ tends to 1, q, r tend to $p - \frac{N(1-m)}{N+1}$ and $\frac{(N+1)p-N}{N} + m$, respectively, which are the bounds for q, r obtained in Theorem 1.1. The existence and regularity of solutions to problem (P) was studied in [6] in the case of $m = 0$ and $p \geq 2$. We point out that our result was obtained for every $p \geq 2 - \frac{1}{N+1}$ and $1 < p < 2 - \frac{1}{N+1}$ (with $\frac{N+2}{(N+1)p-(N-1)} \leq d$) in the case of $m = 0$. From the viewpoint of regularity, Theorem 1.2 improves Theorem 1.9 of [6].

Remark 1.4. The same problem as that of Theorem 1.3 was discussed in [8] and [9] for the case of $m = 0$. However the condition of $p > 2 - \frac{1}{N+1}$ was assumed in [8]. Though Segura de León (see [9]) got the regularity of entropy solution in the framework of Marcinkiewicz space without the restriction of p, his result is not optimal because the same exponents of Sobolev space as that of Theorem 1.3 and [8] cannot be deduced from Segura de León's results even in the case of $p > 2 - \frac{1}{N+1}$.

Remark 1.5. In Theorem 1.2 and Theorem 1.3, we need to assume entropy solution $u \in L^\infty(0, T; L^{(2-p-d+dp)N/(N+p-pd)}(\Omega))$ and $u \in L^\infty(0, T; L^d(\Omega))$, respectively. In fact, the existence of at least an entropy solution having this properties can be obtained by using the same method as that of [6]. However, we mainly study the regularity, not the existence, of entropy solution to problem (P) in this paper.

2. The proof of Theorems 1.1–1.3

In order to prove the main results of this paper, we need the following lemmas.

Lemma 2.1. *If $f \in L^1(Q)$, $u_0 \in L^1(\Omega)$, and u is an entropy solution to problem* (P), *then*

$$\int_{\{h \leq |u| < h+k\}} b(|u|)|Du|^p \, dx \, dt \leq k\Big(\int_{\{|u| \geq h\}} |f| \, dx \, dt + \int_{\{|u_0| \geq h\}} |u_0| \, dx\Big), \ \forall \, k, h > 0. \tag{2.1}$$

Proof: To prove Lemma 2.1, we need to introduce a time-regularizing convolution operator as it is done in [12], [3], [6] and [8]. More precisely, let $\tilde{T}_h(u)$ be zero extension of $T_h(u)$ outside $(0, T)$. Then we define

$$(T_h(u))_\nu(x,t) = \int_{-\infty}^{t} \nu \tilde{T}_h(u) e^{\nu(s-t)} \, ds. \tag{2.2}$$

The property of $(T_h(u))_\nu$ can be seen in [12] and [3]. Let us take a sequence $\{\psi_n\} \subset C_0^\infty(\Omega)$ such that ψ_n converges to u_0 in $L^1(\Omega)$ and consider the function $\phi_{n,\nu}(x,t) = (T_h(u))_\nu + e^{-\nu t} T_h(\psi_n)$. Taking $\phi = \phi_{n,\nu}$ in (1.5), we get

$$\begin{aligned} &\int_\Omega S_k(u(T) - \phi_{n,\nu}(T))\,dx + \int_0^T \langle (\phi_{n,\nu})_t, T_k(u - \phi_{n,\nu})\rangle\,dt \\ &\quad + \int_Q a(x,t,u,Du) DT_k(u - \phi_{n,\nu})\,dx\,dt \\ &\leq \int_\Omega S_k(u_0 - T_h(\psi_n))\,dx + \int_Q f T_k(u - \phi_{n,\nu})\,dx\,dt. \end{aligned} \tag{2.3}$$

Note that $|\phi_{n,\nu}| \leq h$ and $(\phi_{n,\nu})_t = \nu(T_h(u) - \phi_{n,\nu})$. Therefore we have

$$\begin{aligned} &\int_0^T < (\phi_{n,\nu})_t, T_k(u - \phi_{n,\nu}) > \ dt \\ &= \int_Q \nu(T_h(u) - \phi_{n,\nu}) T_k(u - \phi_{n,\nu})\,dx\,dt \\ &= \int_{\{|u| \leq h\}} \nu(u - \phi_{n,\nu}) T_h(u - \phi_{n,\nu})\,dx\,dt \\ &\quad + \int_{\{u > h\}} \nu(h - \phi_{n,\nu}) T_k(u - \phi_{n,\nu})\,dx\,dt \\ &\quad + \int_{\{u < -h\}} \nu(-h - \phi_{n,\nu}) T_k(u - \phi_{n,\nu})\,dx\,dt \geq 0, \ \forall\, n, \nu. \end{aligned} \tag{2.4}$$

Since $S_k(s) \geq 0$, $\forall\, s \in R$, (2.3) implies that

$$\begin{aligned} &\int_Q a(x,t,u,Du) DT_k(u - (T_h(u))_\nu - e^{-\nu t} T_h(\psi_n))\,dx\,dt \\ &\leq \int_Q f T_k(u - (T_h(u))_\nu - e^{-\nu t} T_h(\psi_n))\,dx\,dt + \int_\Omega S_k(u_0 - T_h(\psi_n))\,dx. \end{aligned} \tag{2.5}$$

Since $DT_k(u - \phi_{n,\nu}) = 0$ where $|u| > h + k$, the first integral in (2.5) can be rewritten in the following way:

$$\int_Q a(x,t,T_{h+k}(u), DT_{h+k}(u)) DT_k(u - (T_h(u))_\nu - e^{-\nu t} T_h(\psi_n))\,dx\,dt. \tag{2.6}$$

It is easy to see that, as ν goes to infinity, we have

$$DT_k(u - (T_h(u))_\nu - e^{-\nu t} T_h(\psi_n)) \longrightarrow DT_k(u - T_h(u)) \ \text{ strongly in } \ L^p(Q). \tag{2.7}$$

Let ν tend to infinity in (2.5). We get

$$\begin{aligned}
&\int_Q a(x,t,T_{h+k}(u),DT_{h+k}(u))DT_k(u-T_h(u))\,dx\,dt \\
&\le \int_Q fT_k(u-T_h(u))\,dx\,dt + \int_\Omega S_k(u_0-T_h(\psi_n))\,dx.
\end{aligned} \tag{2.8}$$

Finally we pass to the limit in (2.8) as n tends to infinity, obtaining

$$\begin{aligned}
&\int_Q a(x,t,T_{h+k}(u),DT_{h+k}(u))DT_k(u-T_h(u))\,dx\,dt \\
&\le \int_Q fT_k(u-T_h(u))\,dx\,dt + \int_\Omega S_k(u_0-T_h(u_0))\,dx.
\end{aligned} \tag{2.9}$$

The above inequality can be rewritten in the following way

$$\begin{aligned}
&\int_Q a(x,t,u,Du)DT_k(u-T_h(u))\,dx\,dt \\
&\le k[\int_{\{|u|\ge h\}} |f|\,dx\,dt + \int_{\{|u_0|\ge h\}} |u_0|\,dx].
\end{aligned} \tag{2.10}$$

From (1.1) and (2.10), we can get (2.1). Thus we complete the proof of Lemma 2.1. $\square$

We also need the following embedding theorem.

Lemma 2.2 (see [13, Proposition 3.1]). *If $v \in L^l(0,T;W_0^{1,l}(\Omega)) \cap L^\infty(0,T;L^\rho(\Omega))$ with $l \ge 1$, $\rho \ge 1$, then there exists a constant C depending only on N, l, ρ such that*

$$\int_Q |v|^r\,dx\,dt \le C\|v\|_{L^\infty(0,T;L^\rho(\Omega))}^{\rho l/N} \int_Q |Dv|^l\,dx\,dt, \tag{2.11}$$

where $r = \frac{(N+\rho)l}{N}$.

Lemma 2.3. *Let u be an entropy solution to problem* (P). *Then for any fixed $0<\tau<1$ and large enough $l>1$, we have*

$$\sum_{k=1}^{l} \int_{\{|u|\ge k\}} |f|k^{-\tau}\,dx\,dt \le \frac{1}{1-\tau}\int_Q |f|\,|T_l(u)|^{1-\tau}\,dx\,dt. \tag{2.12}$$

Proof: Let $B_h = \{(x,t) \in Q : h \le |u(x,t)| < h+1\}$. It follows from the formula of Abel's summation that

$$\begin{aligned}
&\sum_{k=1}^{l} \int_{\{|u|\ge k\}} |f| k^{-\tau}\, dx\, dt \\
&= \sum_{k=1}^{l} \sum_{h=k}^{\infty} \int_{B_h} |f| k^{-\tau}\, dx\, dt \\
&= \int_{\{|u|\ge l\}} |f| \sum_{k=1}^{l} k^{-\tau}\, dx\, dt + \sum_{k=1}^{l-1} \int_{B_k} |f| \sum_{h=1}^{k} h^{-\tau}\, dx\, dt \\
&\le \frac{1}{1-\tau} \int_{\{|u|\ge l\}} |f| l^{1-\tau}\, dx\, dt + \frac{1}{1-\tau} \sum_{k=1}^{l-1} \int_{B_k} |f| k^{1-\tau}\, dx\, dt \\
&= \frac{1}{1-\tau} \int_{\{|u|\ge l\}} |f|\, |T_l(u)|^{1-\tau}\, dx\, dt + \frac{1}{1-\tau} \sum_{k=1}^{l-1} \int_{B_k} |f|\, |T_k(u)|^{1-\tau}\, dx\, dt \\
&\le \frac{1}{1-\tau} \int_{\{|u|\ge l\}} |f|\, |T_l(u)|^{1-\tau}\, dx\, dt + \frac{1}{1-\tau} \sum_{k=1}^{l-1} \int_{B_k} |f|\, |T_l(u)|^{1-\tau}\, dx\, dt \\
&= \frac{1}{1-\tau} \int_{\{|u|\ge 1\}} |f|\, |T_l(u)|^{1-\tau}\, dx\, dt \\
&\le \frac{1}{1-\tau} \int_{Q} |f|\, |T_l(u)|^{1-\tau}\, dx\, dt.
\end{aligned}$$

Thus the proof of Lemma 2.3 is complete. □

Remark 2.1. Similarly to the proof of Lemma 2.3, we can obtain

$$\sum_{k=1}^{\infty} \int_{\{|u_0|\ge k\}} |u_0| k^{d-2}\, dx \le \frac{1}{d-1} \int_{\Omega} |u_0|^d\, dx,$$

where $1 < d < 2$.

Proof of Theorem 1.1:

Proof of (i): For any given $k \ge 1$, replacing h and k with k and 1 in Lemma 2.1 respectively, we get

$$\int_{\{k\le|u|<k+1\}} b(|u|)|Du|^p\, dx\, dt \le \int_{\{|u|\ge k\}} |f|\, dx\, dt + \int_{\{|u_0|\ge k\}} |u_0|\, dx. \tag{2.13}$$

Inequality (1.6) implies that

$$\gamma k^m \int_{B_k} |Du|^p\, dx\, dt \le \|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}.\ \forall\, k \ge s_0. \tag{2.14}$$

Thus we get

$$\int_{B_k} |Du|^p \, dx \, dt \le \frac{C_1}{k^m}, \tag{2.15}$$

where $C_1 = \frac{1}{\gamma}(\|f\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)})$. Let $k_0 = [s_0] + 1$, where $[s_0]$ denotes the maximal integer not beyond s_0. Since $m > 1$, we get

$$\int_Q |Du|^p \, dx \, dt \le \int_Q |DT_{k_0}(u)|^p \, dx \, dt + \sum_{k=1}^{\infty} \frac{C_1}{k^m} \le C_2, \tag{2.16}$$

where C_2 is a positive constant. The above estimate is due to the summability of integration domain and the convergence of m-series $(m > 1)$.

In the following, we will denote by C_i analogous constants. It can be deduced that u has zero trace by Theorem 2.1 in [7]. Thus we obtain that u belongs to $L^p(0, T; W_0^{1,p}(\Omega))$. Taking $l = p$, $\rho = 1$, $r = \frac{N+1}{N} p$ in (2.11), we get $u \in L^r(Q)$, $r = \frac{N+1}{N} p$.

Proof of (ii): In the case of $0 \le m < 1$, for any given $k > k_0$, arguing as for (2.16), it can be deduced that

$$\begin{aligned} \int_Q |DT_k(u)|^p \, dx \, dt &= \int_Q |DT_{k_0}(u)|^p \, dx \, dt + \sum_{i=k_0}^{k-1} \frac{C_1}{i^m} \\ &\le \int_Q |DT_{k_0}(u)|^p \, dx \, dt + \sum_{i=1}^{k-1} \frac{C_1}{i^m} \\ &\le C_3(1 + k^{1-m}). \end{aligned} \tag{2.17}$$

Taking $\rho = 1, l = p, r = \frac{(N+1)p}{N}, v = T_k(u)$ in (2.11), we have

$$\begin{aligned} \int_Q |T_k(u)|^{\frac{(N+1)p}{N}} \, dx \, dt &\le C_4 \|T_k(u)\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{p}{N}} \int_Q |DT_k(u)|^p \, dx \, dt \\ &\le C_4 \|u\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{p}{N}} \int_Q |DT_k(u)|^p \, dx \, dt \\ &\le C_5 \int_Q |DT_k(u)|^p \, dx \, dt. \end{aligned} \tag{2.18}$$

However, we get

$$\operatorname{meas}\{|u| > k\} \le k^{-\frac{(N+1)p}{N}} \int_Q |DT_k(u)|^p \, dx \, dt. \tag{2.19}$$

By (2.17)–(2.19), we obtain

$$\text{(2.20)} \qquad \begin{aligned} \operatorname{meas}\{|u|>k\} &\le C_5 k^{-\frac{(N+1)p}{N}} C_3(1+k^{1-m}) \\ &\le C_6 k^{-(p-1+\frac{p}{N}+m)}, \ \forall\, k>k_0. \end{aligned}$$

Thus we get

$$\text{(2.21)} \qquad \sup_{k>0} k(\operatorname{meas}\{|u|>k\})^{\frac{1}{p-1+\frac{p}{N}+m}} \le C_7.$$

Hence we obtain $u \in \mathcal{M}^r(Q)$, $r = p-1+\frac{p}{N}+m$. For any given $l>0$,

$$\text{(2.22)} \qquad \operatorname{meas}\{|DT_k(u)|>\frac{l}{2}\} \le \int_Q \frac{|DT_k(u)|^p}{(\frac{l}{2})^p}\,dx\,dt \le \frac{2^p C_3(1+k^{1-m})}{l^p}.$$

Thus

$$\text{(2.23)} \qquad \begin{aligned} \operatorname{meas}\{|Du|>l\} &\le \operatorname{meas}\{|Du-DT_k(u)|>\frac{l}{2}\} + \operatorname{meas}\{|DT_k(u)|>\frac{l}{2}\} \\ &\le \operatorname{meas}\{|u|>k\} + \operatorname{meas}\{|DT_k(u)|>\frac{l}{2}\} \\ &\le C_8 k^{-(p-1+\frac{p}{N}+m)} + \frac{2^p C_3(1+k^{1-m})}{l^p}. \end{aligned}$$

Taking $k = l^{q/r}$ in (2.23), we get

$$\text{(2.24)} \qquad \operatorname{meas}\{|Du|>l\} \le C_9 l^{-q}, q = p - \frac{N}{N+1}(1-m).$$

Hence $|Du| \in \mathcal{M}^q(Q)$, $q = p - \frac{N}{N+1}(1-m)$.

Proof of (iii): As $m=1$, for large enough $l>k_0$, for every $1 \le q < p$ and $\lambda > 0$, we have

$$\text{(2.25)} \qquad \begin{aligned} \int_Q \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^\lambda}\,dx\,dt &= \int_{\{|u|\le k_0\}} \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^\lambda}\,dx\,dt \\ &\quad + \sum_{k=k_0}^{l-1} \int_{B_k} \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^\lambda}\,dx\,dt \\ &\le \int_{\{|u|\le k_0\}} |Du|^p\,dx\,dt + \sum_{k=k_0}^{l-1} \int_{B_k} \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^\lambda}\,dx\,dt \end{aligned}$$

$$
\begin{aligned}
&= \int_Q |DT_{k_0}(u)|^p \,dx\,dt + \sum_{k=k_0}^{l-1} \int_{B_k} \frac{|Du|^p}{(1+|u|)^\lambda} \,dx\,dt \\
&\le \int_Q |DT_{k_0}(u)|^p \,dx\,dt + \sum_{k=k_0}^{l-1} \frac{1}{(1+k)^\lambda} \frac{C_1}{k} \\
&\le \int_Q |DT_{k_0}(u)|^p \,dx\,dt + \sum_{k=1}^{\infty} \frac{1}{(1+k)^\lambda} \frac{C_1}{k} \\
&\le C_{10}.
\end{aligned}
$$

By Hölder's inequality, we obtain

$$
\begin{aligned}
&\int_Q |DT_l(u)|^q \,dx\,dt = \int_Q \frac{|DT_l(u)|^q}{(1+|T_l(u)|)^{\lambda q/p}} (1+|T_l(u)|)^{\lambda q/p} \,dx\,dt \\
&\le \Big(\int_Q \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^\lambda} \,dx\,dt\Big)^{q/p} \Big(\int_Q (1+|T_l(u)|)^{\lambda q/(p-q)} \,dx\,dt\Big)^{1-q/p} \\
&\le C_{11}\Big[1 + \Big(\int_Q |T_l(u)|^{\lambda q/(p-q)} \,dx\,dt\Big)^{1-q/p}\Big].
\end{aligned}
\tag{2.26}
$$

If we set $\lambda q/(p-q) = q(N+1)/N$, then we have $\lambda = \frac{N+1}{N}(p-q)$. Let $r = \frac{q(N+1)}{N}$, $\rho = 1$, $l = q$ in Lemma 2.2, we get

$$
\begin{aligned}
\int_Q |DT_l(u)|^q \,dx\,dt &\le C_{11}\Big[1 + \Big(\int_Q |T_l(u)|^{q(N+1)/N} \,dx\,dt\Big)^{1-q/p}\Big] \\
&\le C_{12}\Big[1 + \Big(\int_Q |DT_l(u)|^q \,dx\,dt\Big)^{1-q/p}\Big].
\end{aligned}
\tag{2.27}
$$

Hence

$$
\int_Q |DT_l(u)|^q \,dx\,dt \le C_{13} \quad \text{and} \quad \int_Q |T_l(u)|^q \,dx\,dt \le C_{13},
\tag{2.28}
$$

where C_{13} is a positive constant independent of l. Letting $l \to +\infty$ in (2.28), we obtain by Fatou lemma

$$
\int_Q |Du|^q \,dx\,dt \le C_{13} \quad \text{and} \quad \int_Q |u|^q \,dx\,dt \le C_{13}.
\tag{2.29}
$$

The condition $\lambda > 0$ implies that it must be $q < p$ and $\frac{q(N+1)}{N} < p + \frac{p}{N}$.Thus it follows that $u \in L^q(0,T; W_0^{1,q}(\Omega))(1 \le q < p)$ from (2.29) and Theorem 2.1

in [7]. (2.29) and Lemma 2.2 ($r = \frac{q(N+1)}{N}$, $\rho = 1$, $l = q$) imply that $u \in L^r(Q)$, $r < p + \frac{p}{N}$. $\square$

PROOF OF THEOREM 1.2:

Proof of (i): If $m \geq 1 - \frac{(N+2-d)p}{(N+p-pd)d'}$, and $u_0 = 0$, for large enough $l > k_0$, taking $\tau = 1 - \frac{(N+2-d)p}{(N+p-pd)d'}$ in (2.12), (1.6) and (2.12)–(2.13) (here $u_0 = 0$) imply that

$$\begin{aligned}\int_Q |DT_l(u)|^p\,dx\,dt &= \int_{\{|u|\leq k_0\}} |DT_l(u)|^p\,dx\,dt + \sum_{k=k_0}^{l-1}\int_{B_k} |DT_l(u)|^p\,dx\,dt\\ &= \int_{\{|u|\leq k_0\}} |Du|^p\,dx\,dt + \sum_{k=k_0}^{l-1}\int_{B_k} |Du|^p\,dx\,dt\\ &\leq \int_Q |DT_{k_0}(u)|^p\,dx\,dt + \frac{1}{\gamma}\sum_{k=k_0}^{l-1}\int_{\{|u|\geq k\}} |f|k^{-m}\,dx\,dt\\ &\leq C_{14} + \frac{1}{\gamma}\sum_{k=1}^{l}\int_{\{|u|\geq k\}} |f|k^{-[1-\frac{(N+2-d)p}{(N+p-pd)d'}]}\,dx\,dt\\ &\leq C_{14} + C_{15}\int_Q |f|\,|T_l(u)|^{\frac{(N+2-d)p}{(N+p-pd)d'}}\,dx\,dt\\ &\leq C_{16}[1 + (\int_Q |T_l(u)|^{\frac{(N+2-d)p}{(N+p-pd)}}\,dx\,dt)^{\frac{1}{d'}}].\end{aligned} \tag{2.30}$$

Taking $r = \frac{(N+2-d)p}{N+p-pd}$, $\rho = \frac{(2-p-d+pd)N}{N+p-pd}$, $l = p$ in Lemma 2.2, (2.30) yields

$$\int_Q |T_l(u)|^r\,dx\,dt \leq C_{17}\int_Q |DT_l(u)|^p\,dx\,dt \leq C_{18}[1 + (\int_Q |T_l(u)|^r\,dx\,dt)^{1/d'}]. \tag{2.31}$$

Thus there exists a positive constant C_{19} independent of l such that

$$\int_Q |T_l(u)|^r\,dx\,dt + \int_Q |DT_l(u)|^p\,dx\,dt \leq C_{19}. \tag{2.32}$$

Let $l \to +\infty$ in (2.32). By Fatou lemma, it follows that $u \in L^p(0,T;W^{1,p}(\Omega)) \cap L^r(Q)$, $r = \frac{(N+2-d)p}{N+p-pd}$. Furthermore, the conclusion that u has zero trace can be got by Lemma 2.1 in [7].

Proof of (ii): Taking $0<\lambda<1-m$, let $\tau=m+\lambda$ in (2.12). Hölder's inequality and (2.12) imply that

(2.33)

$$\begin{aligned}
&\int_Q |DT_l(u)|^q\,dx\,dt \le \int_Q \frac{|DT_l(u)|^q}{(1+|T_l(u)|)^{\lambda\frac{q}{p}}}(1+|T_l(u)|)^{\lambda\frac{q}{p}}\,dx\,dt\\
&\le \Big(\int_Q \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^{\lambda}}\,dx\,dt\Big)^{\frac{q}{p}}\Big(\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big)^{1-\frac{q}{p}}\\
&= \Big[\int_{\{|u|\le k_0\}} \frac{|DT_l(u)|^p}{(1+|T_l(u)|)^{\lambda}}\,dx\,dt+\sum_{k=k_0}^{l-1}\int_{B_k}\frac{|DT_l(u)|^p}{(1+|T_l(u)|)^{\lambda}}\,dx\,dt\Big]^{\frac{q}{p}}\\
&\quad\times\Big[\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big]^{1-\frac{q}{p}}\\
&\le \Big[\Big(\int_{\{|u|\le k_0\}} |Du|^p\,dx\,dt\Big)^{\frac{q}{p}}+\Big(\sum_{k=k_0}^{l-1}\int_{B_k}\frac{|Du|^p}{(1+|u|)^{\lambda}}\,dx\,dt\Big)^{\frac{q}{p}}\Big]\\
&\quad\times\Big[\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big]^{1-\frac{q}{p}}\\
&= \Big[\Big(\int_Q |DT_{k_0}(u)|^p\,dx\,dt\Big)^{\frac{q}{p}}+\Big(\sum_{k=k_0}^{l-1}\int_{B_k}\frac{|Du|^p}{(1+|u|)^{\lambda}}\,dx\,dt\Big)^{\frac{q}{p}}\Big]\\
&\quad\times\Big(\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big)^{1-\frac{q}{p}}\\
&\le \Big[C_{20}+\Big(\sum_{k=k_0}^{l}\frac{1}{\gamma}\int_{\{|u|\ge k\}}\frac{|f|}{k^{m+\lambda}}\,dx\,dt\Big)^{\frac{q}{p}}\Big]\Big[\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big]^{1-\frac{q}{p}}\\
&\le \Big[C_{20}+C_{21}\Big(\int_Q |f|\,|T_l(u)|^{1-m-\lambda}\,dx\,dt\Big)^{\frac{q}{p}}\Big]\Big[\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big]^{1-\frac{q}{p}}\\
&\le \Big[C_{20}+C_{22}\Big(\int_Q |T_l(u)|^{(1-m-\lambda)d'}\,dx\,dt\Big)^{\frac{q}{pd'}}\Big]\Big[\int_Q (1+|T_l(u)|)^{\lambda\frac{q}{p-q}}\,dx\,dt\Big]^{1-\frac{q}{p}}.
\end{aligned}$$

Set $(1-m-\lambda)d'=\lambda q/(p-q)=\frac{(N+2-d)q}{N+p-pd}$ and $r=\frac{(N+2-d)q}{N+p-pd}$, $\rho=\frac{(2-p-d+pd)N}{N+p-pd}$, $l=q$ in Lemma 2.2 (here $v=T_l(u)$). (2.11) and (2.33) yield

$$\int_Q |T_l(u)|^r\,dx\,dt+\int_Q |DT_l(u)|^q\,dx\,dt\le C_{23}, \tag{2.34}$$

where C_{23} is a positive constant independent of l, $q=d[p-\frac{N+p-pd}{N+2-d}(1-m)]$ and $r=d[\frac{p(N+2-d)}{N+p-pd}-1+m]$. Let $l\to+\infty$ in (2.34) and by Lemma 2.1 in [7], it yields $u\in L^q(0,T;W_0^{1,q}(\Omega))\cap L^r(Q)$ by Fatou lemma. Furthermore, to take

$q \geq 1$ for every $0 \leq m < 1 - \frac{(N+2-d)p}{(N+p-pd)d'}$, we must have $d \geq \frac{N+2}{(N+1)p-(N-1)}$. Thus if we choose $p \geq 2 - \frac{1}{N+1}$, then we can deduce that $\frac{N+2}{(N+1)p-(N-1)} \leq 1$. Hence as $p \geq 2 - \frac{1}{N+1}$, the above conclusion is satisfied for every $1 < d < \frac{(N+2)p}{(N+2)p-N}$. If we take $1 < p < 2 - \frac{1}{N+1}$, then we get $\frac{N+2}{(N+1)p-(N-1)} > 1$, thus as $1 < p < 2 - \frac{1}{N+1}$, we must restrict $\frac{N+2}{(N+1)p-(N-1)} \leq d < \frac{(N+2)p}{(N+2)p-N}$. □

Proof of Theorem 1.3:

Proof of (i): Note that $f = 0$, for any given $m \geq 2 - d$. (1.6), (2.13) (here $f = 0$) and Remark 2.1 imply that

$$\begin{aligned}
\int_Q |Du|^p \, dx\, dt &= \int_Q |DT_{k_0}(u)|^p \, dx\, dt + \sum_{k=k_0}^{\infty} \int_{B_k} |Du|^p \, dx\, dt \\
&\leq C_{14} + \frac{1}{\gamma} \sum_{k=k_0}^{\infty} \int_{\{|u_0| \geq k\}} \frac{|u_0|}{k^m} \, dx \\
&\leq C_{14} + \frac{1}{\gamma} \sum_{k=1}^{\infty} \int_{\{|u_0| \geq k\}} \frac{|u_0|}{k^{2-d}} \, dx \\
&\leq C_{14} + C_{24} \int_\Omega |u_0|^d \, dx \\
&\leq C_{25}.
\end{aligned} \tag{2.35}$$

Thus we obtain $u \in L^p(0,T; W_0^{1,p}(\Omega))$ and taking $\rho = d$, $l = p$, $r = \frac{N+d}{N}p$, we get $u \in L^r(Q)$, $r = \frac{N+d}{N}p$.

Proof of (ii): The same as that of Theorem 1.2(ii), we omit the details. □

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