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Extraresolvability of balleans

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Abstract. A ballean is a set endowed with some family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. We introduce and study a new cardinal invariant of a ballean, the extraresolvability, which is an asymptotic reflection of the corresponding invariant of a topological space.

Keywords: ball structure, ballean, resolvability, extraresolvability

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1. Uniform spaces and balleans

A ball structure is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a ball of radius α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii*. Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\},$$
$$B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha),$$
$$B^*(A, \alpha) = \bigcup_{a \in A} B^*(a, \alpha).$$

A ball structure $\mathcal{B} = (X, P, B)$ is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,
$$B^*(x, \alpha') \subseteq B(x, \alpha),\ B(x, \beta') \subseteq B^*(x, \beta);$$
- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$
- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
$$B(B(x,\alpha),\beta) \subseteq B(x,\gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family
$$\left\{ \bigcup_{x\in X} B(x,\alpha) \times B(x,\alpha) : \alpha \in P \right\}$$
is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x,U) = \{y \in X : (x,y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

A ball structure is said to be a *ballean* (or a *coarse structure*) if it is upper symmetric and upper multiplicative. For motivation to study balleans as the asymptotic counterparts of the uniform topological spaces see [5], [9], [10], [14].

Now we define the mappings which play the part of uniformly continuous mappings on the ballean stage. Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,
$$f(B_1(x,\alpha)) \subseteq B_2(f(x),\beta).$$
A bijection $f : X_1 \to X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings. If $X_1 = X_2$ and the identity mapping id : $X_1 \to X_2$ is an asymorphism, we identify $\mathcal{B}_1$ and $\mathcal{B}_2$, and write $\mathcal{B}_1 = \mathcal{B}_2$. For each ballean $\mathcal{B} = (X, P, B)$, replacing every ball $B(x,\alpha)$ with $B(x,\alpha) \cap B^*(x,\alpha)$, we get the same ballean. Therefore, **in what follows, we assume that** $B(x,\alpha) = B^*(x,\alpha)$ **for all** $x \in X$**,** $\alpha \in P$**.**

Let $\mathcal{B} = (X, P, B)$ be a ballean. We say that $\mathcal{B}$ is *connected* if, for any two points $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x,\alpha)$. **In what follows, all balleans under consideration are supposed to be connected.**

A subset $V \subseteq X$ is called *bounded* if there exist $x \in X$ and $\alpha \in P$ such that $V \subseteq B(x,\alpha)$. A ballean $\mathcal{B}$ is called *bounded* if its support is bounded.

For a ballean $\mathcal{B}$ we define a preordering $\leq$ on its set P of radii by the rule: $\alpha \leq \beta$ if and only if $B(x,\alpha) \subseteq B(x,\beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\alpha' \in P$ such that $\alpha \leq \alpha'$. A *cofinality* $\mathrm{cf}(\mathcal{B})$ of $\mathcal{B}$ is the minimal cardinality of cofinal subsets of P.

Every metric space (X, d) determines the metric ballean $\mathcal{B}(X, d) = (X, \mathbb{R}^+, B_d)$, where $B_d(x,r) = \{y \in X : d(x,y) \leq r\}$. A ballean $\mathcal{B}$ is called *metrizable* if $\mathcal{B}$ is asymorphic to $\mathcal{B}(X, d)$ for some metric space (X, d). By [9, Theorem 9.1], a ballean $\mathcal{B}$ is metrizable if and only if $\mathrm{cf}(\mathcal{B}) \leq \aleph_0$.

2. Types of subsets of a ballean

Let $\mathcal{B} = (X, P, B)$ be a ballean. We say that a subset A of X is

- *large* if there exists $\alpha \in P$ such that $X = B(A, \alpha)$;
- *small* if $X \setminus B(A, \alpha)$ is large for every $\alpha \in P$;
- *thick* if $\mathrm{int}(A, \alpha) \neq \emptyset$ for every $\alpha \in P$, where $\mathrm{int}(A, \alpha) = \{x \in X : B(x, \alpha) \subseteq A\}$;
- *extralarge* if $\mathrm{int}(A, \alpha)$ is large for every $\alpha \in P$;
- *piecewise large* if there exists $\beta \in P$ such that $\mathrm{int}(B(A, \beta), \alpha) \neq \emptyset$ for every $\alpha \in P$;
- *pseudodiscrete* if, for every $\alpha \in P$, there exists a bounded subset V of X such that $B(a, \alpha) \cap A = \{a\}$ for every $a \in A \setminus V$.

We shall use the following elementary observations from [9, Chapter 12].

1. For a subset S of X, the following properties are equivalent: S is small, S is not piecewise large, $X \setminus S$ is extralarge, $(X \setminus S) \cap L$ is large for every large subset L of X.
2. A subset A of X is thick if and only if $A \cap L \neq \emptyset$ for every large subset L of X.
3. If the subsets $X_1, X_2, \dots, X_n$ of X are extralarge, then $X_1 \cap \dots \cap X_n$ is extralarge. If the subsets $S_1, \dots, S_n$ of X are small, then $S_1 \cup \dots \cup S_n$ is small. If a piecewise large subset A of X is finitely partitioned $A = A_1 \cup \dots \cup A_n$, then at least one cell A_i of the partition is piecewise large.

These observations give a foundation for the following uniform spaces-balleans vocabulary:

dense subset	large subset
nowhere dense subset	small subset
subset with nonempty interior	thick subset
subset with dense interior	extralarge subset
somewhere dense subset	piecewise large subset
discrete subset	pseudodiscrete subset

Using this vocabulary, we get the following cardinal invariants of a ballean:

density $(\mathcal{B}) = \min\{|L| : L$ is a large subset of $X\}$,
cellularity $(\mathcal{B}) = \sup\{|F| : F$ is a disjoint family of thick subsets of $X\}$,
spread $(\mathcal{B}) = \sup\{|Y|_{\mathcal{B}} : Y$ is a pseudodiscrete subset of $X\}$, where
$|Y|_{\mathcal{B}} = \min\{|Y \setminus V| : V$ is a bounded subset of $X\}$.

For interrelations between these invariants see [6], [13].

3. Resolvability

A topological space X is called resolvable [8] if X has two disjoint dense subsets. For a cardinal κ, a topological space X is called κ-resolvable [2] if X contains

κ pairwise disjoint dense subsets. For resolvability of topological spaces and topological groups see the surveys [3], [4], [11].

Given a cardinal κ, we say that a ballean $\mathcal{B} = (X, P, B)$ is *κ-resolvable* if X can be partitioned to κ large subsets. The *resolvability* of $\mathcal{B}$ is the cardinal

$$\operatorname{res}(\mathcal{B}) = \sup\{\kappa : \mathcal{B} \text{ is } \kappa\text{-resolvable}\}.$$

Clearly, $\operatorname{res}(\mathcal{B}) \leq \Delta(\mathcal{B})$, where $\Delta(\mathcal{B}) = \min\{|Y| \colon Y \text{ is a thick subset of } X\}$.

We say that a subset Y of X is *κ-crowded* if there exists $\alpha \in P$ such that $|B(y, \alpha) \cap Y| \geq \kappa$ for every $y \in Y$. A ballean $\mathcal{B}$ is called κ-crowded if its support X is κ-crowded. The *crowdedness* of $\mathcal{B}$ is the cardinal

$$\operatorname{cr}(\mathcal{B}) = \sup\{\kappa : X \text{ is } \kappa\text{-crowded}\}.$$

The following two theorems are from [12].

Theorem 1. *For every ballean $\mathcal{B} = (X, P, B)$, the following statements hold:*

(i) *if $\mathcal{B}$ is κ-crowded, then $\mathcal{B}$ is κ-resolvable;*
(ii) $\operatorname{cr}(\mathcal{B}) \leq \operatorname{res}(\mathcal{B}) \leq \operatorname{cr}(\mathcal{B}) \cdot \operatorname{cf}(\mathcal{B})$;
(iii) *if κ is a finite cardinal and $\mathcal{B}$ is κ-resolvable, then $\mathcal{B}$ is κ-crowded.*

Theorem 2. *Let (X, d) be a metric space, $\mathcal{B} = \mathcal{B}(X, d)$. Then* $\operatorname{res}(\mathcal{B}) = \operatorname{cr}(\mathcal{B})$ *and X can be partitioned to* $\operatorname{cr}(\mathcal{B})$ *large subsets.*

4. Extraresolvability

A topological space X is called extraresolvable [7], [1], if there exists a family $\mathcal{F}$ of dense subsets of X such that $|\mathcal{F}| > \Delta(X)$, where $\Delta(X) = \min\{|U| : U \text{ is a nonempty open subset of } X\}$, and $F \cap F'$ is nowhere dense whenever $F, F' \in \mathcal{F}$ and $F \neq F'$.

Given a cardinal κ, we say that a ballean $\mathcal{B} = (X, P, B)$ is *κ-extraresolvable* if there exists a family $\mathcal{F}$ of large subsets of X such that $|\mathcal{F}| = \kappa$ and $F \cap F'$ is small whenever F, F' are distinct elements of $\mathcal{F}$. The *extraresolvability* of $\mathcal{B}$ is the cardinal

$$\operatorname{exres}(\mathcal{B}) = \sup\{\kappa : \mathcal{B} \text{ is } \kappa\text{-extraresolvable}\}.$$

Clearly, $\operatorname{res}(\mathcal{B}) \leq \operatorname{exres}(\mathcal{B})$. We note also that $\operatorname{res}(\mathcal{B}) = \operatorname{exres}(\mathcal{B}) = |X|$ for every bounded ballean $\mathcal{B}$.

Lemma 1. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $\alpha \in P$, $n \in \mathbb{N}$. Assume that there exists a piecewise large subset Y of X such that the family $\{B(y, \alpha) : y \in Y\}$ is disjoint and $|B(y, \alpha)| \leq n$ for every $y \in Y$. If $F_1, \dots, F_{n+1}$ are subsets of X such that $B(F_i, \alpha) = X$, $i \in \{1, \dots, n+1\}$, then there exist distinct $i, j \in \{1, \dots, n+1\}$ such that $F_i \cap F_j$ is piecewise large.*

Proof: We assume on the contrary that each subset $F_i \cap F_j$ is small and put

$$Z = \bigcup_{i \neq j} (F_i \cap F_j).$$

Then Z is small and $(F_i \setminus Z) \cap (F_j \setminus Z) = \emptyset$, $i \neq j$. Since Y is piecewise large and Z is small, there exists $y \in Y$ such that $B(y, \alpha) \cap Z = \emptyset$. Since $B(F_i, \alpha) = X$, we have $F_i \cap B(y, \alpha) \neq \emptyset$. Since $|B(y, \alpha)| \leq n$, there are distinct $i, j \in \{1, \dots, n+1\}$ such that $F_i \cap F_j \cap B(y, \alpha) \neq \emptyset$. We take any $y' \in F_i \cap F_j \cap B(y, \alpha)$. Then $y' \in Z$ contradicting to the choice of y. $\square$

Theorem 3. *Let $\mathcal{B} = (X, P, B)$ be a ballean. If $\mathrm{cr}(\mathcal{B})$ is finite then*

$$\mathrm{cr}(\mathcal{B}) = \mathrm{res}(\mathcal{B}) = \mathrm{exres}(\mathcal{B}).$$

Proof: Let $\mathrm{cr}(\mathcal{B}) = n$. By Theorem 1(i), $n \leq \mathrm{res}(\mathcal{B})$. Let $F_1, \dots, F_{n+1}$ be distinct large subsets of X. We show that $F_i \cap F_j$ is piecewise large for some distinct $i, j \in \{1, \dots, n+1\}$ so $\mathrm{exres}(\mathcal{B}) \leq n$. For every $\alpha \in P$, we put

$$X_\alpha = \{x \in X : |B(x, \alpha)| \leq n\}.$$

We assume that X_α is small for some $\alpha \in P$. Then $X \setminus B(X_\alpha, \alpha)$ is large. We pick $\beta \in P$ such that

$$\bigcup \{B(x, \beta) : x \in X \setminus B(X_\alpha, \alpha)\} = X.$$

Then we choose $\gamma \in P$ such that $B(B(x, \beta), \beta) \subseteq B(x, \gamma)$ for each $x \in X$. We take an arbitrary $y \in X$ and choose $x \in X \setminus B(X_\alpha, \alpha)$ such that $y \in B(x, \beta)$. Then $B(x, \beta) \subseteq B(y, \gamma)$. It follows that $|B(y, \gamma)| > n$ for every $y \in X$ contradicting $\mathrm{cr}(\mathcal{B}) = n$. Hence, X_α is piecewise large for every $\alpha \in P$.

We choose $\alpha \in P$ such that $B(F_i, \alpha) = X$ for each $i \in \{1, \dots, n+1\}$. Then we take a subset $Y \subseteq X_\alpha$ such that the family $\{B(y, \alpha) : y \in Y\}$ is maximal disjoint. By the above paragraph, X_α is piecewise large so Y is piecewise large. Since $|B(y, \alpha)| \leq n$ for each $y \in Y$, we can apply Lemma 1 to conclude that $F_i \cap F_j$ is not small for some distinct $i, j \in \{1, \dots, n+1\}$. $\square$

Let $\mathcal{B} = (X, P, B)$ be a ballean. Given any $\alpha \in P$ and a cardinal κ, we put

$$X(\alpha, \kappa) = \{x \in X : |B(x, \alpha)| \leq \kappa\}.$$

Lemma 2. *Let $\mathcal{B} = (X, P, B)$ be an unbounded ballean. Assume that, for every $\alpha \in P$, there exists a natural number n such that $X(\alpha, n)$ is piecewise large. Let $\mathcal{F}$ be a family of large subsets of X such that $|\mathcal{F}| > \mathrm{cf}(\mathcal{B})$. Then there exists an infinite subfamily $\mathcal{F}'$ of $\mathcal{F}$ such that $F \cap F'$ is piecewise large for all $F, F' \in \mathcal{F}'$. In particular,* $\mathrm{exres}(\mathcal{B}) \leq \mathrm{cf}(\mathcal{B})$.

Proof: Let P' be a cofinal subset of P such that $|P'| = \mathrm{cf}(\mathcal{B})$. For every $\alpha \in P$, we put

$$\mathcal{F}_\alpha = \{F \in \mathcal{F} : B(F, \alpha) = X\}.$$

Then $\mathcal{F} = \bigcup_{\alpha \in P'} \mathcal{F}_\alpha$. If every family $\mathcal{F}_\alpha$, $\alpha \in P'$, is finite then $|\mathcal{F}| \leq \mathrm{cf}(\mathcal{B}|)$ contradicting the assumption. Hence, there exists $\alpha \in P'$ such that $\mathcal{F}_\alpha$ is infinite. We choose a natural number n such that $X(\alpha, n)$ is piecewise large. Let Y be a subset of $X(\alpha, n)$ such that the family $\{B(y, \alpha) : y \in Y\}$ is maximal disjoint. Since $X(\alpha, n)$ is piecewise large, Y is piecewise large. Let $F_1, \dots, F_n + 1 \in \mathcal{F}_\alpha$. By Lemma 1, there are distinct $i, j \in \{1, \dots, n+1\}$ such that $F_i \cap F_j$ is piecewise large.

We consider a complete graph Γ with the set of vertices $\mathcal{F}_\alpha$. We color an edge $\{F, F'\}$ of Γ in yellow if $F \cap F'$ is piecewise large, otherwise we color this edge in blue. By Ramsey theorem, there exists an infinite subfamily $\mathcal{F}'$ of $\mathcal{F}_\alpha$ such that the complete subgraph Γ' determined by $\mathcal{F}'$ is monochrome. By above paragraph, Γ' must be yellow. Hence, $F \cap F'$ is piecewise large for all $F, F' \in \mathcal{F}'$. $\square$

Theorem 4. *Let (X, d) be an unbounded metric space, $\mathcal{B} = \mathcal{B}(X, d)$. Assume that* $\mathrm{cr}(\mathcal{B}) = \aleph_0$ *and, for every natural number m, there exists a natural number n such that $X(m, n)$ is piecewise large. Then*

$$\mathrm{res}(\mathcal{B}) = \mathrm{exres}(\mathcal{B}) = \aleph_0.$$

Proof: By Lemma 2, $\mathrm{exres}(\mathcal{B}) \leq \aleph_0$. By Theorem 2, $\mathrm{res}(\mathcal{B}) = \aleph_0$. $\square$

Corollary. *Let (X, d) be an unbounded metric space, $\mathcal{B} = \mathcal{B}(X, d)$. Assume that* $\mathrm{cr}(\mathcal{B}) = \aleph_0$ *and, for every natural number m, there exists a natural number n such that $|B(x, m)| \leq n$ for every $x \in X$. Then*

$$\mathrm{res}(\mathcal{B}) = \mathrm{exres}(\mathcal{B}) = \aleph_0.$$

Lemma 3. *Let $\mathcal{B} = (X, P, B)$ be a ballean, Y be a countable large subset of X, κ be an infinite cardinal. Assume that there exists $\alpha \in P$ such that the family $\{B(y, \alpha) : y \in Y\}$ is disjoint and $|B(y, \alpha)| \geq \kappa$ for each $y \in Y$. Then there exists a disjoint family $\mathcal{F}$ of countable subsets of X such that $|\mathcal{F}| = \kappa^{\aleph_0}$ and $F \cap F'$ is finite for all distinct $F, F' \in \mathcal{F}$. In particular,* $\mathrm{exres}(\mathcal{B}) \geq \kappa^{\aleph_0}$.

Proof: Let $Y = \{y_n : n \in \omega\}$. For each $n \in \omega$, we choose some subset $Z_n = \{z(n, \gamma_0, \gamma_1, \dots, \gamma_n) : \gamma_i \in \kappa\}$ of distinct elements from $B(y_n, \alpha)$. Then we fix some symbol $v \notin X$ and construct a homogeneous tree T of local degree κ with the root v and the set of vertices $\{v\} \cup \bigcup_{n\in\omega} Z_n$. At the first step we connect v with all the vertices $\{z(0,\gamma_0) : \gamma_0 \in \kappa\}$. At the second step we connect each vertex $z(0,\gamma_0)$ with all the vertices $\{z(1,\gamma_0,\gamma_1) : \gamma_1 \in \kappa\}$. At the third step we connect each vertex $z(1,\gamma_0,\gamma_1)$ with all the vertices $\{z(2,\gamma_0,\gamma_1,\gamma_2) : \gamma_2 \in \kappa\}$, and so on. After ω steps we get the tree T. Each ray in T starting at v determines a subset F of X consisting of all vertices on this ray except v. We denote by $\mathcal{F}$ the family of all obtained subsets. By the construction of T, we have $F \cap B(y,\alpha) \neq \emptyset$ for all $y \in Y, F \in \mathcal{F}$. Since Y is large, each member of $\mathcal{F}$ is large. Clearly, $|\mathcal{F}| = \kappa^{\aleph_0}$ and $F \cap F'$ is finite for all distinct $F, F' \in \mathcal{F}$. $\square$

Lemma 4. *Let (X,d) be a metric space, $\mathcal{B} = \mathcal{B}(X,d)$. Assume that there exists $n \in \mathbb{N}$ such that $X(n,k)$ is small for each $k \in \mathbb{N}$, and $\bigcup_{k\in\mathbb{N}} X(n,k)$ is large. Then* $\operatorname{exres}(\mathcal{B}) \geq 2^{\aleph_0}$.

Proof: We put $Y = \bigcup_{k\in\mathbb{N}} X(n,k)$ and choose a subset Z of Y such that the family $\{B(z,n) : z \in Z\}$ is maximal disjoint. Clearly, Z is large. For every $k \in \mathbb{N}$, we put

$$Z_k = \{z \in Z : |B(z,n)| = k\}.$$

Since each subset $X(n,k), k \in \mathbb{N}$ is small and Z is large, for every $m \in \omega$, there exists $k \in \mathbb{N}$ such that $Z_k \neq \emptyset$ and $k > m$. Hence, we can choose an increasing sequence $(k_m)_{m\in\omega}$ of natural numbers such that $Z_{k_m} \neq \emptyset$ and $k_m \geq 2^m$.

For each $m \in \omega$, we choose the pairwise disjoint subsets $Z(m,1), Z(m,2), \ldots, Z(m,2^m)$ of X such that for each $z \in Z_{k_m} \cup \cdots \cup Z_{k_{m+1}-1}$, we have

$$|B(z,m) \cap Z(m,i)| = 1,\ Z(m,i) \subseteq B(Z_{k_m} \cup \cdots \cup Z_{k_{m+1}-1}, n),\ i \in \{1,\ldots,2^m\}.$$

Then we construct a binary tree T with the root $Z(0,1)$ and the set of vertices $\{Z(m,i) : m \in \omega, i \in \{1,\ldots,2^m\}\}$. We define the edges of T as follows. At the first step we define the edges $\{Z(0,1), Z(1,1)\}$, $\{Z(0,1), Z(1,12)\}$. At the second step we define the edges

$$\{Z(1,1), Z(2,1)\}, \{Z(1,1), Z(2,2)\}$$
$$\{Z(1,2), Z(2,3)\}, \{Z(1,2), Z(2,4)\},$$

and so on.

Every ray in T starting at the root $Z(0,1)$ determines a subset S of X which is a union of all vertices of T (as the subsets of X) on this ray. By the construction of T, S is large and the intersection of any two distinct subsets S, S' of this form is small. Since there are $2^{\aleph_0}$ distinct rays in T, we conclude $\operatorname{exres}(\mathcal{B}) \geq 2^{\aleph_0}$. $\square$

For a ballean $\mathcal{B} = (X,P,B)$ and a subset Y of X, we put $\mathcal{B}_Y = (Y,P,B_Y)$, where $B_Y(y,\alpha) = B(y,\alpha) \cap Y$.

Theorem 5. *Let (X,d) be an unbounded countable metric space, $\mathcal{B}=\mathcal{B}(X,d)$. Assume that there exists $n\in\mathbb{N}$ such that $X(n,k)$ is small for each $k\in\mathbb{N}$. Then* $\mathrm{exres}(\mathcal{B})=2^{\aleph_0}$.

PROOF: We put $Y=\bigcup_{k\in\mathbb{N}}X(n,k)$, $Z=X\setminus Y$ and construct a family $\mathcal{F}'$ of large subsets of X such that $|\mathcal{F}'|=2^{\aleph_0}$ and $F\cap F'$ is small for all distinct $F,F'\in\mathcal{F}'$. We consider three cases.

Case $\mathcal{B}_Y$ is bounded, $\mathcal{B}_Z$ is unbounded. By Lemma 3, there exists a family $\mathcal{F}$, $|\mathcal{F}|=2^{\aleph_0}$, of large subsets of $\mathcal{B}_Z$ such that $F\cap F'$ is finite for all distinct $F,F'\in\mathcal{F}$. We put $\mathcal{F}'=\{Y\cup F: F\in\mathcal{F}\}$.

Case $\mathcal{B}_Z$ is bounded, $\mathcal{B}_Y$ is unbounded. By Lemma 4, there exists a family $\mathcal{F}$, $|\mathcal{F}|=2^{\aleph_0}$, of large subsets of $\mathcal{B}_Y$ such that $F\cap F'$ is small for all distinct $F,F'\in\mathcal{F}$. We put $\mathcal{F}'=\{Z\cup F: F\in\mathcal{F}\}$.

Case $\mathcal{B}_Y,\mathcal{B}_Z$ are unbounded. Applying Lemmas 3 and 4 to $\mathcal{B}_Z$ and $\mathcal{B}_Y$, we get two corresponding families $\mathcal{F}_1$, $\mathcal{F}_2$, $|\mathcal{F}_1|=|\mathcal{F}_2|=2^{\aleph_0}$ of large subsets of $\mathcal{B}_Z$ and $\mathcal{B}_Y$. Let $\mathcal{F}_1=\{F_\lambda:\lambda\in 2^{\aleph_0}\}$, $\mathcal{F}_2=\{F'_\lambda:\lambda\in 2^{\aleph_0}\}$. We put $\mathcal{F}'=\{F_\lambda\cup F'_\lambda: \lambda\in 2^{\aleph_0}\}$. □

It follows from Theorems 3, 4, 5 that, for a countable metric space (X,d), $\mathrm{exres}\,\mathcal{B}(X,d)$ could be either a natural number, or $\aleph_0$, or $2^{\aleph_0}$. It is easy to construct an example for each case.

Theorem 6. *Let κ be an infinite regular cardinal such that $2^\gamma<\kappa$ for each cardinal $\gamma<\kappa$. Then there exists a ballean $\mathcal{B}=(X,P,B)$ such that $|X|=\kappa$,* $\mathrm{res}(\mathcal{B})=\kappa$ *and* $\mathrm{exres}(\mathcal{B})=2^\kappa$.

PROOF: We denote by $\mathcal{S}$ the family of all subsets of X of cardinality $<\kappa$. Let P be the set of all mappings $f:X\to\mathcal{S}$ such that, for every $x\in X$, we have $x\in f(x)$ and

$$|\{y\in X: x\in f(y)\}|<\kappa.$$

Given any $x\in X$ and $\alpha\in P$, we put $B(x,f)=f(x)$ and consider the ball structure $\mathcal{B}=(X,P,B)$. Since $B^*(x,f)=\{y\in X: x\in f(y)\}$, $\mathcal{B}$ is upper symmetric. Since κ is regular, $|B(B(x,f),g)|<\kappa$ so $\mathcal{B}$ is upper multiplicative. Hence, $\mathcal{B}$ is a ballean.

Let Y be a subset of X. If $|Y|<\kappa$, by regularity of κ, we conclude that Y is bounded so Y is small. We assume that $|Y|=\kappa$ and show that Y is large. We choose a subset Z of Y such that $|Z|=|X\setminus Y|$ and fix some bijection $g:Z\to X\setminus Y$. Then we define $f\in P$ by the rule: $f(x)=\{x,g(x)\}$ for each $x\in Z$, and $f(x)=\{x\}$ for each $x\in X\setminus Z$. Then $B(Y,f)=X$.

To conclude the proof, it suffices to point out a family $\mathcal{F}$ of subsets of X such that $|\mathcal{F}|=2^\kappa$ and $|F\cap F'|<\kappa$ for all distinct $F,F'\in\mathcal{F}$. To this end we use the standard construction. By the assumption $2^\gamma<\kappa,\gamma<\kappa$, we identify X with the set of vertices of the binary tree T of height κ, and denote by $\mathcal{F}$ the family of all rays starting at the root of T. □

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