A generalization of a generic theorem in the theory of cardinal invariants of topological spaces

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Abstract. The main goal of this paper is to establish a technical result, which provides an algorithm to prove several cardinal inequalities and relative versions of cardinal inequalities related to the well-known Arhangel'skii's inequality: If X is a T_2-space, then $|X| \leq 2^{L(X)\chi(X)}$. Moreover, we will show relative versions of three well-known cardinal inequalities.

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1. Introduction

Among the best known theorems in cardinal functions are those which give an upper bound on the cardinality of a space in terms of other cardinal invariants. In [10] Hodel classified the bounds on the cardinality of a space in two categories: *easy* and *difficult* to prove. For instance, the following inequalities are in the difficult category:

(1) (Arhangel'skii) If X is a T_2-space, then $|X| \leq 2^{L(X)\chi(X)}$.
(2) (Hajnal-Juhász) If X is a T_2-space, then $|X| \leq 2^{c(X)\chi(X)}$.
(3) (Charlesworth) If X is a T_1-space, then $|X| \leq psw(X)^{L(X)\psi(X)}$.

The proofs of several inequalities in the difficult category have a common construction that is inspired by Arhangel'skii's original proof of inequality (1). This suggests the general problem of finding a result which captures this common core. Theorems 3.1 and 3.3 in [8] are tailored to prove cardinal function inequalities related to Arhangel'skii's inequality (1), and moreover, Theorem 3.1 (in [8]) captures a common construction to several inequalities which are either a generalization or a variation of Arhangel'skii's inequality (1). Arhangel'skii [2] has a much more general result, his theorem is an *algorithm* for proving relative versions of cardinal inequalities and captures the common construction of several inequalities in the difficult category too.

In the first part of this paper, following the ideas of Arhangel'skii [2] and Hodel [8], we will establish a common generalization of several results in [8] and [9]. In the second part we will use our main result (Theorem 3.5) to show relative versions of three cardinal function inequalities.

2. Notation and terminology

We refer the reader to [9] and [11] for definitions and terminology on cardinal functions not explicitly given. Let L, wL d, χ, ψ and ψ_c denote the following standard cardinal functions: Lindelöf degree, weak Lindelöf number, density, character, pseudocharacter and closed pseudocharacter, respectively.

Let X be a topological space and let Y be a subspace of X. In what follows, $\overline{A}$ is the closure of A in X. The closure of a subset A of Y in Y is denoted by $\mathrm{cl}_Y(A)$. For any set X and any cardinal κ, $[X]^{\leq\kappa}$ denotes the collection of all subsets of X with cardinality $\leq \kappa$; $[X]^{<\kappa}$ is defined analogously.

We shall use the notation and terminology employed in [2]. For convenience of the reader, we repeat some of the definitions contained in that paper.

Definition 2.1. Let X be a topological space and let Y be a subspace of X.

(1) $c(Y, X)$ is the smallest infinite cardinal κ such that the cardinality of every disjoint family of non-empty open subsets of X, each of which intersects Y, does not exceed κ.
(2) $L(Y, X)$ is the smallest infinite cardinal κ such that from each open covering of X one can choose a subfamily $\mathcal{V}$ of cardinality $\leq \kappa$ covering Y.
(3) Let κ be an infinite cardinal, $\pi\chi(y, Y, X) \leq \kappa$ for $y \in Y$, if there is a family $\mathcal{U}_y$ of open subsets of X such that $|\mathcal{U}_y| \leq \kappa$, every neighborhood of y in X contains some $U \in \mathcal{U}_y$, and $U \cap Y \neq \emptyset$ for each $U \in \mathcal{U}_y$. We define $\pi\chi(Y, X)$, as the smallest infinite cardinal κ such that $\pi\chi(y, Y, X) \leq \kappa$ for all $y \in Y$.

Definition 2.2. A space X is *Hausdorff* on the subspace Y (or Y is *Hausdorff* in X) if every two distinct points of Y can be separated by disjoint neighborhoods in X.

Let τ and κ infinite cardinals such that $\kappa < \mathrm{cf}(\tau)$ and we put $\mu = \tau^\kappa$. Let $\mathcal{L}$ be the family of subsets of Y of cardinality not greater than μ, that is, $\mathcal{L} = [Y]^{\leq\mu}$.

A *τ-long increasing sequence in $\mathcal{L}$* is a transfinite sequence $\{F_\alpha : \alpha < \tau\}$ of elements of $\mathcal{L}$ such that $F_\alpha \subseteq F_\beta$ if $\alpha < \beta < \tau$.

A *sensor* is a pair $(\mathcal{A}, \mathcal{F})$, where $\mathcal{A}$ is a family of subset of Y and $\mathcal{F}$ is a collection of families of subsets of X.

We assume that with each sensor $s = (\mathcal{A}, \mathcal{F})$ a subset $\Theta(s)$ of X is associated, called the Θ-*closure* of s.

Definition 2.3. A sensor $s = (\mathcal{A}, \mathcal{F})$ will be called *small* if:

(1) $|\mathcal{A}| \leq \kappa$ and $|A| \leq \kappa$ for every $A \in \mathcal{A}$;
(2) $|\mathcal{F}| \leq \kappa$ and $|\mathcal{C}| \leq \kappa$ for every $\mathcal{C} \in \mathcal{F}$, and
(3) $Y \setminus \Theta(s) \neq \emptyset$.

Let H be a subset of Y and $\mathcal{G}$ a family of subsets of X. A sensor $(\mathcal{A},\mathcal{F})$ is said to be *generated* by the pair $(H,\mathcal{G})$, if $A\subseteq H$ for each $A\in\mathcal{A}$, and $\mathcal{C}\subseteq\mathcal{G}$, for each $\mathcal{C}\in\mathcal{F}$.

Let $\mathcal{Q}$ be the set of all families $\mathcal{G}$ of subsets of X such that $|\mathcal{G}|\leq\mu$. If g is a mapping of $\mathcal{L}$ into $\mathcal{Q}$, $\mathcal{E}\subseteq\mathcal{L}$, then $\mathcal{U}_g(\mathcal{E})=\bigcup\{g(F):F\in\mathcal{E}\}$.

Let g be a mapping of $\mathcal{L}$ into $\mathcal{Q}$, and let $\mathcal{E}$ be a subfamily of $\mathcal{L}$. A sensor s will be called *good for* $\mathcal{E}$, if it is generated by the pair $(\bigcup\mathcal{E},\mathcal{U}_g(\mathcal{E}))$ and $\bigcup\mathcal{E}\subseteq\Theta(s)$.

A *propeller* (with respect to (g,Θ)) *in* $\mathcal{L}$ is a τ-long increasing sequence $\mathcal{E}$ in $\mathcal{L}$ such that no small sensor s is good for $\mathcal{E}$.

3. Main result

Definition 3.1. Let X be a set (non-empty) and let τ, κ be infinite cardinals. An operator $c:\mathcal{P}(X)\to\mathcal{P}(X)$ will be called (τ,κ)-*closure* if:

(1) $A\subseteq c(A)$ for each $A\in\mathcal{P}(X)$;
(2) if $A\subseteq B$ then $c(A)\subseteq c(B)$, for each $A,B\in\mathcal{P}(X)$, and
(3) if $|A|\leq\tau^\kappa$ then $|c(A)|\leq\tau^\kappa$, for each $A\in\mathcal{P}(X)$.

Remark 3.2. It is clear that if $\tau=\kappa^+$ then $\tau^\kappa=2^\kappa$; hence, in this case, condition (3) in the previous definition establishes:
If $|A|\leq 2^\kappa$, then $|c(A)|\leq 2^\kappa$.

Example 3.3. Let X be a Hausdorff space and let τ and κ be infinite cardinals with $\kappa<\mathrm{cf}(\tau)$. It is not difficult to show that if $\psi_c(X)t(X)\leq\kappa$ then the operator $c(A)=\overline{A}$ is a (τ,κ)-closure operator (see 4.3 in [10]).

Example 3.4. In [4], Bella and Cammaroto introduced the notion of θ-closure of a subset A of X, denoted $[A]_\theta$[1], and proved that: If X is a Urysohn space and $A\subseteq X$, then $|[A]_\theta|\leq|A|^{\chi(X)}$. It follows that if τ and κ are infinite cardinals with $\kappa<\mathrm{cf}(\tau)$ and $\chi(X)\leq\kappa$, then the operator $c(A)=[A]_\theta$ is a (τ,κ)-closure operator.

The proof of Theorem 3.5 below follows the same pattern as the proof of Theorem 1 in [2], therefore some of the details are omitted.

Theorem 3.5. *Let X and Y be sets with $Y\subseteq X$, and let τ and κ be infinite cardinals such that $\kappa<\mathrm{cf}(\tau)$. If $c:\mathcal{P}(X)\to\mathcal{P}(X)$ is a (τ,κ)-closure operator, then for every function $g:\mathcal{L}\to\mathcal{Q}$, there exists a family $\{E_\alpha:\alpha\in\tau\}\subseteq\mathcal{L}$ such that:*

(1) *for each $0<\alpha<\tau$, $\bigcup\{c(E_\beta)\cap Y:\beta<\alpha\}\subseteq E_\alpha$, and*
(2) *$\mathcal{E}=\{c(E_\alpha)\cap Y:\alpha\in\tau\}$ is a propeller in $\mathcal{L}$.*

[1]Let X be a topological space and let $A\subseteq X$. A point $x\in X$ is a *θ-limit point* of A if $\overline{V}\cap A\neq\emptyset$ for every open neighborhood V of x. The *θ-closure* of A is the set $[A]_\theta=\{x:x\in A$ or x is a θ-limit point of $A\}$.

Proof: Let $g : \mathcal{L} \to \mathcal{Q}$ be a function. We construct an increasing family $\{E_\alpha : \alpha < \tau\}$ in $\mathcal{L}$ by transfinite recursion such that:

(a) $|E_\alpha| \leq \mu$, $0 \leq \alpha < \tau$;

(b) $\mathcal{U}_\alpha = \bigcup\{g(c(E_\beta \cap Y)) : \beta < \alpha\}$, $0 < \alpha < \tau$,

(c) if s is a small sensor generated by $(\bigcup\{c(E_\beta) \cap Y : \beta < \alpha\}, \mathcal{U}_\alpha)$, then $E_\alpha \cap (Y \setminus \Theta(s)) \neq \emptyset$.

Fix $0 < \alpha < \tau$ and assume that $E_\beta \in \mathcal{L}$ and $\mathcal{U}_\beta$ are already defined such that (1) and (2) hold for each $\beta \in \alpha$. Put $H_\alpha = \bigcup\{c(E_\beta) \cap Y : \beta < \alpha\}$ and $\mathcal{U}_\alpha = \bigcup\{g(c(E_\beta) \cap Y) : \beta < \alpha\}$. It is clear that $|H_\alpha| \leq \mu$ and $|\mathcal{U}_\alpha| \leq \mu$.

For each small sensor s generated by $(H_\alpha, \mathcal{U}_\alpha)$, choose one point $m(s) \in Y \setminus \Theta(s)$ and let A_α be the set of points chosen in this way. Let $E_\alpha = H_\alpha \cup A_\alpha$, and note that $E_\alpha \subseteq Y$ and $|E_\alpha| \leq \mu$. This completes the construction.

The proof is complete if $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha < \tau\}$ is a propeller in $\mathcal{L}$. To see this, let $P = \bigcup \mathcal{E}$ and suppose there is a small sensor $s = (\mathcal{A}, \mathcal{F})$ generated by the pair $(P, \mathcal{U}_g(\mathcal{E}))$ such that $P \subseteq \Theta(s)$. Since $\kappa < \operatorname{cf}(\tau)$, there exists $\alpha_0 < \tau$ such that $A \subseteq H_{\alpha_0}$ for each $A \in \mathcal{A}$, and $\mathcal{B} \subseteq \mathcal{U}_{\alpha_0}$, for each $\mathcal{B} \in \mathcal{F}$. Hence the sensor $s = (\mathcal{A}, \mathcal{F})$ is generated by the pair $(H_{\alpha_0}, \mathcal{U}_{\alpha_0})$. Therefore $m(s) \in c(E_{\alpha_0}) \subseteq P \subseteq \Theta(s)$, which contradicts the choice of $m(s)$. □

4. Some corollaries of the main theorem

Unless stated otherwise, the operator c is the identity operator. In [9] Hodel establishes three theorems of combinatorial set theory. As will be seen later, two of these results are also a consequence of our theorem.

Among the consequences of the following result, there are the de Groot's inequality: For $X \in T_2$, $|X| \leq 2^{hL(X)}$, and the Ginsburg-Woods's inequality: For $X \in T_1$, $|X| \leq 2^{e(X)\Delta(X)}$.

Corollary 4.1 ([9]). *Let X be a set and let κ be an infinite cardinal. For each $x \in X$ let $\mathcal{B}_x = \{V(\gamma, x) : \gamma < \kappa\}$ be a collection of subsets of X such that for each $\gamma < \kappa$, $x \in V(\gamma, x)$. Assume that*

(1) if $x \neq y$, there exists $\gamma < \kappa$ such that $y \notin V(\gamma, x)$,

(2) for each $\gamma < \kappa$ and each $A \subseteq X$, there exists $B \subseteq A$ with $|B| \leq \kappa$ such that $A \subseteq \bigcup\{V(\gamma, x) : x \in B\}$.

Then $|X| \leq 2^\kappa$.

Proof: Let $\tau = \kappa^+$, $\mu = 2^\kappa$ and $\mathcal{L} = [X]^{\leq\mu}$. For every sensor $s = (\emptyset, \{\mathcal{C}_\gamma : \gamma < \kappa\})$ we put $\Theta(s) = \bigcup_{\gamma<\kappa} \bigcup \mathcal{C}_\gamma$, and $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$ for every $F \in \mathcal{L}$. By Theorem 3.5, there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that $\mathcal{E} = \{E_\alpha : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$ and for every $0 < \alpha < \kappa^+$, $\bigcup\{E_\beta : \beta < \alpha\} \subseteq E_\alpha$. Let $P = \bigcup \mathcal{E}$. Clearly $|P| \leq 2^\kappa$.

The proof is complete if $X \subseteq P$. Suppose not and fix $p \in X \setminus P$. Let $A_\gamma = \{x \in P : p \notin V(\gamma, x)\}$ for each $\gamma \in \kappa$. By (2), there is $B_\gamma \in [A_\gamma]^{\leq\kappa}$ such that

$A_\gamma \subseteq \bigcup\{V(\gamma,x) : x \in B_\gamma\}$, for each $\gamma \in \kappa$. Denote $\mathcal{C}_\gamma = \{V(\gamma,x) : x \in B_\gamma\}$ for each $\gamma \in \kappa$, and let $s = (\emptyset, \{\mathcal{C}_\gamma : \gamma \in \kappa\})$. It is clear that p is not in the Θ-closure of the sensor s, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. $\square$

Let κ be an infinite cardinal, and let X be a set. Suppose that for each $x \in X$, $\mathcal{V}_x$ is a family of subsets of X which contains x. For $L \subseteq X$, let $L^* = \{x : x \in X, V \cap L \neq \emptyset$, for all $V \in \mathcal{V}_x\}$. This operator was defined by Hodel in [9]. We shall use the following result from [9].

Theorem 4.2. *Let κ be an infinite cardinal, let X be a set. If for each $x \in X$, $\mathcal{V}_x = \{V_\gamma(x) : \gamma < \kappa\}$ is a family of subsets of X which contains x such that if $x \neq y$, there exists $\gamma \in \kappa$ such that $V_\gamma(x) \cap V_\gamma(y) = \emptyset$. Then*

(1) $|L^*| \leq |L|^\kappa$;
(2) *if $L = \bigcup_{\alpha<\kappa^+} E^*_\alpha$, where $\{E_\alpha : 0 \leq \alpha < \kappa^+\}$ is a sequence of subsets of X with $\bigcup_{\beta<\alpha} E^*_\beta \subseteq E_\alpha$ for all $\alpha < \kappa^+$, then $L^* = L$.*

Corollary 4.3 ([9]). *Let κ be an infinite cardinal and let X be a set. For each $x \in X$ let $\mathcal{B}_x = \{V(\gamma,x) : \gamma < \kappa\}$ be a collection of subsets of X such that for each $\gamma < \kappa$, $x \in V(\gamma,x)$. Assume that*

(1) *if $x \neq y$, there exists $\gamma < \kappa$ such that $V(\gamma,x) \cap V(\gamma,y) = \emptyset$,*
(2) *if $\mathcal{V}$ is a subcollection of $\{V(\gamma,x) : \gamma < \kappa, x \in X\}$ which cover X, then there is a subcollection of $\mathcal{V}$ of cardinality $\leq \kappa$ which cover X.*

Then $|X| \leq 2^\kappa$.

Proof: Let $\tau = \kappa^+$, $\mu = 2^\kappa$ and $\mathcal{L} = [X]^{\leq\mu}$. For every sensor $s = (\emptyset, \{\mathcal{F}\})$ we put $\Theta(s) = \bigcup \mathcal{F}$, and $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$ for every $F \in \mathcal{L}$. By Theorem 4.2, the operator $c(F) = F^*$ is a (κ^+,κ)-closure operator, hence by Theorem 3.5 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$, such that for every $0 < \alpha < \kappa^+$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$. Let $P = \bigcup \mathcal{E}$ and note that $|P| \leq 2^\kappa$. By Theorem 4.2, note that $P = P^*$.

The proof is complete if $X \subseteq P$. Suppose not and fix $p \in X \setminus P$. For each $x \in P$, choose $\gamma_x < \kappa$ such that $p \notin V(\gamma_x, x)$, and for each $x \in X \setminus P$, choose $\gamma_x < \kappa$ such that $V(\gamma_x,x) \cap P = \emptyset$. The collection $\{V(\gamma_x,x) : x \in X\}$ covers X, so by (2) there exists $B \in [X]^{\leq\kappa}$ such that $\{V(\gamma_x,x) : x \in B\}$ covers X. Let $B_0 = B \cap P$ and let $\mathcal{F} = \{V_{\gamma_x} : x \in B_0\}$. It is clear that p is not in the Θ-closure of the sensor $s = (\emptyset, \{\mathcal{F}\})$, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. $\square$

Hodel's inequalities [9] follow immediately: $|X| \leq 2^{L(X)H\psi(X)}$ and $|X| \leq 2^{c(X)H\psi(X)}$ for every Hausdorff space X.

A separating cover $\mathcal{V}$ for X is a cover for X having the property that for any $x, y \in X$ with $x \neq y$, there is a $V \in \mathcal{V}$ such that $x \in V$ and $y \notin V$.

Corollary 4.4 ([9]). *Let κ, γ be infinite cardinals, let X be a set, let $\mathcal{V}$ be a separating cover of X. Assume that*

(1) $\operatorname{ord}(x, \mathcal{V}) \leq \gamma$, *for all $x \in X$,*
(2) *if $\mathcal{V}_0$ is a subcollection of $\mathcal{V}$ which covers X, then some subcollection of $\mathcal{V}_0$ of cardinality at most κ covers X.*

Then $|\mathcal{V}| \leq \gamma^\kappa$.

Proof: Let $\tau = \gamma^\kappa = \mu$ and note that $\kappa < \operatorname{cf}(\tau)$. For each $x \in X$ denote $\mathcal{B}_x = \{V \in \mathcal{V} : x \in V\}$. For every sensor $s = (\emptyset, \{\mathcal{F}\})$ we put $\Theta(s) = \bigcup \mathcal{F}$, and for every $F \in \mathcal{L} = [X]^{\leq \mu}$ we put $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$. By Theorem 3.5 there is a propeller $\mathcal{E} = \{E_\alpha : \alpha \in \tau\}$ in $\mathcal{L}$. Let $P = \bigcup \mathcal{E}$.

Let us show that $V \cap P \neq \emptyset$ for every $V \in \mathcal{V}$. Let $V_0 \in \mathcal{V}$ and suppose that $V_0 \cap P = \emptyset$. Let $p \in V_0$ and let $\mathcal{W} = \{V : V \in \mathcal{V}, p \notin V\}$. Since $\mathcal{V}$ is separating, $\mathcal{V}_0 = \mathcal{W} \cup \{V_0\}$ covers X. By (2), there exists $\mathcal{H} \in [\mathcal{V}_0]^{\leq \kappa}$ such that $\mathcal{H}$ covers X. Let $\mathcal{H}_0 = \{V : V \in \mathcal{H}, V \cap P \neq \emptyset\}$. It is clear that p is not in the Θ-closure of the sensor $s = (\emptyset, \{\mathcal{H}_0\})$, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. □

The following result is inspired by Theorem 3.1 in [8].

Corollary 4.5. *Let X be a set, κ, τ be infinite cardinals such that $\kappa < \operatorname{cf}(\tau)$ and let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a (τ, κ)-closure operator on X, and for each $x \in X$ let $\mathcal{B}_x = \{V(\gamma, x) : \gamma < \kappa\}$ be a collection of subsets of X such that for each $\gamma < \kappa$, $x \in V(\gamma, x)$. Assume the following:*

(1) *if $x \in c(H)$, then there exists $A \subseteq H$ with $|A| \leq \kappa$ such that $x \in c(A)$;*
(2) *if $H \neq \emptyset$, $c(H) = H$, and $p \notin H$, then there exists $A \subseteq H$ with $|A| \leq \kappa$ and a function $f : A \to \kappa$ such that $H \subseteq \bigcup_{x \in A} V(f(x), x)$ and $q \notin \bigcup_{x \in A} V(f(x), x)$.*

Then $|X| \leq \tau^\kappa$.

Proof: Let $\mu = \tau^\kappa$ and $\mathcal{L} = [X]^{\leq \mu}$. We put $\Theta((\emptyset, \{\mathcal{F}\})) = \bigcup \mathcal{F}$, and $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$, for $F \in \mathcal{L}$. By Theorem 3.5 there exists $\{E_\alpha : \alpha \in \tau\} \subseteq \mathcal{L}$ such that for every $0 < \alpha < \tau$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \tau\}$ is a propeller in $\mathcal{L}$. Let $P = \bigcup \mathcal{E}$ and note that $|P| \leq \mu$. By (1) and since c is an (τ, κ)-closure operator, we have $c(P) = P$.

The proof is complete if $X \subseteq P$. Suppose not and fix $p \in X \setminus P$. By (2) there exists $A \in [P]^{\leq \kappa}$ and a function $f : A \to \kappa$ such that $P \subseteq \bigcup_{x \in A} V(f(x), x)$ and $p \notin \bigcup_{x \in A} V(f(x), x)$, it is clear that p is not in the Θ-closure of the sensor $s = (\emptyset, \{\{V(f(x), x) : x \in A\}\})$, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. □

Observe that Corollary 4.5 captures the common construction for the following generalizations of Arhangel'skii's inequality (see inequality (1) in our introduction).

(1) (Hodel [9]) If X is a Hausdorff space then $|X| \leq 2^{H\psi(X)L(X)}$.
(2) (Hodel [4]) If X is a Hausdorff space then $|X| \leq 2^{t(X)\psi_c(X)aL(X)}$.
(3) (Stavrova [12]) If X is a Urysohn space then $|X| \leq 2^{U\psi(X)aL_c(X)}$.
(4) (Alas [1]) If X is a Urysohn space then $|X| \leq 2^{\chi(X)wL_c(X)}$.

However, it is possible to prove the previous inequalities using Theorem 3.5. For example we will sketch the proof of (4). Recall that $wL_c(X)$ is the smallest infinite cardinal κ such that for every closed subset F of X and every collection $\mathcal{U}$ of open sets in X that covers F, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $F \subseteq \overline{\bigcup \mathcal{V}}$.

Proof of (4): Let $\chi(X)wL_c(X) = \kappa$, let $\tau = \kappa^+$, $\mu = 2^\kappa$ and $\mathcal{L} = [X]^{\leq\mu}$. For every $x \in X$ let $\mathcal{B}_x$ be a local base of x in X with $|\mathcal{B}_x| \leq \kappa$. We put $\Theta((\emptyset, \{\mathcal{F}\})) = \overline{\bigcup \mathcal{F}}$, and $g(F) = \bigcup\{\mathcal{B}_x : x \in F\}$, for each $F \in \mathcal{L}$. Since the operator $c(A) = [A]_\theta$, where $[A]_\theta$ is the θ-closure of A (see Example 3.4), is a (κ^+, κ)-closure operator, by Theorem 3.5 there exists a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that for each $0 < \alpha < \kappa^+$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ and $\mathcal{E} = \{c(E_\alpha) : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$. Let $P = \bigcup\{c(E_\alpha) : \alpha < \kappa^+\}$. Then $|P| \leq 2^\kappa$. Note that $c(P) = P$.

The proof is complete if $X \subseteq P$. Suppose not and fix $p \in X \setminus P$. There is $V \in \mathcal{V}_p$ such that $\overline{V} \cap P = \emptyset$, because P is θ-closed. For each $x \in P$ fix $V_x \in \mathcal{V}_x$ such that $V_x \subseteq X \setminus \overline{V}$. Then $\{V_x : x \in P\}$ is a collection of open subsets in X which covers P, hence since P is closed in X, there exists $B \in [P]^{\leq\kappa}$ such that $P \subseteq \overline{\bigcup_{x\in B} V_x}$. It is clear that p is not in the Θ-closure of the sensor $s = (\emptyset, \{\{V_x : x \in B\}\})$, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. □

In recent years there has been considerable interest in relative versions of cardinal functions inequalities; see, for example, [2], [6], and [12]. The following result due to Hodel gives a unified approach to several such results related to Arhangel'skii's inequality (see inequality (1) in our introduction). As we will see soon this result is a consequence of our main theorem.

Corollary 4.6 ([8]). *Let X be a set, let $Y \subseteq X$, κ be a cardinal infinite and for each $x \in X$ let $\mathcal{B}_x = \{V(\gamma, x) : \gamma < \kappa\}$ be a collection of subsets of X such that $x \in V(\gamma, x)$ for all $\gamma < \kappa$. Assume the following holds:*

(1) *given $\alpha, \beta < \kappa$, there exists $\gamma < \kappa$ such that $V(\gamma, x) \subseteq V(\beta, x) \cap V(\alpha, x)$;*
(2) *if $x \neq y$, then there exist $\alpha, \beta < \kappa$ such that $V(\alpha, x) \cap V(\beta, y) = \emptyset$;*
(3) *if $f : X \to \kappa$, then there exists $A \subseteq X$ with $|A| \leq \kappa$ such that $Y \subseteq \bigcup_{x\in A} V(f(x), x)$.*

Then $|Y| \leq 2^\kappa$.

Proof: Let $\tau = \kappa^+$, $\mu = 2^\kappa$ and $\mathcal{L} = [Y]^{\leq\mu}$. We put $\Theta((\emptyset, \{\mathcal{F}\})) = \bigcup \mathcal{F}$ and $g(F) = \bigcup\{\mathcal{B}_x : x \in c(F)\}$, for every $F \in \mathcal{L}$. It follows from Theorem 4.2 that $c(C) = C^*$ is a (κ^+, κ)-closure operator. By Theorem 3.5 there is a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ such that for every $0 < \alpha < \kappa^+$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$

and $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$. Let $P = \bigcup \mathcal{E}$. Then $|P| \leq 2^\kappa$, therefore $|c(P)| \leq 2^\kappa$. Let us show that $Y \subseteq c(P)$.

Suppose not and fix $p \in Y \setminus c(P)$. For each $x \in c(P)$, choose γ_x such that $p \notin V(\gamma_x, x)$, and for each $x \in X \setminus c(P)$, choose γ_x such that $V(\gamma_x, x) \cap (\bigcup\{E_\alpha : \alpha < \kappa^+\}) = \emptyset$. Let $f : X \to \kappa$ be the function $f(x) = \gamma_x$. Then by (3) there exists $A \in [X]^{\leq\kappa}$ such that $Y \subseteq \bigcup\{V(f(x), x) : x \in A\}$. Let $B = A \cap c(P)$. It is clear that p is not in the Θ-closure of the sensor $s = (\emptyset, \{\{V(f(x), x) : x \in B\}\})$, while $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction.

□

5. Relative version for some cardinal function inequalities

In this section we will use our main result (Theorem 3.5) to establish relative versions for three well known cardinal inequalities. The first one is a relative version of a generalization of the Arhangel'skii's inequality (see [3]).

Corollary 5.1. *Let X be a T_1-space such that:*

(1) $t(X) \leq \kappa$;
(2) *for every* $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$;
(3) $\psi(X) \leq 2^\kappa$.

If $Y \subseteq X$ with $L(Y, X) \leq \kappa$, then $|Y| \leq 2^\kappa$.

Proof: Let $\tau = \kappa^+$, $\mu = 2^\kappa$ and let $\mathcal{L} = [Y]^{\leq\mu}$. For every $x \in X$ fix $\mathcal{B}_x$ a local pseudobase of x in X such that $|\mathcal{B}_x| \leq 2^\kappa$. We put $\Theta((\emptyset, \{\mathcal{F}\})) = \overline{\bigcup \mathcal{F}}$, and $g(F) = \bigcup\{\mathcal{B}_x : x \in c(F)\}$, for $F \in \mathcal{L}$. Consider the operator $c(A) = \mathrm{cl}_X(A)$ and note that by (2) c is a (κ^+, κ)-closure operator, hence by Theorem 3.5, there is a family $\{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$, such that for every $0 < \alpha < \kappa^+$, $\bigcup\{c(E_\beta) : \beta < \alpha\} \subseteq E_\alpha$ and $\mathcal{E} = \{c(E_\alpha) \cap Y : \alpha \in \kappa^+\}$ is a propeller in $\mathcal{L}$. Let $H = \bigcup \mathcal{E}$. It is clear that $|H| \leq 2^\kappa$, hence $|c(H)| \leq 2^\kappa$. Moreover it is not difficult to prove that if $x \in c(H)$, then there exists $\alpha < \tau$ such that $x \in c(E_\alpha)$.

The proof is complete if $Y \subseteq c(H)$. Suppose not and fix $p \in Y \setminus c(H)$. For each $x \in c(H)$, choose $U_x \in \mathcal{B}_x$ such that $p \notin U_x$. It is clear that $\mathcal{V} = \{U_x : x \in c(H)\} \cup \{X \setminus c(H)\}$ is an open cover of X, since $L(Y, X) \leq \kappa$, there exists $A \in [c(H)]^{\leq\kappa}$ such that $Y \subseteq \bigcup\{U_x : x \in A\} \cup (X \setminus c(H))$. Clearly $H \subseteq \bigcup\{U_x : x \in A\}$ and $p \notin \bigcup\{U_x : x \in A\}$. Let $s = (\emptyset, \{\{U_x : x \in A\}\})$. We see that s is a small sensor good for $\mathcal{E}$. Which is a contradiction. □

Some consequences of Corollary 5.1 are:

(a) [3] Let X be a T_1-space such that
  (1) $t(X) \leq \kappa$, $L(X) \leq \kappa$,
  (2) for every $A \in [X]^{\leq 2^\kappa}$, $|\overline{A}| \leq 2^\kappa$,
  (3) $\psi(X) \leq 2^\kappa$.

Then $|X| \leq 2^{\kappa}$.

(b) [2] If X is a T_2-space and $Y \subseteq X$ then $|Y| \leq 2^{L(Y,X)\chi(X)}$.

(c) [6] If X is a T_2-space then $|X \setminus X_0| \leq 2^{L(X,X_0)\psi_c(X)t(X)}$.

For presenting our next result we need the following definition.

Definition 5.2. Let X be a *Hausdorff* space on the subspace Y. The *Hausdorff pseudocharacter* of X on Y, denoted $H\psi(Y,X)$, is the smallest infinite cardinal κ such that for every $y \in Y$ there is a collection $\mathcal{U}_y$ of open neighborhoods of y in X with $|\mathcal{U}_y| \leq \kappa$ such that if $x, y \in Y$ and $x \neq y$, there exists $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ with $U \cap V = \emptyset$.

Lemma 5.3 ([2]). *If $c(Y,X) \leq \kappa$, then for every family $\mathcal{U}$ of open subsets of X, there exists $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $(\bigcup\mathcal{U}) \cap Y \subseteq \overline{\bigcup\mathcal{V}}$.*

Corollary 5.4. *If X is Hausdorff onto Y, then $|Y| \leq 2^{c(Y,X)\cdot H\psi(Y,X)}$.*

Proof: Let $\kappa = c(Y,X)H\psi(Y,X)$, $\tau = \kappa^+$, $\mu = 2^{\kappa}$ and $\mathcal{L} = [Y]^{\leq\mu}$. Let $\mathcal{U}_y$ be a family of open neighborhoods of y in X, for every $y \in Y$, with $|\mathcal{U}_y| \leq \kappa$ such that if $x, y \in Y$, $x \neq y$, then there are $U \in \mathcal{U}_x$ and $V \in \mathcal{U}_y$ such that $U \cap V = \emptyset$. We put $\Theta((\emptyset,\mathcal{F})) = \bigcup\{\overline{\bigcup\mathcal{C}} : \mathcal{C} \in \mathcal{F}\}$ and let $g(E) = \bigcup\{\mathcal{U}_x : x \in E\}$ for every $E \in \mathcal{L}$. By Theorem 3.5, there is a family $\mathcal{E} = \{E_\alpha : \alpha \in \kappa^+\} \subseteq \mathcal{L}$ which is a propeller in $\mathcal{L}$. Let $P = \bigcup\mathcal{E}$. It is clear that $|P| \leq 2^{\kappa}$.

The proof is complete if $Y \subseteq P$. Suppose not and fix $p \in Y \setminus P$. For every $V \in \mathcal{U}_p$, let $\mathcal{F}_V = \{W \in \mathcal{U}_{g(\mathcal{E})} : W \cap V = \emptyset\}$ and note that $P \subseteq \bigcup\{\bigcup\mathcal{F}_V : V \in \mathcal{U}_p\}$. Since $c(Y,X) \leq \kappa$, by Lemma 5.3, there is $\mathcal{G}_V \in [\mathcal{F}_V]^{\leq\kappa}$ such that $\bigcup\mathcal{F}_V \cap Y \subseteq \overline{\bigcup\mathcal{G}_V}$, for each $V \in \mathcal{U}_p$. Let $\mathcal{F} = \{\mathcal{G}_V : V \in \mathcal{U}_p\}$. Hence $P \subseteq \Theta((\emptyset,\mathcal{F}))$ and $p \notin \Theta((\emptyset,\mathcal{F}))$. We see that $s = (\emptyset,\mathcal{F})$ is a small sensor good for $\mathcal{E}$. A contradiction. □

Some consequences of Corollary 5.4 are:

(i) ([10]) If X is a T_2-space then $|X| \leq 2^{c(X)\cdot H\psi(X)}$.

(ii) ([9]) If X is a T_2-space then $|X| \leq 2^{c(X)\chi(X)}$.

In [12] D. Stavrova established the following result: *If X is a Hausdorff space then* $|X \setminus X_0| \leq 2^{L(X,X_0)\cdot H\psi(X)}$. At the moment the authors do not know the answer to the next question.

Question 5.5. Suppose that X is a Hausdorff onto Y. Is it true that $|Y| \leq 2^{L(Y,X)\cdot H\psi(Y,X)}$?

Corollary 5.6. *Let X be a T_2-space. Then $|Y| \leq \pi\chi(Y,X)^{c(Y,X)\psi_c(X)}$.*

Proof: Let $\gamma = \pi\chi(Y,X)$, $\kappa = c(Y,X)\psi_c(X) = \kappa$, and let $\tau = \gamma^{\kappa}$. Note that $\kappa < \mathrm{cf}(\tau)$. Let $\mathcal{L} = [Y]^{\leq\tau}$. For every $y \in Y$, we fix $\mathcal{B}_y$ as in Definition 2.1(3), with $|\mathcal{B}_x| \leq \gamma$. We put $\Theta((\emptyset,\mathcal{F})) = \bigcup\{\overline{\bigcup\mathcal{C}} : \mathcal{C} \in \mathcal{F}\}$ and $g(F) = \bigcup\{\mathcal{B}_y : y \in F\}$. By

Theorem 3.5, there is $\mathcal{E} = \{E_\alpha : \alpha < \tau\}$ which is a propeller in $\mathcal{L}$. Let $P = \bigcup \mathcal{E}$. It is clear that $|P| \leq \tau$.

The proof is complete if $Y \subseteq P$. Suppose not and fix $p \in Y \setminus P$. Let $\mathcal{U}_y = \{V_\alpha : \alpha < \kappa\}$ be a family of open neighborhoods of p in X such that $\bigcap\{\overline{V}_\alpha : \alpha < \kappa\} = \{p\}$. For each $\alpha < \kappa$, let $W_\alpha = X \setminus \overline{V}_\alpha$ and $\mathcal{F}_\alpha = \{V : V \in \mathcal{B}_z$, with $z \in P \cap W_\alpha$ and $V \subseteq W_\alpha\}$. Now, since $c(Y, X) \leq \kappa$, by Lemma 5.3, for each $\alpha \in \kappa$ there is $\mathcal{G}_\alpha \in [\mathcal{F}_\alpha]^{\leq \kappa}$ such that $\bigcup \mathcal{F}_\alpha \cap Y \subseteq \overline{\bigcup \mathcal{G}_\alpha}$. Let $\mathcal{F} = \{\mathcal{G}_\alpha : \alpha \in \kappa\}$ and let $s = (\emptyset, \mathcal{F})$. It is not difficult to prove that $p \notin \Theta(s)$ and $P \subseteq \Theta(s)$. We see that s is a small sensor good for $\mathcal{E}$. A contradiction. $\square$

Some consequences of Corollary 5.6 are:

(a) ([13]) If X is a T_2 space then $|X| \leq \pi\chi(X)^{c(X)\psi_c(X)}$.
(b) ([10]) If X is a T_3 space then $|X| \leq 2^{c(X)\pi\chi(X)\psi(X)}$.

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