On semiregular digraphs of the congruence $x^k \equiv y \pmod n$

Lawrence Somer, Michal Křížek

Abstract. We assign to each pair of positive integers n and $k \geq 2$ a digraph $G(n,k)$ whose set of vertices is $H = \{0, 1, \dots, n-1\}$ and for which there is a directed edge from $a \in H$ to $b \in H$ if $a^k \equiv b \pmod n$. The digraph $G(n,k)$ is semiregular if there exists a positive integer d such that each vertex of the digraph has indegree d or 0. Generalizing earlier results of the authors for the case in which $k = 2$, we characterize all semiregular digraphs $G(n,k)$ when $k \geq 2$ is arbitrary.

Keywords: Chinese remainder theorem, congruence, group theory, dynamical system, regular and semiregular digraphs

Classification: 11A07, 11A15, 05C20, 20K01

1. Introduction

This paper extends results given in the works [2] and [6] which provide an interesting connection between number theory, graph theory and group theory. In the papers [4] and [5] we investigated properties of the iteration digraph representing a dynamical system occurring in number theory.

For $n \geq 1$ let

$$H = \{0, 1, \dots, n-1\}$$

and let f be a map of H into itself. The *iteration digraph* of f is a directed graph whose vertices are elements of H and such that there exists exactly one directed edge from x to $f(x)$ for all $x \in H$. For a fixed integer $k \geq 2$ and for each $x \in H$ let $f(x)$ be the remainder of x^k modulo n, i.e.,

$$f(x) \in H \quad \text{and} \quad x^k \equiv f(x) \pmod n. \tag{1.1}$$

From here on, whenever we refer to the iteration digraph of f, we assume that the mapping f is as given in (1.1), see Figure 1. Each pair of natural numbers n and $k \geq 2$ has a specific iteration digraph corresponding to it.

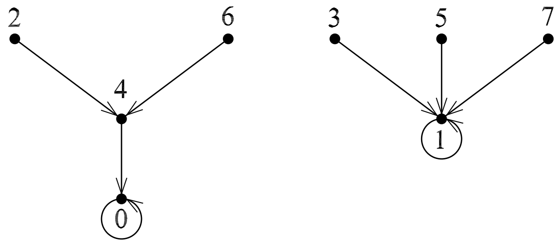

Figure 1. The iteration digraph corresponding to $n = 8$ and $k = 2$.

We identify the vertex a of H with its residue modulo n. For brevity we will make statements such as $\gcd(a, n) = 1$, treating the vertex a as a number. Moreover, when we refer, for instance, to the vertex a^k, we identify it with the remainder $f(a) \in H$ given by (1.1). In this paper we will often identify the vertex n with the vertex 0 for convenience.

For particular values of n and k, we denote the iteration digraph of f by $G(n, k)$. It is obvious that $G(n, k)$ with n vertices also has exactly n directed edges.

Let $\omega(n)$ denote the number of distinct primes dividing $n \geq 2$ and let the prime power factorization of n be given by

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}, \tag{1.2}$$

where $p_1 < p_2 < \cdots < p_r$ are primes and $\alpha_i > 0$, i.e., $r = \omega(n)$. For $n = 1$ we set $\omega(1) = 0$.

A *component* of the iteration digraph is a subdigraph which is a maximal connected subgraph of the associated nondirected graph.

The *indegree* of a vertex $a \in H$ of $G(n, k)$, denoted by $\operatorname{indeg}_n(a)$, is the number of directed edges coming into a, and the *outdegree* of a is the number of directed edges leaving the vertex a. We frequently will simply write $\operatorname{indeg}(a)$ when it is understood that a is a vertex in $G(n, k)$. By the definition of f, the outdegree of each vertex of $G(n, k)$ is equal to 1.

It is clear that each component has a unique cycle, since each vertex of the component has outdegree 1 and the component has only a finite number of vertices. Cycles of length 1 are called fixed points.

Remark 1.1. Recall that a graph is *regular* if all its vertices have the same degree. We say that the digraph $G(n, k)$ is *regular* if each of its vertices have the same indegree. The digraph $G(n, k)$ is said to be *semiregular* if there exists a positive integer d such that each vertex of $G(n, k)$ either has indegree d or 0. Note that the set of semiregular digraphs $G(n, k)$ includes the subset of regular digraphs.

Clearly, $G(n,k)$ is regular only if $G(n,k)$ has no vertices of indegree 0. Since each component of $G(n,k)$ has a unique cycle, we see that $G(n,k)$ is regular if and only if each component of $G(n,k)$ is a cycle and each vertex of $G(n,k)$ has indegree 1. Since any vertex of indegree 0 is a noncycle vertex and there is a path from any noncycle vertex to the cycle in its component, we see that $G(n,k)$ is regular if and only if each vertex of positive indegree has indegree equal to 1. Noting that each vertex of $G(n,k)$ has outdegree 1, we observe that $G(n,k)$ is regular as a digraph if and only if $G(n,k)$ is regular as an undirected graph. Figure 2 provides an example of a regular digraph, while Figure 3 gives an example of a semiregular digraph which is not regular.

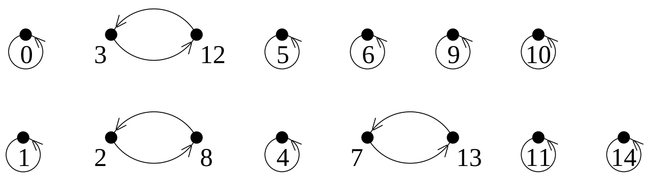

Figure 2. The iteration digraph corresponding to $n = 15$ and $k = 3$.

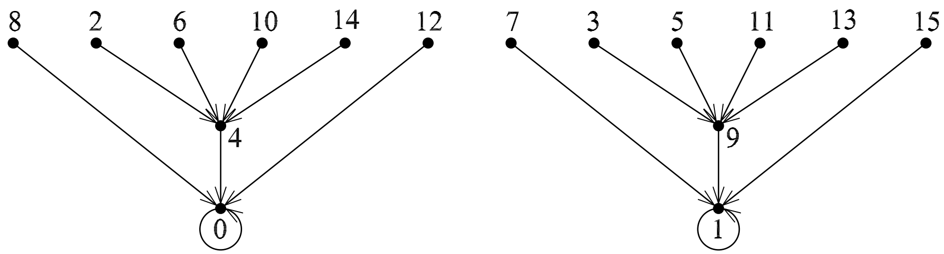

Figure 3. The iteration digraph corresponding to $n = 16$ and $k = 2$.

In [4] all semiregular digraphs $G(n,k)$ were characterized when $k = 2$. In this paper, given a fixed integer $k \geq 2$, we find all semiregular and regular digraphs $G(n,k)$. Further, we specify two particular subdigraphs of $G(n,k)$. Let $G_1(n,k)$ be the induced subdigraph of $G(n,k)$ on the set of vertices which are coprime to n and $G_2(n,k)$ be the induced subdigraph on the remaining vertices not coprime with n. We observe that $G_1(n,k)$ and $G_2(n,k)$ are disjoint and that $G(n,k) = G_1(n,k) \cup G_2(n,k)$, that is, no edge goes between $G_1(n,k)$ and $G_2(n,k)$. For example, the second component of Figure 4 is $G_1(12,2)$ whereas the remaining three components make up $G_2(12,2)$. It is clear that 0 is always a fixed point of $G_2(n,k)$. If $n > 1$ then 1 and $n-1$ are always vertices of $G_1(n,k)$.

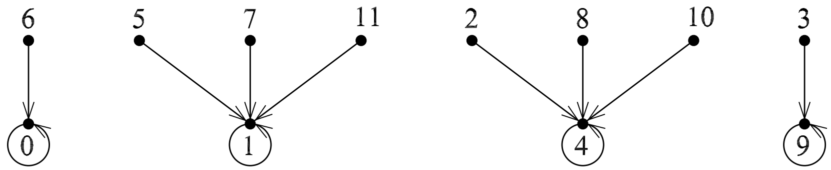

Figure 4. The iteration digraph corresponding to $n = 12$ and $k = 2$.

In Theorems 4.1 and 4.3, we will show that $G_1(n, k)$ is always semiregular. In Theorem 4.4 we will also determine when $G_2(n, k)$ is semiregular. Observe that in Figure 1, the subdigraph $G_2(8, 2)$ is semiregular but $G(8, 2)$ is not semiregular. Note further that in Figures 4 and 5, $G_2(n, k)$ is not semiregular, but each of its components is semiregular. We will characterize later those digraphs for which each of the components of $G_2(n, k)$ is semiregular.

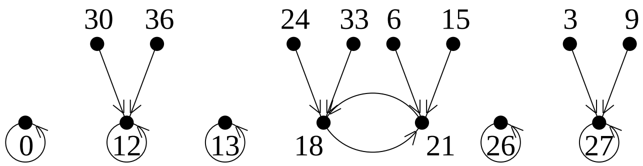

Figure 5. The iteration subdigraph $G_2(39, 3)$.

Let $N(n, k, a)$ denote the number of incongruent solutions of the congruence

$$x^k \equiv a \pmod{n}.$$

Then obviously

$$N(n, k, a) = \mathrm{indeg}_n(a). \tag{1.3}$$

It follows from (1.3) and Theorem 2.20 in [3] that if n has the factorization given in (1.2), then

$$\mathrm{indeg}_n(a) = N(n, k, a) = \prod_{i=1}^{r} N(p_i^{\alpha_i}, k, a) = \prod_{i=1}^{r} \mathrm{indeg}_{q_i}(a), \tag{1.4}$$

where $q_i = p_i^{\alpha_i}$.

2. Properties of the Carmichael lambda-function

Before proceeding further, we need to review some properties of the Carmichael lambda-function $\lambda(n)$, which modifies the Euler totient function $\phi(n)$.

Definition 2.1. Let n be a positive integer. Then the *Carmichael lambda-function* $\lambda(n)$ is defined as follows:

$$\begin{aligned} \lambda(1) &= 1 = \phi(1), \\ \lambda(2) &= 1 = \phi(2), \\ \lambda(4) &= 2 = \phi(4), \\ \lambda(2^k) &= 2^{k-2} = \tfrac{1}{2}\phi(2^k) \quad \text{for} \quad k \geq 3, \\ \lambda(p^k) &= (p-1)p^{k-1} = \phi(p^k) \quad \text{for any odd prime} \quad p \quad \text{and} \quad k \geq 1, \\ \lambda(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}) &= \operatorname{lcm}[\lambda(p_1^{k_1}),\ \lambda(p_2^{k_2}), \ldots, \lambda(p_r^{k_r})], \end{aligned}$$

where $p_1, p_2, \ldots, p_r$ are distinct primes and $k_i \geq 1$ for all $i \in \{1, \ldots, r\}$.

It immediately follows from Definition 2.1 that

$$\lambda(n) \mid \phi(n)$$

for all n and that $\lambda(n) = \phi(n)$ if and only if $n \in \{1, 2, 4, q^k, 2q^k\}$, where q is an odd prime and $k \geq 1$.

The following theorem generalizes the well-known Euler's theorem which says (see [1, p. 20]) that $a^{\phi(n)} \equiv 1 \pmod{n}$ if and only if $\gcd(a, n) = 1$. It shows that $\lambda(n)$ is the smallest possible order modulo n.

Theorem 2.2 (Carmichael). *Let $a, n \in \mathbb{N}$. Then*

$$a^{\lambda(n)} \equiv 1 \pmod{n}$$

if and only if $\gcd(a, n) = 1$. *Moreover, there exists an integer g such that*

$$\operatorname{ord}_n g = \lambda(n),$$

where $\operatorname{ord}_n g$ *denotes the multiplicative order of g modulo n.*

For the proof see [1, p. 21].

3. Results on the indegree

We will need the following theorems concerning the indegrees of vertices in $G_1(n, k)$ and $G_2(n, k)$ in order to prove our main results on semiregularity.

Theorem 3.1. *Let n have the factorization given in* (1.2) *and let a be a vertex of positive indegree in $G_1(n, k)$. Then*

$$\operatorname{indeg}(a) = \varepsilon \prod_{i=1}^{r} \gcd(\lambda(p_i^{\alpha_i}), k),$$

where $\varepsilon = 2$ if $2 \mid k$ and $8 \mid n$, and $\varepsilon = 1$ otherwise.

This is proved in [6, pp. 231–232].

Theorem 3.2. *Let n have the factorization given in* (1.2), *let a be a vertex of positive indegree in $G_2(n,k)$, and let*

$$a = Q \prod_{i=1}^{r} p_i^{\beta_i},$$

where $\gcd(Q,n) = 1$, *$\beta_i \geq 0$ for $1 \leq i \leq r$, and $\beta_i \geq 1$ for at least one value of i. Then for $i = 1,2,\ldots,r$ either $\beta_i \geq \alpha_i$, or both $\beta_i < \alpha_i$ and $\beta_i = kt_i$ for some nonnegative integer t_i. Moreover,*

$$\operatorname{indeg}(a) = \prod_{i=1}^{r} A_i B_i,$$

where

$$A_i = \begin{cases} p_i^{\alpha_i - \lceil \alpha_i/k \rceil} & \text{if } \beta_i \geq \alpha_i, \\ p_i^{(k-1)t_i} & \text{if } 0 \leq \beta_i < \alpha_i, \end{cases}$$

and

$$B_i = \varepsilon_i \gcd(\lambda(p_i^{\alpha_i - \min(\alpha_i,\beta_i)}), k),$$

where $\varepsilon_i = 2$ if $p_i = 2$, $2 \mid k$ and $\alpha_i - \beta_i \geq 3$, and $\varepsilon_i = 1$ otherwise.

Proof: By the Chinese remainder theorem, $\operatorname{indeg}(a) > 0$ if and only if for $i = 1,2,\ldots,r$ there exists an integer b_i, a nonnegative integer t_i and an integer c_i coprime to p_i such that

$$b_i^k \equiv (p_i^{t_i} c_i)^k \equiv p_i^{kt_i} c_i^k \equiv a \equiv p_i^{\beta_i}(a/p_i^{\beta_i}) \pmod{p_i^{\alpha_i}}. \tag{3.1}$$

If $\beta_i \geq \alpha_i$, then $b_i \equiv 0 \pmod{p_i^{\alpha_i}}$ satisfies congruence (3.1). Now suppose that $\beta_i < \alpha_i$. Then congruence (3.1) is satisfied only if $kt_i = \beta_i$.

By (1.4), the remainder of our assertion will follow if we can show that

$$N(p_i^{\alpha_i}, k, a) = A_i B_i$$

for $i = 1,2,\ldots,r$. First suppose that $\beta_i \geq \alpha_i$. Then

$$N(p_i^{\alpha_i}, k, a) = N(p_i^{\alpha_i}, k, 0) = p_i^{\alpha_i - \lceil \alpha_i/k \rceil} = A_i = A_i B_i.$$

Now suppose that $\beta_i < \alpha_i$. Let b_i be a residue such that $b_i^k \equiv a \pmod{p_i^{\alpha_i}}$. By (3.1),

$$b_i \equiv p_i^{t_i} c_i \pmod{p_i^{\alpha_i}},$$

where t_i is a nonnegative integer and c_i is an integer such that $\gcd(c_i, p_i) = 1$ and

$$c_i^k \equiv a/p_i^{\beta_i} \pmod{p_i^{\alpha_i - \beta_i}}.$$

Moreover, since $(\mathbb{Z}/p_i^{\alpha_i})^*$ is a group under multiplication, there exists an integer d_i such that

$$d_i^k \equiv b_i^k \equiv a \equiv p_i^{kt_i} c_i^k \pmod{p_i^{\alpha_i}} \tag{3.2}$$

if and only if

$$d_i \equiv p_i^{t_i} c_i e_i \pmod{p_i^{\alpha_i}} \tag{3.3}$$

for some integer e_i such that

$$e_i^k \equiv 1 \pmod{p_i^{\alpha_i - kt_i}}. \tag{3.4}$$

Furthermore,

$$p_i^{t_i} c_i e_i \equiv p_i^{t_i} c_i e_i' \pmod{p_i^{\alpha_i}} \tag{3.5}$$

if and only if

$$e_i \equiv e_i' \pmod{p_i^{\alpha_i - t_i}}. \tag{3.6}$$

We note that if $e_i^k \equiv 1 \pmod{p_i^{\alpha_i - kt_i}}$ and $e_i \equiv e_i' \pmod{p_i^{\alpha_i - t_i}}$, then $(e_i')^k \equiv 1 \pmod{p_i^{\alpha_i - kt_i}}$. It now follows from (3.2)–(3.6) that

$$N(p_i^{\alpha_i}, k, a) = p_i^{(\alpha_i - t_i) - (\alpha_i - kt_i)} C_i = p_i^{(k-1)t_i} C_i = A_i C_i,$$

where C_i denotes the number of solutions to the congruence

$$x^k \equiv 1 \pmod{p_i^{\alpha_i - kt_i}}.$$

By Theorem 3.1, $C_i = B_i$, and we obtain the required result. $\square$

An even more complicated version of Theorem 3.2 is proved in [6, pp. 236–237].

4. On regularity and semiregularity of digraphs

We now present our main theorems.

Theorem 4.1. *Let $n \geq 1$ and $k \geq 2$ be integers. Then*

(i) *$G_1(n,k)$ is regular if and only if* $\gcd(\lambda(n),k) = 1$*;*

(ii) *$G_2(n,k)$ is regular if and only if either n is square-free and* $\gcd(\lambda(n),k) = 1$*, or $n = p$, where p is a prime;*

(iii) *$G(n,k)$ is regular if and only if n is square-free and* $\gcd(\lambda(n),k) = 1$.

Proof: We suppose that n has the factorization given in (1.2).

(i) By Remark 1.1 and Theorem 3.1, it suffices to show that

$$\prod_{i=1}^{r} \gcd(\lambda(p_i^{\alpha_i}),k) = 1. \tag{4.1}$$

However, (4.1) is satisfied if and only if $\gcd(\lambda(n),k) = 1$.

(ii) First suppose that n is not square-free and $q^2 \mid n$ for some prime q. Then by Theorem 3.2, $q \mid \operatorname{indeg}(0)$, and consequently $\operatorname{indeg}(0) > 1$. Thus $G_2(n,k)$ is not regular in this case.

Now suppose that n is square-free and $n = p$. Then $G_2(n,k)$ consists solely of the fixed point p and $G_2(n,k)$ is regular.

We next suppose that $n = p_1p_2\cdots p_r$, where $r \geq 2$. By Theorem 3.2 and Remark 1.1 the subdigraph $G_2(n,k)$ is regular if and only if for each vertex $a \in G_2(n,k)$,

$$\prod_{i=1}^{r} A_iB_i = 1, \tag{4.2}$$

where A_i and B_i are defined as in Theorem 3.2. Equation (4.2) holds if and only if $A_i = B_i = 1$ for $i = 1,2,\ldots,r$. If $a \equiv 0 \pmod{p_i}$, then $A_i = B_i = 1$. We further note that if p_i is any prime such that $1 \leq i \leq r$, then there exists a vertex $a \in G_2(n,k)$ such that $a \not\equiv 0 \pmod{p_i}$. In this case, $\alpha_i = 1$, $\beta_i = 0$, $t_i = 0$, $A_i = 1$, and

$$B_i = \gcd(\lambda(p_i),k).$$

Hence, $G_2(n,k)$ is regular if and only if

$$\gcd(\lambda(p_i),k) = 1 \tag{4.3}$$

for $i = 1,2\ldots,r$. However, (4.3) holds if and only if

$$\gcd(\lambda(n),k) = 1.$$

The result now follows.

(iii) This is a consequence of (i) and (ii). □

Remark 4.2. Part (i) of Theorem 4.1 was also proved in [6, p. 232].

Theorem 4.3. *Let $k \geq 2$ be an integer and let $n \geq 2$ have the canonical factorization given in* (1.2). *If* $\gcd(\lambda(n), k) > 1$, *then $G_1(n,k)$ is semiregular but not regular. If $a \in G_1(n,k)$ and* $\operatorname{indeg}(a) > 0$, *then*

$$\operatorname{indeg}(a) = \varepsilon \prod_{i=1}^{r} \gcd(\lambda(p_i^{\alpha_i}), k), \tag{4.4}$$

where $\varepsilon = 2$ if $2 \mid k$ and $8 \mid n$, and $\varepsilon = 1$ otherwise.

Proof: By Theorem 4.1, $G_1(n,k)$ is not regular if $\gcd(\lambda(n), k) > 1$. By Theorem 3.1, $G_1(n,k)$ is semiregular and (4.4) holds. □

Theorems 4.1 and 4.3 completely specify when the digraph $G_1(n,k)$ is either regular or semiregular. Theorem 4.4 will determine exactly when the digraphs $G_2(n,k)$ and $G(n,k)$ are semiregular. We can then use Theorem 4.1 to separate out the cases in which $G_2(n,k)$ and $G(n,k)$ are also regular. We use the notation $\prod_{i=1}^{0} a_i$ to denote that the corresponding product is empty and set equal to 1 by convention.

Theorem 4.4. *Let $k \geq 2$ be a fixed integer with the factorization*

$$k = Q \prod_{i=1}^{\ell} p_i^{\alpha_i}, \tag{4.5}$$

where each p_i is a prime such that $\gcd(p_i - 1, k) = 1$ and in addition, $\ell \geq 1$, $\alpha_i \geq 1$, $\gcd(Q, p_1 p_2 \cdots p_\ell) = 1$, and $\gcd(q-1, k) > 1$ for each prime q dividing Q. Let $n \geq 2$ have the prime power factorization

$$n = \prod_{i=1}^{\ell} p_i^{\beta_i} \prod_{i=1}^{m} q_i^{\gamma_i} \prod_{i=1}^{s} h_i^{\delta_i},$$

where $\beta_i \geq 0$, $m \geq 0$, $s \geq 0$, $\gamma_i \geq 1$, $\delta_i \geq 1$, $\gcd(q_i(q_i - 1), k) = 1$ for $i = 1, 2, \ldots, r$, and $\gcd(h_i - 1, k) > 1$ for $i = 1, 2, \ldots, s$.

(i) $G_2(n,k)$ is semiregular if and only if one of the following conditions holds:

- (a) *$n = \prod_{i=1}^{\ell} p_i^{\beta_i} \prod_{i=1}^{m} q_i$ for $0 \leq \beta_i \leq \alpha_i + 1$ and $\omega(n) \geq 2$,*
- (b) *$n = p_i^{\beta_i}$ for some $i \in \{1, 2, \ldots, \ell\}$, where $1 \leq \beta_i \leq k + \alpha_i + 1$ and p_i is odd,*
- (c) *$n = q_1^{\gamma_1}$ for $1 \leq \gamma_1 \leq k + 1$,*
- (d) *$n = h_i^{\delta_i}$ for $1 \leq \delta_i \leq k$,*
- (e) *$n = 2^{\beta_1}$ for $\beta_1 \in \{1, 2, 3, 4, 6\}$ when $k = 2$,*
- (f) *$n = 2^{\beta_1}$ for $1 \leq \beta_1 \leq 9$ when $k = 2^2$,*
- (g) *$n = 2^{\beta_1}$ for $1 \leq \beta_1 \leq k + \alpha_1 + 2$ when $p_1 = 2$ and $k \geq 6$,*

(ii) *$G(n,k)$ is semiregular if and only if one of the following conditions holds:*

(a) $n = \prod_{i=1}^{\ell} p_i^{\beta_i} \prod_{i=1}^{m} q_i$ *for* $0 \le \beta_i \le \alpha_i + 1$ *and* $m \ge 0$ *when* p_i *is odd for each* $i \in \{1, 2, \dots, \ell\}$,

(b) $n = 2^{\beta_1}$ *for* $\beta_1 \in \{1, 2, 4\}$ *when* $k = 2$,

(c) $n = 2^{\beta_1}$ *for* $1 \le \beta_1 \le 5$ *when* $k = 2^2$,

(d) $n = 2^{\beta_1}$ *for* $1 \le \beta_1 \le \alpha_1 + 2$ *when* $p_1 = 2$ *and* $k \ge 6$.

Remark 4.5. Note that in the hypotheses of Theorem 4.4, there exists at least one prime p_1 dividing k such that $\gcd(p_1 - 1, k) = 1$. Simply choose p_1 to be the least prime dividing k. We further observe that if $2 \mid k$, there does not exist a prime q_i such that $\gcd(q_i(q_i - 1), k) = 1$. We finally notice that in Theorem 4.4, we allow both the possibility that h_i does divide k and also the possibility that h_i does not divide k, where $1 \le i \le s$.

Proof of Theorem 4.4: (i) The necessity and sufficiency of condition (e) for the case in which $k = 2$ were shown in [4]. For the remainder of the proof of (i), we assume that $k \ne 2$ and treat only conditions (a)–(d) and (f)–(g).

Let q be a prime. If $1 \le \beta \le k$, then clearly $G_2(q^\beta, k)$ is semiregular, since the only vertex in $G_2(q^\beta, k)$ having positive indegree is the vertex 0. From here on, when we consider digraphs $G_2(n,k)$ we assume that either $\omega(n) \ge 2$ or n is of the form q^β for $\beta \ge k + 1$.

We note for future reference that if $n = q^\beta$, where q is a fixed prime and the positive integer β varies, then the function

$$\mathrm{indeg}(q^\beta) = N(q^\beta, k, q^\beta) = q^{\beta - \lceil \beta/k \rceil}$$

is nondecreasing as β increases. We will also frequently make use of the facts that both $N(n,k,0) > 0$ and $N(n,k,1) > 0$ for all n and k, and in addition $N(p^\alpha, k, p^{jk}) > 0$ when p is a prime and $\alpha > jk$.

First assume that $\omega(n) \ge 2$. We show that $G_2(n,k)$ is semiregular if and only if $G(q^{\nu_q(n)}, k)$ is semiregular for every prime q dividing n, where $\nu_q(n)$ is the exponent β such that $q^\beta \mid n$ but $q^{\beta+1} \nmid n$, that is $q^{\nu_q(n)} \| n$. For each prime q dividing n, let $q(n) = q^{\nu_q(n)}$. Since, by (1.4),

$$\mathrm{indeg}_n(a) = \prod_{q \mid n} \mathrm{indeg}_{q(n)}(a)$$

for each vertex $a \in G_2(n,k)$, we see that $G_2(n,k)$ is semiregular if $G(q^{\nu_q(n)}, k)$ is semiregular for each prime q dividing n.

Now suppose that $q \mid n$ and $G(q^{\nu_q(n)}, k)$ is not semiregular. Then there exist nonnegative integers a and b, each having positive indegree in $G(q^{\nu_q(n)}, k)$, such that $\mathrm{indeg}_{q(n)}(a) \ne \mathrm{indeg}_{q(n)}(b)$. Let $n = q^{\nu_q(n)} M$, where $M > 1$ and $q \nmid M$. By the Chinese remainder theorem, we can find vertices a_1 and a_2 in $G_2(n,k)$ such that $a_1 \equiv a \pmod{q^{\nu_q(n)}}$, $a_1 \equiv 0 \pmod{M}$, and $a_2 \equiv b \pmod{q^{\nu_q(n)}}$, $a_2 \equiv 0 \pmod{M}$. Then

$$\operatorname{indeg}_n(a_1) = \operatorname{indeg}_{q(n)}(a)\operatorname{indeg}_M(0) \neq \operatorname{indeg}_n(a_2) = \operatorname{indeg}_{q(n)}(b)\operatorname{indeg}_M(0),$$

and $G_2(n,k)$ is not semiregular.

Note that the above arguments also show that when $\omega(n) \geq 2$, $G_2(n,k)$ is semiregular if and only if $G(n,k)$ is semiregular.

We now prove that no prime h_1 divides n when $\omega(n) \geq 2$ and $G_2(n,k)$ is semiregular. Suppose that $h_1^{\delta_1} \| n$, where $\delta_1 \geq 1$. Note that by definition, $h_1 \neq 2$. Then by Theorems 3.1 and 3.2,

$$N(h_1^{\delta_1}, k, 0) = h_1^{\delta_1 - \lceil \delta_1/k \rceil}$$

and

$$N(h_1^{\delta_1}, k, 1) = \gcd(\lambda(h_1^{\delta_1}), k) = \gcd(h_1^{\delta_1 - 1}(h_1 - 1), k).$$

Since $\gcd(h_1 - 1, k) > 1$, there exists a prime p such that $p \mid \gcd(h_1 - 1, k)$. Hence, $p \mid N(h_1^{\delta_1}, k, 1)$, but $p \nmid N(h_1^{\delta_1}, k, 0)$. Thus, $G(h_1^{\delta_1}, k)$ is not semiregular, which implies that $G_2(n,k)$ is not semiregular. Consequently, if $G_2(n,k)$ is semiregular and $\omega(n) \geq 2$, then $\gcd(q-1, k) = 1$ for each prime q dividing n. Thus, $p_i \neq 2$ for $1 \leq i \leq \ell$ if $G_2(n,k)$ is semiregular and $\omega(n) \geq 2$.

Next suppose that $G_2(n,k)$ is semiregular and $q_i^2 \mid n$ for some $i \in \{1, 2, \ldots, m\}$. Then

$$N(q_i^{\gamma_i}, k, 1) = \gcd(\lambda(q_i^{\gamma_i}), k) = \gcd(q_i^{\gamma_i - 1}(q_i - 1), k) = 1,$$

whereas

$$q_i \mid N(q_i^{\gamma_i}, k, q_i^{\gamma_i}) = q_i^{\gamma_i - \lceil \gamma_i/k \rceil}.$$

Hence, $G(q_i^{\gamma_i}, k)$ is not semiregular, which again implies that $G_2(n,k)$ is not semiregular.

We observe by Theorem 4.1 that $G(q_i, k)$ is regular and thus semiregular for $1 \leq i \leq m$. We now show that $G(p_i^{\beta_i}, k)$ is semiregular for $1 \leq i \leq \ell$ when p_i is odd and $1 \leq \beta_i \leq \alpha_i + 1$. This will establish the sufficiency of condition (a) when $\omega(n) \geq 2$. Clearly, if $\beta_i \leq \alpha_i + 1$, then $\beta_i < p_i^{\alpha_i} \leq k$ for p_i an odd prime. Then $\operatorname{indeg}(a) > 0$ for $a \in G_2(p_i^{\beta_i}, k)$ if and only if $a = 0$. If $c \in G_1(p_i^{\beta_i}, k)$ and $\operatorname{indeg}(c) > 0$, then by Theorems 3.1 and 3.2,

$$\operatorname{indeg}(c) = \gcd(\lambda(p_i^{\beta_i}), k) = \gcd(p_i^{\beta_i - 1}(p_i - 1), k) = p_i^{\beta_i - 1} = \operatorname{indeg}(0),$$

and $G(p_i^{\beta_i}, k)$ is semiregular.

At this point, we assume that $p_i^{\alpha_i+2}|n$. By our earlier observation, $p_i \neq 2$. Noting that $\gcd(p_i-1,k)=1$ and $\beta_i \geq \alpha_i+2$, we see by (4.5) that

$$
\begin{aligned}
N(p_i^{\beta_i},k,1) &= \gcd(\lambda(p_i^{\beta_i}),k) = \gcd(p_i^{\beta_i-1}(p_i-1),k)\\
&= p_i^{\alpha_i} < N(p_i^{\alpha_i+2},k,p_i^{\alpha_i+2}) = p_i^{\alpha_i+2-\lceil(\alpha_i+2)/k\rceil}\\
&= p_i^{\alpha_i+1} \leq N(p_i^{\beta_i},k,p_i^{\beta_i}).
\end{aligned}
\tag{4.6}
$$

In the last equality in (4.6), we made use of the fact that if $p_i^{\alpha_i}\|k$, where $\alpha_i \geq 1$, then $\alpha_i+2 \leq p_i^{\alpha_i} \leq k$ when p_i is an odd prime. Thus, $G(p_i^{\beta_i},k)$ is not semiregular in this case. We have now established the necessity of condition (a) when $\omega(n) \geq 2$.

We assume from here on that $\omega(n)=1$. First suppose that $n=h_1^{\delta_1}$, where $\delta_1 \geq k+1$. Let p be a prime such that $p \mid \gcd(h_1-1,k)$. Then by Theorem 3.2,

$$p \nmid N(h_1^{\delta_1},k,h_1^{\delta_1}) = h_1^{\delta_1-\lceil\delta_1/k\rceil},$$

whereas

$$p \mid N(h_1^{\delta_1},k,h_1^k) = h_1^{k-1}\gcd(\lambda(h_1^{\delta_1-k}),k) = h_1^{k-1}\gcd(h_1^{\delta_1-k-1}(h_1-1),k).$$

Thus, $G_2(h_1^{\delta_1},k)$ is not semiregular in this case. We have now established condition (d).

Now assume that $n=q_1^{\gamma_1}$, where $\gamma_1 \geq k+2$. Then by Theorems 3.1 and 3.2,

$$
\begin{aligned}
N(q_1^{\gamma_1},k,q_1^k) &= q_1^{k-1}\gcd(\lambda(q_1^{\gamma_1-k}),k) = q_1^{k-1}\gcd(q_1^{\gamma_1-k-e_1}(q_1-1),k)\\
&= q_1^{k-1} < N(q_1^{k+2},k,q_1^{k+2}) = q_1^{k+2-\lceil(k+2)/k\rceil}\\
&= q_1^k \leq N(q_1^{\gamma_1},k,q_1^{\gamma_1}),
\end{aligned}
\tag{4.7}
$$

where $e_1=2$ if $q_1=2$ and $\gamma_1-k\geq 3$ and $e_1=1$ otherwise. The last equality in (4.7) follows from the fact that $k+2 \leq 2k$, since $k \geq 2$. Thus $G_2(q_1^{\gamma_1},k)$ is not semiregular in this case.

We note that $G_2(q_1^{\gamma_1},k)$ is semiregular when $\gamma_1 = k+1$. Observe that $\operatorname{indeg}(a)>0$ for $a \in G_2(q_1^{k+1},k)$ only if $q_1^k\|a$ or $a \equiv 0 \pmod{q_1^{k+1}}$. Then

$$
\begin{aligned}
N(q_1^{k+1},k,q^k) &= q_1^{k-1}\gcd(\lambda(q_1^{k+1-k}),k) = q_1^{k-1}\gcd(q_1-1,k) = q_1^{k-1}\\
&= N(q_1^{k+1},k,q_1^{k+1}) = q_1^{k+1-\lceil(k+1)/k\rceil} = q_1^{k-1}.
\end{aligned}
$$

Hence, $G_2(q_1^{k+1},k)$ is semiregular by Theorem 3.2. We have now established condition (c).

Further, assume that $n = p_i^{\beta_i}$, where $i \in \{1, 2, \ldots, \ell\}$ and either p_i is odd or $p_i = 2$ and $k \geq 6$. First suppose that $\beta_i \geq k + \alpha_i + 2 + \mu(p_i)$, where $\mu(p_i) = 0$ if p_i is odd and $\mu(p_i) = 1$ if $p_i = 2$. Note that $\alpha_i + 2 + \mu(p_i) \leq p_i^{\alpha_i} \leq k$ if p_i is odd and $\alpha_i + 2 + \mu(p_i) < k$ if $p_i = 2$. Then by Theorem 3.2,

$$\begin{aligned} N(p_i^{\beta_i}, k, p^k) &= p_i^{k-1}\varepsilon_i \gcd(\lambda(p_i^{\beta_i - k}), k) = p_i^{k-1}\varepsilon_i \gcd(p_i^{\beta_i - k - \varepsilon_i}(p_i - 1), k) \\ &= p_i^{k+\alpha_i-1} < N(p_i^{k+\alpha_i+2+\mu(p_i)}, k, p_i^{k+\alpha_i+2+\mu(p_i)}) \\ &= p_i^{k+\alpha_i+2+\mu(p_i) - \lceil (k+\alpha_i+2+\mu(p_i))/k \rceil} \\ &= p_i^{k+\alpha_i+\mu(p_i)} \leq N(p_i^{\beta_i}, k, p_i^{\beta_i}), \end{aligned} \tag{4.8}$$

where $\varepsilon_i = 2$ if $p = 2$ and $\beta_i - k \geq 3$, and $\varepsilon_i = 1$ otherwise. Therefore, $G_2(p_i^{\beta_i}, k)$ is not semiregular in this case.

Now suppose that $k + 1 \leq \beta_i \leq k + \alpha_i + 1 + \mu(p_i)$. Since $k < \beta_i \leq k + \alpha_i + 1 + \mu(p_i) < 2k$ for $i \in \{1, 2, \ldots, \ell\}$, we see that $\operatorname{indeg}(a) > 0$ for $a \in G_2(p_i^{\beta_i}, k)$ only if $p_i^k \| a$ or $a \equiv 0 \pmod{p_i^{\beta_i}}$. Similarly to (4.8) we get

$$\begin{aligned} N(p_i^{\beta_i}, k, p^k) &= p_i^{k-1}\varepsilon_i \gcd(p_i^{\beta_i - k - \varepsilon_i}(p_i - 1), k) \\ &= p_i^{k-1}\varepsilon_i p_i^{\beta_i - k - \varepsilon_i} = p_i^{\beta_i - 2} = N(p_i^{\beta_i}, k, p_i^{\beta_i}) \\ &= p_i^{\beta - \lceil \beta_i/k \rceil} = p_i^{\beta_i - 2}, \end{aligned}$$

where ε_i is defined as before. Thus $G_2(p_i^{\beta_i}, k)$ is semiregular in this instance. Conditions (b) and (g) are now established.

It only remains to show that when $k = 4$, then $G_2(n, 4)$ is semiregular if and only if condition (f) holds. First suppose that $k = 4$ and $n = 2_1^{\beta}$, where $\beta_1 \geq 10$. Then, by Theorem 3.2,

$$\begin{aligned} N(2^{\beta_i}, 4, 2^4) &= 2^3 \cdot 2 \cdot \gcd(\lambda(2^{\beta_1 - 4}), 2^2) = 2^3 \cdot 2 \cdot 2^2 = 2^6 < N(2^{10}, 4, 2^{10}) \\ &= 2^{10 - \lceil 10/4 \rceil} = 2^7 \leq N(2^{\beta_1}, 4, 2^{\beta_1}) \end{aligned}$$

and $G_2(2^{\beta_1}, 4)$ is not semiregular.

Finally, we show that $G_2(2^{\beta_1}, 4)$ is semiregular when $5 \leq \beta_1 \leq 9$. First assume that $5 \leq \beta_1 \leq 8$. Then $\operatorname{indeg}(a) > 0$ for $a \in G_2(2^{\beta_1}, 4)$ only if $2^4 \| a$ or $a \equiv 0 \pmod{2^{\beta_1}}$. Observe that

$$\begin{aligned} N(2^{\beta_1}, 4, 2^4) &= 2^3 \cdot \varepsilon_1 \cdot \gcd(\lambda(2^{\beta_1 - 4}), 2^2) = 2^3 \cdot \varepsilon_1 \cdot 2^{\beta_1 - 4 - \varepsilon_1} = 2^{\beta_1 - 2} \\ &= 2^{\beta_1 - \lceil \beta_1/4 \rceil} = N(2^{\beta_1}, 4, 2^{\beta_1}) \end{aligned}$$

where $\varepsilon_1 = 2$ if $\beta_1 - 4 \geq 3$ and $\varepsilon_1 = 1$ otherwise. Therefore, $G_2(2^{\beta_1}, 4)$ is semiregular in this case.

Now assume that $\beta_1 = 9$. Then $\mathrm{indeg}(a) > 0$ for $a \in G_2(2^9, 4)$ only if $2^4 \| a$, or $2^8 \| a$, or $a \equiv 0 \pmod{2^9}$. Then by Theorem 3.2,

$$\begin{aligned} N(2^9, 4, 2^4) &= 2^3 \cdot 2 \cdot \gcd(\lambda(2^{9-4}), 2^2) = 2^3 \cdot 2 \cdot 2^2 = 2^6 \\ &= N(2^9, 4, 2^8) = 2^{3 \cdot 2} \gcd(\lambda(2^{9-8}), 2^2) = 2^6 \gcd(1, 4) = 2^6 = 2^{9 - \lceil 9/4 \rceil} \\ &= N(2^9, 4, 2^9), \end{aligned}$$

and $G_2(2^9, 4)$ is also semiregular. Condition (f) is now established and part (i) is proved.

(ii) Note that $G(n, k)$ is semiregular if and only if $G_1(n, k)$ and $G_2(n, k)$ are both semiregular, and for any two vertices $a \in G_1(n, k)$ and $b \in G_2(n, k)$ having positive indegree, $\mathrm{indeg}(a) = \mathrm{indeg}(b)$. Part (ii) now follows from the proof of part (i) of this theorem and from Theorems 4.1 and 4.3. □

5. Digraphs for which some components are semiregular

We saw in Theorems 4.1 and 4.3 that $G_1(n, k)$ is always semiregular for any n and k. By Theorem 4.4, $G_2(n, k)$ is, in general, not semiregular. Theorems 5.1 and 5.4 below present cases in which some but not necessarily all of the components of $G_2(n, k)$ are semiregular or regular. We also determine when all of the components of $G_2(n, k)$ are semiregular even if $G_2(n, k)$ is not itself necessarily semiregular. By our comments above, if each component of $G_2(n, k)$ is semiregular, so is each component of $G(n, k)$. Clearly, $G(n, k)$ is regular if and only if each component of $G(n, k)$ is regular.

Before presenting Theorems 5.1 and 5.4, we need to define some subdigraphs of $G_2(n, k)$ as given in [6]. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_r\}$ be the set of prime divisors of $n \geq 2$ and consider a partition of this set given by $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint and $\mathcal{P}_1$ is nonempty. Let $G^*_{\mathcal{P}_1}(n, k)$ be the subdigraph of $G(n, k)$ induced by the vertices which are multiples of $\prod_{p \in \mathcal{P}_1} p$ and which are also relatively prime to all primes $q \in \mathcal{P}_2$. Let ℓ be a prime and m a positive integer. Noting that $\gcd(a, \ell^m) > 1$ if and only if $\gcd(a^k, \ell^m) > 1$, we see by the Chinese remainder theorem that $G^*_{\mathcal{P}_1}(n, k)$ is a union of components of $G_2(n, k)$ for all nonempty subsets $\mathcal{P}_1$ of $\mathcal{P}$. It is also evident that $G_2(n, k)$ is the disjoint union of $G^*_{\mathcal{P}_1}(n, k)$ as $\mathcal{P}_1$ ranges over all nonempty subsets of $\mathcal{P}$. One further sees that if $a \in G^*_{\mathcal{P}_1}(n, k)$, then $a \in G_2(p^{\nu_p(n)}, k)$ for each prime $p \in \mathcal{P}_1$ and $a \in G_1(q^{\nu_q(n)}, k)$ for each prime $q \in \mathcal{P}_2$. Moreover, if a is a cycle vertex of $G^*_{\mathcal{P}_1}(n, k)$ and $p \in \mathcal{P}_1$, then $a \equiv 0 \pmod{p^{\nu_p(n)}}$. This follows since if a is part of a t-cycle in $G^*_{\mathcal{P}_1}(n, k)$ and $p \in \mathcal{P}_1$, then $a^{k^t} \equiv a \pmod{p^{\nu_p(n)}}$, which implies that

$$a^{k^t} - a = a(a^{k^t - 1} - 1) \equiv 0 \pmod{p^{\nu_p(n)}}.$$

Since $\gcd(a, a^{k^t-1} - 1) = 1$ and $p \mid a$, we see that $p^{\nu_p(n)} \mid a$.

Theorem 5.1. *The digraph $G^*_{\mathcal{P}_1}(n,k)$ is semiregular if and only if $G_2(p^{\nu_p(n)}, k)$ is semiregular for each prime $p \in \mathcal{P}_1$.*

Proof: First suppose that $G_2(p^{\nu_p(n)}, k)$ is semiregular for each $p \in \mathcal{P}_1$. Let a and b be vertices in $G^*_{\mathcal{P}_1}(n,k)$ such that $\operatorname{indeg}(a) > 0$ and $\operatorname{indeg}(b) > 0$. Then both a and b are vertices in $G_2(p^{\nu_p(n)}, k)$ for $p \in \mathcal{P}_1$, and a and b are both vertices in $G_1(q^{\nu_q(n)}, k)$ for $q \in \mathcal{P}_2$. Since $G_1(q^{\nu_q(n)}, k)$ is semiregular for all $q \in \mathcal{P}_2$ by Theorems 4.1 and 4.3, we see by (1.4) that $\operatorname{indeg}_n(a) = \operatorname{indeg}_n(b)$. Thus, $G^*_{\mathcal{P}_1}(n,k)$ is semiregular.

We now prove that if any component of $G^*_{\mathcal{P}_1}(n,k)$ is semiregular, then the digraph $G_2(p^{\nu_p(n)}, k)$ is semiregular for each prime $p \in \mathcal{P}_1$. This is a somewhat stronger result than the converse implication. Let C be a semiregular component in $G^*_{\mathcal{P}_1}(n,k)$. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha_i}$$

and $Q_i = p_i^{\alpha_i}$. By way of contradiction, we can assume without loss of generality that $p_1 \in \mathcal{P}_1$ and $G_2(p_1^{\alpha_1}, k)$ is not semiregular. By relabeling the primes dividing n if necessary, we can also assume that $p_1, p_2, \ldots, p_s \in \mathcal{P}_1$ and $p_{s+1}, p_{s+2}, \ldots, p_r \in \mathcal{P}_2$.

Suppose that a_1 and b_1 are vertices in $G_2(p_1^{\alpha_1}, k)$ having positive indegree such that $\operatorname{indeg}_{Q_1}(a_1) \neq \operatorname{indeg}_{Q_1}(b_1)$. Since p_1 divides both a_1 and b_1, there exists a least nonnegative integer h such that

$$a_1^{k^h} \equiv b_1^{k^h} \equiv 0 \pmod{p_1^{\alpha_1}}.$$

Let c be a cycle vertex in C. Let c_h be the cycle vertex in C which is h vertices before c, that is, $c^{k^h} \equiv c \pmod n$. Note that $c \equiv 0 \pmod{Q_i}$ for $i = 1, 2, \ldots, s$ and $\gcd(c_h, Q_i) = \gcd(c, Q_i) = 1$ for $i = s+1, s+2, \ldots, r$. By the Chinese remainder theorem, we can find vertices a_2 and b_2 in $G^*_{\mathcal{P}_1}(n,k)$ such that $a_2 \equiv a_1 \pmod{Q_1}$, $b_2 \equiv b_1 \pmod{Q_1}$, $a_2 \equiv b_2 \equiv 0 \pmod{Q_i}$ for $2 \le i \le s$, and $a_2 \equiv b_2 \equiv c_h \pmod{Q_i}$ for $s+1 \le i \le r$. Then

$$a_2^{k^h} \equiv b_2^{k^h} \equiv 0 \pmod{Q_i}$$

for $1 \le i \le s$, and

$$a_2^{k^h} \equiv b_2^{k^h} \equiv c_h^{k^h} \equiv c \pmod{Q_i}$$

for $s+1 \le i \le r$. Applying the Chinese remainder theorem again, one sees that $a_2^{k^h} \equiv b_2^{k^h} \equiv c \pmod n$, and both a_2 and b_2 are vertices in the component C.

By (1.4),

$$\operatorname{indeg}_n(a_2) = \operatorname{indeg}_{Q_1}(a_1) \prod_{i=2}^{s} \operatorname{indeg}_{Q_i}(0) \prod_{i=s+1}^{r} \operatorname{indeg}_{Q_i}(c_h)$$

and

$$\operatorname{indeg}_n(b_2) = \operatorname{indeg}_{Q_1}(b_1) \prod_{i=2}^{s} \operatorname{indeg}_{Q_i}(0) \prod_{i=s+1}^{r} \operatorname{indeg}_{Q_i}(c_h).$$

Since 0 is a cycle vertex in $G_2(Q_i, k)$ for $2 \le i \le s$ and c_h is a cycle vertex in $G_1(Q_i, k)$ for $s+1 \le i \le r$, we see that both the vertices a_2 and b_2 have positive indegree in the component C and $\operatorname{indeg}_n(a_2) \ne \operatorname{indeg}_n(b_1)$. Thus, the component C is not semiregular, which is a contradiction. The result now follows. □

By the proof and the discussion preceding Theorem 5.1, we have the following two immediate corollaries.

Corollary 5.2. *The digraph $G^*_{\mathcal{P}_1}(n, k)$ is semiregular if and only if at least one of its components is semiregular.*

Corollary 5.3. *Each component of $G(n, k)$ is semiregular if and only if the digraph $G_2(p^{\nu_p(n)}, k)$ is semiregular for each prime p dividing n.*

Theorem 5.4. *Let $n = n_1 n_2$, where*

$$n_1 = \prod_{p \in \mathcal{P}_1} p^{\nu_p(n)} \quad \text{and} \quad n_2 = \prod_{p \in \mathcal{P}_2} p^{\nu_p(n)}.$$

Then $G^*_{\mathcal{P}_1}(n, k)$ is regular if and only if n_1 is square-free and $\gcd(\lambda(n_2), k) = 1$.

Proof: First suppose that n_1 is square-free and $\gcd(\lambda(n_2), k) = 1$. Then $\nu_p(n) = 1$ for each prime $p \in \mathcal{P}_1$ and thus by Theorem 4.1(ii), $G_2(p^{\nu_p(n)}, k)$ is regular for each $p \in P_1$. Moreover, by the definition of the Carmichael lambda-function, $\lambda(p^{\nu_p(n)}) \mid \lambda(n_2)$ and hence, $\gcd(\lambda(p^{\nu_p(n)}), k) = 1$ for each prime $p \in \mathcal{P}_2$. Therefore, by Theorem 4.1, $G_1(p^{\nu_p(n)}, k)$ is regular for each $p \in P_2$. Let a be a vertex in $G^*_{\mathcal{P}_1}(n, k)$. Then $a \in G_2(p^{\nu_p(n)}, k)$ for $p \in \mathcal{P}_1$ and $a \in G_1(p^{\nu_p(n)}, k)$ for $p \in P_2$. By (1.4),

$$\operatorname{indeg}_n(a) = \prod_{p \in \mathcal{P}_1} N(p^{\nu_p(n)}, k, a) \cdot \prod_{p \in \mathcal{P}_2} N(p^{\nu_p(n)}, k, a) = \prod_{p \in \mathcal{P}_1} 1 \cdot \prod_{p \in \mathcal{P}_2} 1 = 1.$$

Consequently, we see by Remark 1.1 that $G^*_{\mathcal{P}_1}(n, k)$ is regular.

We now suppose that C is a regular component in $G^*_{\mathcal{P}_1}(n, k)$. We will show that n_1 is square-free and $\gcd(\lambda(n_2), k) = 1$. We can assume without loss of generality that $p_1, p_2, \dots, p_s \in \mathcal{P}_1$ and $p_{s+1}, p_{s+2}, \dots, p_r \in \mathcal{P}_2$. Let c be a cycle vertex of C. Then $c \equiv 0 \pmod{p^{\nu_p(n)}}$ for each $p \in \mathcal{P}_1$. Let the factorization of n be as given in (1.2). Then

$$\mathrm{indeg}_n(c) = 1 = \prod_{i=1}^{s} N(p_i^{\alpha_i}, k, 0) \cdot \prod_{i=s+1}^{r} N(p_i^{\alpha_i}, k, c).$$

Hence, $N(p_i^{\alpha_i}, k, 0) = 1$ for $1 \le i \le s$ and $N(p_i^{\alpha_i}, k, c) = 1$ for $s+1 \le i \le r$. If $\alpha_i \ge 2$ for some $i \in \{1, 2, \dots, s\}$, then by Theorem 3.2,

$$N(p_i^{\alpha_i}, k, 0) = p_i^{\alpha_i - \lceil \alpha_i / k \rceil} \ge p > 1,$$

which is a contradiction. Thus, $\alpha_i = 1$ for $1 \le i \le s$, and consequently, n_1 is square-free. Since $N(p_i^{\alpha_i}, k, c) = 1$ for $s+1 \le i \le r$, it follows from Theorem 3.1 that $\gcd(\lambda(p_i^{\alpha_i}), k) = 1$ for $s+1 \le i \le r$. Since

$$n_2 = \prod_{i=s+1}^{r} p_i^{\alpha_i},$$

it follows from the definition of λ that

$$\lambda(n_2) \mid \prod_{i=s+1}^{r} \lambda(p_i^{\alpha_i}).$$

Hence, $\gcd(\lambda(n_2), k) = 1$. □

By the proof of Theorem 5.4 we have the following corollary.

Corollary 5.5. *The digraph $G^*_{\mathcal{P}_1}(n, k)$ is regular if and only if at least one component of $G^*_{\mathcal{P}_1}(n, k)$ is regular.*

Acknowledgment. This paper was supported by Institutional Research Plan nr. AV0Z 10190503 and Grant nr. 1P05ME749 of the Ministry of Education of the Czech Republic.

References

[1] Křížek M., Luca F., Somer L., *17 Lectures on Fermat Numbers: From Number Theory to Geometry*, CMS Books in Mathematics, vol. 9, Springer, New York, 2001.

[2] Lucheta C., Miller E., Reiter C., *Digraphs from powers modulo p*, Fibonacci Quart. **34** (1996), 226–239.

[3] Niven I., Zuckerman H.S., Montgomery H.L., *An Introduction to the Theory of Numbers*, fifth edition, John Wiley & Sons, New York, 1991.

[4] Somer L., Křížek M., *On a connection of number theory with graph theory*, Czechoslovak Math. J. **54** (2004), 465–485.
[5] Somer L., Křížek M., *Structure of digraphs associated with quadratic congruences with composite moduli*, Discrete Math. **306** (2006), 2174–2185.
[6] Wilson B., *Power digraphs modulo n*, Fibonacci Quart. **36** (1998), 229–239.

Department of Mathematics, Catholic University of America, Washington, D.C. 20064, U.S.A.

E-mail: somer@cua.edu

Institute of Mathematics, Academy of Sciences, Žitná 25, CZ–115 67 Prague 1, Czech Republic

E-mail: krizek@math.cas.cz

(*Received* April 3, 2006)