Approximations by regular sets and Wiener solutions in metric spaces

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Abstract. Let X be a complete metric space equipped with a doubling Borel measure supporting a weak Poincaré inequality. We show that open subsets of X can be approximated by regular sets. This has applications in nonlinear potential theory on metric spaces. In particular it makes it possible to define Wiener solutions of the Dirichlet problem for p-harmonic functions and to show that they coincide with three other notions of generalized solutions.

Keywords: axiomatic potential theory, capacity, corkscrew, Dirichlet problem, doubling, metric space, nonlinear, p-harmonic, Poincaré inequality, quasiharmonic, quasisuperharmonic, quasiminimizer, quasisuperminimizer, regular set, Wiener solution

Classification: Primary 31C45; Secondary 31D05, 49J27

1. Introduction

If Ω is a nonempty bounded open set in $\mathbb{R}^n$, $f \in C(\partial\Omega)$ and $p > 1$, then there exists a unique bounded p-harmonic function u with the boundary data f (in a weak sense), see e.g. Theorem 9.25 in Heinonen–Kilpeläinen–Martio [12]. If, moreover, Ω has sufficiently smooth boundary then

$$\lim_{\Omega \ni x \to x_0} u(x) = f(x_0) \quad \text{for all} \quad x_0 \in \partial\Omega. \tag{1.1}$$

Sets satisfying this condition for all $f \in C(\partial\Omega)$ are called *regular*. By the Wiener criterion, a nonempty bounded open set $\Omega \subset \mathbb{R}^n$ is regular with $1 < p < n$ if and only if for all $x_0 \in \partial\Omega$,

$$\int_0^1 \left(\frac{C_p(B(x_0,t) \setminus \Omega)}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t} = \infty, \tag{1.2}$$

where C_p is the p-capacity in $\mathbb{R}^n$, see Wiener [25] ($p = 2$), Maz'ya [19] and Kilpeläinen–Malý [15].

In particular, this implies that Euclidean domains, whose complements have a corkscrew (see the definition below) at every boundary point (such as balls and

The authors were supported by the Swedish Research Council.

polyhedra), are regular for all $p > 1$. This provides us with an abundance of regular sets in $\mathbb{R}^n$ and makes it possible to approximate every Euclidean domain by regular ones. This is frequently used in potential theory, in particular when studying p-superharmonic functions and balayage, see e.g. Chapters 7 and 8 in Heinonen–Kilpeläinen–Martio [12]. The possibility to approximate by regular sets is also one of the axioms in the axiomatic potential theory, see e.g. Chapter 16 in [12].

If Ω is not regular, then (1.1) fails for some $f \in C(\partial\Omega)$, i.e. the Dirichlet problem cannot be solved in the classical sense for general boundary data $f \in C(\partial\Omega)$. Thus, other notions of solutions are required, which led Perron [20] and Wiener [24] to their definitions of generalized solutions of the Dirichlet problem. In particular, Wiener's construction is based on approximations by regular sets.

During the last decade, potential theory and p-(super)harmonic functions have been developed in the setting of doubling metric measure spaces supporting a p-Poincaré inequality. This theory unifies, and has applications in, several areas of analysis, such as weighted Sobolev spaces, calculus on Riemannian manifolds and Carnot groups, subelliptic differential operators and potential theory on graphs. Several results concerning solubility of the Dirichlet (boundary value) problem for p-harmonic functions have been extended to this setting in e.g. Cheeger [10], Shanmugalingam [22] and Björn–Björn–Shanmugalingam [5] and [6]. Conditions, similar to (1.2), guaranteeing regularity of boundary points have also been proved, see e.g. Björn–MacManus–Shanmugalingam [8] and J. Björn [7], but there are hardly any concrete examples of regular sets in metric spaces. It can even happen that a ball in a reasonable metric space is not regular, see Example 3.1. This lack of regular sets has been one of the reasons why some traditional methods could not be used directly in metric spaces.

In this paper, we show how open sets in metric spaces can be approximated by bounded regular sets, i.e. we prove the following result. (See Section 2 for the definitions.)

Theorem 1.1. *Let X be a complete metric space endowed with a complete doubling Borel measure which supports a weak p-Poincaré inequality. Let $\Omega \subset X$ be nonempty and open. If X is bounded, assume moreover that $\Omega \neq X$. Then there exist bounded open sets $\Omega_1 \Subset \Omega_2 \Subset \cdots$, regular for p-quasisuperharmonic functions and such that $\Omega = \bigcup_{j=1}^\infty \Omega_j$.*

This shows that there is an abundance of regular sets, thus opening for various applications. One of them is the definition of *Wiener solutions* of the Dirichlet problem for p-harmonic functions with continuous boundary data on metric measure spaces, inspired by the construction in Wiener [24]. The results in Björn–Björn–Shanmugalingam [5], [6] provide us with three fundamentally different definitions of solutions to the Dirichlet problem for p-harmonic functions with continuous boundary data. In Section 4, we show that Wiener solutions exist, are unique and coincide with the three other types of solutions. These results also hold for $\mathcal{A}$-harmonic functions as defined on p. 57 of Heinonen–Kilpeläinen–Martio [12] with the usual degenerate ellipticity assumptions (3.3)–(3.7) on p. 56 of [12].

Theorem 1.1 makes it also possible to apply the axiomatic potential theory to this setting (at least in the case of Cheeger p-harmonic functions, where we have the sheaf property), see Section 5.

Another application of Theorem 1.1 has been given recently in A. Björn [2], where it was shown that two different types of p-superharmonic functions, used in Kinnunen–Martio [16] and [17], coincide with the classical definition (in e.g. Heinonen–Kilpeläinen–Martio [12]).

Note that in contrast to the Euclidean setting, where balls and polyhedra form a universal supply of regular domains, here we do not have at hand such a general family of regular sets. Instead, our construction of approximating regular sets depends on the local geometry of X and Ω. Nevertheless, we have the following consequence of Theorem 1.1.

Corollary 1.2. *Let X be as in Theorem 1.1 and $x \in X$. Then there exists a basis of neighbourhoods of x, which are regular for p-quasisuperharmonic functions.*

2. Notation and preliminaries

We assume throughout the paper that $X = (X, d, \mu)$ is a *complete* metric space endowed with a metric d and a complete Borel measure μ which is *doubling*, i.e. there exists a constant $C > 0$ such that for all balls $B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X,

$$0 < \mu(B(x_0, 2r)) \le C\mu(B(x_0, r)) < \infty.$$

In [13], Heinonen and Koskela introduced upper gradients as a substitute for the modulus of the usual gradient. It has many useful properties similar to those of the usual gradient.

Definition 2.1. A nonnegative Borel function g on X is an *upper gradient* of an extended real-valued function u on X if for all nonconstant rectifiable curves $\gamma : [0, l_\gamma] \to X$, parameterized by the arc length ds,

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \le \int_\gamma g\, ds \tag{2.1}$$

whenever both $u(\gamma(0))$ and $u(\gamma(l_\gamma))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

Let also $1 < p < \infty$. We further assume that X supports a *weak p-Poincaré inequality*, i.e. that there exist constants $C > 0$ and $\lambda \ge 1$ such that for all balls

$B = B(x_0, r) \subset X$, all measurable functions u on X and all upper gradients g of u,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \, d\mu \le C(\operatorname{diam} B) \biggl(\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \biggr)^{1/p}, \tag{2.2}$$

where $u_B := \mu(B)^{-1} \int_B u \, d\mu$ and $\lambda B = B(x_0, \lambda r)$.

By Keith–Zhong [14], if X supports a weak p-Poincaré inequality, then it supports a weak q-Poincaré inequality for some $q < p$, which was earlier a standard assumption. There are many spaces satisfying our assumptions, see e.g. A. Björn [2] for a list of examples and Hajłasz–Koskela [11] or Heinonen–Koskela [13] for more detailed descriptions. The following Sobolev type spaces were introduced in Shanmugalingam [21].

Definition 2.2. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$ and

$$\|u\|_{N^{1,p}(X)} = \biggl(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \biggr)^{1/p}$$

with the infimum taken over all upper gradients g of u.

Cheeger [10] gives a different definition of Sobolev spaces, which leads to the same space and yields a notion of a vector-valued gradient Du, see Theorems 4.38 and 4.47 in [10]. This will be used in Section 5.

By Corollary 3.7 in Shanmugalingam [22], every $u \in N^{1,p}(X)$ has a *minimal p-weak upper gradient* g_u (unique up to sets of measure zero), which satisfies (2.1) for p-almost every curve and $g_u \le g$ μ-a.e. for all upper gradients g of u. (For the definition of "p-almost every curve" see e.g. Definition 2.1 in Shanmugalingam [21].)

From now on, Ω will always be a nonempty open set in X. We say that $f \in N^{1,p}_{\mathrm{loc}}(\Omega)$ if $f \in N^{1,p}(\Omega')$ for every open $\Omega' \Subset \Omega$, where by the latter we mean that the closure of Ω' is a compact subset of Ω. Let also

$$N^{1,p}_0(\Omega) = \{u|_\Omega : u \in N^{1,p}(X) \ \text{ and } \ u = 0 \ \text{ in } \ X \setminus \Omega\}.$$

Definition 2.3. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is *p-harmonic* in Ω if it is continuous and minimizes the p-energy integral, i.e. it satisfies

$$\int_{\phi \ne 0} g_u^p \, d\mu \le \int_{\phi \ne 0} g_{u+\phi}^p \, d\mu \quad \text{for all} \quad \phi \in \operatorname{Lip}_c(\Omega), \tag{2.3}$$

where $\mathrm{Lip}_c(\Omega)$ is the space of all Lipschitz functions with compact support in Ω.

A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is *p-quasiharmonic* in Ω if it is continuous and for some $Q \geq 1$ satisfies

$$\int_{\phi \neq 0} g_u^p \, d\mu \leq Q \int_{\phi \neq 0} g_{u+\phi}^p \, d\mu \quad \text{for all} \quad \phi \in \mathrm{Lip}_c(\Omega).$$

Definition 2.4. The *p-capacity* of a set $E \subset X$ is the number

$$C_p(E) = \inf \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u = 1$ on E.

If Ω is bounded and $C_p(X \setminus \Omega) > 0$, then for every $f \in C(\partial\Omega)$, there exists a unique bounded p-harmonic function $H_\Omega f = Hf$ in Ω such that

$$\lim_{\Omega \ni x \to x_0} Hf(x) = f(x_0) \quad \text{outside a set of } p\text{-capacity zero,} \tag{2.4}$$

see Theorem 6.1 and Corollary 6.2 in Björn–Björn–Shanmugalingam [6] together with Theorem 3.9 in Björn–Björn–Shanmugalingam [5].

Definition 2.5. Let Ω be bounded with $C_p(X \setminus \Omega) > 0$. A point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni x \to x_0} Hf(x) = f(x_0) \quad \text{for all} \quad f \in C(\partial\Omega).$$

If all $x_0 \in \partial\Omega$ are regular, then Ω is *regular*.

In view of the results in A. Björn [3] and J. Björn [7] we consider also the following more general notions of regularity.

Definition 2.6. Let Ω be bounded with $C_p(X \setminus \Omega) > 0$.

A point $x_0 \in \partial\Omega$ is *regular for p-quasiharmonic functions* if for all $f \in C(\partial\Omega) \cap N^{1,p}(X)$ and all p-quasiharmonic u in Ω with $u - f \in N^{1,p}_0(\Omega)$, we have

$$\lim_{\Omega \ni x \to x_0} u(x) = f(x_0).$$

A point $x_0 \in \partial\Omega$ is *regular for p-(quasi)superharmonic functions* if for all $f \in C(\partial\Omega) \cap N^{1,p}(X)$ and all p-(quasi)superharmonic u in Ω with $u - f \in N^{1,p}_0(\Omega)$, we have

$$\liminf_{\Omega \ni y \to x_0} u(y) \geq f(x_0).$$

If all $x_0 \in \partial\Omega$ are regular for p-(quasi)(super)harmonic functions, then Ω is *regular for p-(quasi)(super)harmonic functions*.

We refer the reader to Kinnunen–Martio [17] (or A. Björn [3]) for the definition of p-quasisuperharmonicity. When saying that a set is regular in any of the above senses, we automatically assume that it is nonempty bounded open and has complement with positive p-capacity.

It is immediate that regularity for p-quasisuperharmonic functions implies regularity for p-quasiharmonic functions which in turn implies regularity. It is not known whether the converse implications hold, see A. Björn [3, Section 5], for a discussion on the first implication and its converse. On the other hand, regularity and regularity for p-superharmonic functions are equivalent, see Theorem 6.1 in Björn–Björn [4].

There are several capacitary conditions sufficient for regularity of boundary points. Theorem 5.1 in Björn–MacManus–Shanmugalingam [8] implies a condition similar to (1.2) guaranteeing regularity in linearly locally connected metric measure spaces. See Corollary 7.3 in Björn–Björn [4] for a precise formulation. At the same time, Theorem 2.13 and Remark 2.15 in J. Björn [7] provide us with the following sufficient condition.

Proposition 2.7. *Assume that $X \setminus \Omega$ has a* corkscrew *at x_0, i.e. that there exist $c > 0$ and $\rho_0 > 0$ such that for all $0 < \rho \le \rho_0$, the set $B(x_0, \rho) \setminus \Omega$ contains a ball with radius $c\rho$. Then $x_0 \in \partial\Omega$ is regular for p-quasisuperharmonic functions.*

Moreover, if $f \in C(\partial\Omega) \cap N^{1,p}(X)$ is Hölder continuous at x_0, and u is p-quasiharmonic in Ω with $u - f \in N_0^{1,p}(\Omega)$, then also u is Hölder continuous at x_0.

The corkscrew condition is more restrictive than the condition in [8], but it is sufficient for our purposes and does not assume that X is linearly locally connected. Moreover, it applies to p-quasiharmonic functions and not only to p-harmonic functions. In A. Björn [3], it was observed that the proof in [7] shows that the corkscrew condition guarantees regularity for p-quasisuperharmonic functions as well.

3. Approximations by regular sets

We begin this section by giving an example of a metric space satisfying our assumptions, in which a ball needs not be regular.

Example 3.1. Consider the cone

$$X = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j \ge 0 \ \text{ for all } \ j = 1, \dots, n\},$$

equipped with the Euclidean metric and the Lebesgue measure. Then X is a complete metric space with a doubling measure and a 1-Poincaré inequality. This can be easily verified by direct calculation (use e.g. the reflections $\tilde{u}(x_1, \dots, x_n) := u(|x_1|, \dots, |x_n|)$). Let $x = (1, \dots, 1) \in X$, $r = \sqrt{n}$ and $B = B(x, r)$. Then the origin is an isolated boundary point with zero p-capacity and is not regular for any $1 < p < n$.

This example can be iterated in the following way to obtain a sequence of shrinking balls which are not regular: Let T_j, $j = 1, 2, \dots$, be the closed isosceles triangles in $\mathbb{R}^2$ with bases $[2^{-j}, 2^{1-j}] \subset \mathbb{R}$ and heights 2^{1-j}. Let

$$X = [0,1] \times [-1,0] \cup \bigcup_{j=1}^{\infty} T_j \subset \mathbb{R}^2,$$

equipped with the Euclidean metric and 2-dimensional Lebesgue measure. It is not difficult to verify that X is a uniform domain in $\mathbb{R}^2$, i.e. there exists $C > 0$ such that every pair of points $x, y \in X$ can be connected by a curve γ of arc length at most $C|x - y|$ and such that for all $z \in \gamma$,

$$\operatorname{dist}(z, \mathbb{R}^2 \setminus X) \geq C^{-1} \min\{l(\gamma_{xz}), l(\gamma_{yz})\},$$

where $l(\gamma_{xz})$ and $l(\gamma_{yz})$ are the arc lengths of the subcurves of γ connecting z to x and y, respectively. Theorem 4.4 in Björn–Shanmugalingam [9] then implies that X is doubling and supports a 1-Poincaré inequality. Now for $r_j = 5 \cdot 2^{-j-1}$, the balls $B(0, r_j)$ are not regular, since $\partial B(0, r_j)$ contains the isolated boundary point $x_j = (3 \cdot 2^{-j-1}, 2^{1-j})$.

Open problem 3.2. How many irregular balls centred at one point can there be? Does there always exist a base of regular balls? By Example 2.16 in Björn [7], if X is a geodesic space such that all geodesic curves are "open" in the sense that they do not have a first and a last point, then every ball in X is regular.

In this section, we show that open sets in metric spaces can be approximated from inside by bounded regular open sets, i.e. we prove Theorem 1.1. Our construction is based on the following notion.

Definition 3.3. The *inner metric* on X is

$$d'(x, y) = \inf l_\gamma,$$

where l_γ is the arc length of γ and the infimum is taken over all curves γ joining x and y in X. The distance taken with respect to the inner metric d' will be denoted dist'.

By Theorem 17.1 in Cheeger [10], our assumptions imply that X is *quasiconvex*, i.e. that every pair of points in X can be joined by a curve whose length does not exceed a constant multiple of their distance. Hence

$$d(x, y) \leq d'(x, y) \leq L d(x, y), \tag{3.1}$$

where L depends only on the doubling constant of μ and the constants in the Poincaré inequality.

Proposition 3.4. *Let Ω be bounded with nonempty complement $X \setminus \Omega$. Let $\delta > 0$ and assume that the set*

$$\Omega' = \{x \in \Omega : \operatorname{dist}'(x, X \setminus \Omega) > \delta\}$$

is nonempty. Then the complement $X \setminus \Omega'$ has a corkscrew at every boundary point. In particular, Ω' is regular for p-quasi(*super*)*harmonic functions* (*and hence regular*).

Proof: Let $x_0 \in \partial\Omega'$ and $0 < \rho < \delta$ be arbitrary. Find $y \in X \setminus \Omega$ and a curve $\gamma : [0, l_\gamma] \to X$, parameterized by its arc length, such that $\gamma(0) = x_0$, $\gamma(l_\gamma) = y$ and $l_\gamma < \delta + \rho/3$. Let $z = \gamma(2\rho/3)$ and $L \geq 1$ be as in (3.1). We shall show that $B(z, \rho/3L) \subset B(x_0, \rho) \setminus \Omega'$, i.e. that $X \setminus \Omega$ has a corkscrew at x_0. Clearly, $d(x_0, z) \leq 2\rho/3$, i.e. $B(z, \rho/3L) \subset B(x_0, \rho)$. Let $x \in \Omega'$ be arbitrary. Then

$$\begin{aligned} Ld(x, z) &\geq d'(x, z) \geq d'(x, y) - d'(z, y) \\ &\geq \operatorname{dist}'(x, X \setminus \Omega) - (l_\gamma - 2\rho/3) > \delta - (\delta + \rho/3 - 2\rho/3) = \rho/3, \end{aligned}$$

i.e. $B(z, \rho/3L) \cap \Omega'$ is empty and $B(z, \rho/3L) \subset B(x_0, \rho) \setminus \Omega'$. □

Remark 3.5. (i) The proof of Proposition 3.4 only uses the quasiconvexity of X, not the Poincaré inequality or the doubling property. Thus, $X \setminus \Omega'$ has a corkscrew at every boundary point even if X is only quasiconvex and does not support a Poincaré inequality. On the other hand, the Poincaré inequality and the doubling condition are the standard assumptions for the theory of p-harmonic functions on metric spaces and are thus natural for Theorem 1.1.

(ii) The proof of Proposition 3.4 shows that $X \setminus \Omega'$ has a uniform corkscrew at all boundary points, i.e. that the numbers c and ρ_0 in the definition of corkscrew do not depend on x_0. Together with the pointwise estimates in J. Björn [7], this shows that if $f \in C^\alpha(\partial\Omega')$, then $Hf \in C^\beta(\overline{\Omega}')$ for some $\beta > 0$ independent of f.

Proof of Theorem 1.1: If Ω is bounded, then the theorem follows from Proposition 3.4 by taking $\delta = 1/j$, $j = 1, 2, \dots$. If Ω is unbounded, fix $z_0 \in \Omega$ and let

$$\Omega'_j = \{x \in \Omega : d'(x, z_0) < j\} \quad \text{and} \quad \Omega_j = \{x \in \Omega'_j : \operatorname{dist}'(x, X \setminus \Omega'_j) > 1/j\}.$$

Then $\Omega_j \Subset \Omega'_j \Subset \Omega_{j+1}$, $j = 2, 3 \dots$, and Proposition 3.4 implies that each Ω_j is regular, which concludes the proof. □

Proof of Corollary 1.2: Let $B_j = B(x, 1/j)$, $j = 1, 2, \dots$. Using Proposition 3.4, we can find open sets U_j, regular for p-quasisuperharmonic functions, so that $B_{j+1} \subset U_j \subset B_j$. □

4. Wiener solutions

Assume in this section that Ω is a nonempty bounded open set with $C_p(X\setminus\Omega) > 0$. As mentioned in the introduction, if Ω is not regular, then the Dirichlet problem cannot be solved in the classical sense for a general continuous boundary function $f \in C(\partial\Omega)$. (Classical in the sense that the boundary values are really attained at all boundary points. We still consider weak solutions of the equation when our minimization problem corresponds to a partial differential equation.)

Omitting most details let us here just mention that on metric spaces the first type of generalized solution of the Dirichlet problem (for arbitrary $f \in C(\partial\Omega)$) was given by Definition 3.6 in Björn–Björn–Shanmugalingam [5]. A second alternative, Perron solutions, was given in Björn–Björn–Shanmugalingam [6, Definition 3.11], where it was also shown that these two types of generalized solutions always coincide with Hf, defined by (2.4), see Theorem 6.1 and Corollary 6.2 in [6] and Theorem 3.9 in [5].

Theorem 1.1 gives us yet another possibility of defining generalized solutions to the Dirichlet problem.

Definition 4.1. Let $f \in C(\partial\Omega)$. A *Wiener solution* u of the Dirichlet problem in Ω with boundary values f is obtained by the following construction: Extend f in any way to a continuous function (also called f) on $\overline{\Omega}$, let $\Omega_1 \Subset \Omega_2 \Subset \dots \Subset \Omega$ be regular sets such that $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$, and let

$$u = \lim_{j\to\infty} H_{\Omega_j} f.$$

Observe that since Ω_j is regular, the solutions $H_{\Omega_j} f$ are classical solutions of the corresponding boundary value problems.

Theorem 4.2. *Let $f \in C(\partial\Omega)$. Then there exists a Wiener solution of the Dirichlet problem in Ω with boundary values f, and moreover all Wiener solutions of the Dirichlet problem in Ω with boundary values f coincide.*

PROOF: Let us first look at existence. The first step, the extension of f, is directly obtained by Tietze's extension theorem. The next step is to approximate Ω by regular sets, which is obtained by Theorem 1.1. Finally one needs to show that the limit $\lim_{j\to\infty} H_{\Omega_j} f$ exists everywhere in Ω. We combine the existence and uniqueness parts of the proof and make it into a theorem of its own below. $\square$

Theorem 4.3. *Let $\Omega_1 \subset \Omega_2 \subset \dots \subset \Omega = \bigcup_{j=1}^{\infty} \Omega_j$ be open sets and let $f \in C(\overline{\Omega})$. Then*

$$\lim_{j\to\infty} H_{\Omega_j} f = Hf.$$

PROOF: To show this we will use the fact that

$$Hf(x) = \inf_{u\in\mathcal{U}_f} u(x), \quad x \in \Omega, \tag{4.1}$$

where $\mathcal{U}_f = \mathcal{U}_f(\Omega)$ is the set of all p-superharmonic functions u on Ω bounded below such that

$$\liminf_{\Omega \ni y \to x} u(y) \geq f(x) \quad \text{for all} \quad x \in \partial\Omega,$$

which is part of the definition of Perron solutions. We refer the reader to Björn–Björn–Shanmugalingam [6, Definition 3.10], (or A. Björn [2]) for the definition of p-superharmonic functions; here it is enough to know that p-superharmonic functions are lower semicontinuous.

Let $u \in \mathcal{U}_f$ and $\varepsilon > 0$. Extend u to $\overline{\Omega}$, by letting

$$u(x) = \liminf_{\Omega \ni y \to x} u(y), \quad x \in \partial\Omega,$$

which makes u lower semicontinuous on $\overline{\Omega}$. Let further

$$A = \{x \in \overline{\Omega} : u(x) + \varepsilon > f(x)\},$$

which is an open set (in the relative topology), by the lower semicontinuity of $u - f$. The set A contains $\partial\Omega$ by assumption. By compactness, there is some k such that $A \cup \Omega_k = \overline{\Omega}$, and hence $\partial\Omega_k \subset A$. It follows that

$$(u + \varepsilon)|_{\Omega_j} \in \mathcal{U}_f(\Omega_j) \quad \text{for} \quad j \geq k,$$

and thus that $\limsup_{j\to\infty} H_{\Omega_j} f \leq u + \varepsilon$. Letting $\varepsilon \to 0$ and taking infimum over all $u \in \mathcal{U}_f$, shows that

$$\limsup_{j\to\infty} H_{\Omega_j} f \leq Hf.$$

Applying this also to $-f$ we obtain

$$Hf = -H(-f) \leq -\limsup_{j\to\infty} H_{\Omega_j}(-f) = \liminf_{j\to\infty} H_{\Omega_j} f \leq \limsup_{j\to\infty} H_{\Omega_j} f \leq Hf. \qquad \square$$

Theorem 4.3 shows that one could define Wiener solutions also with respect to nonregular exhaustions of Ω. However, that would defy the purpose of Wiener solutions, that to define Wiener solutions we only need to use classical solutions of boundary value problems. Nevertheless, Theorem 4.3 is an interesting stability result.

5. Applications in axiomatic potential theory

Linear axiomatic theory for harmonic functions dates back to the middle of the last century, see e.g. Bauer [1]. Nonlinear axiomatic theory for p-harmonic functions has been developed in Lehtola [18]. Here, we follow the presentation from Chapter 16 in Heinonen–Kilpeläinen–Martio [12].

Let X be as before and assume, moreover, that it is unbounded. Then the following hold.

(a) For every nonempty open $\Omega \subset X$ and every compact $K \subset \Omega$, there exists a regular set Ω' such that $K \subset \Omega' \Subset \Omega$. This follows from our Theorem 1.1. Then for every $f \in C(\partial\Omega')$, there exists a unique function $Hf \in C(\overline{\Omega}')$ which is p-harmonic in Ω' and such that $Hf = f$ on $\partial\Omega'$. Moreover, if $f_1, f_2 \in C(\partial\Omega')$ and $f_1 \le f_2$, then $Hf_1 \le Hf_2$ in Ω'. This follows directly from (4.1).

(b) If $u_1 \le u_2 \le \cdots$, is a sequence of p-harmonic functions in a domain Ω and $u_j(x) \le M$ for all j and some $x \in \Omega$, then the function $u = \lim_{j\to\infty} u_j$ is p-harmonic in Ω. This is Proposition 5.1 from Shanmugalingam [23].

(c) If u is p-harmonic in Ω and $\lambda \in \mathbb{R}^n$, then both λu and $u+\lambda$ are p-harmonic in Ω.

This means that Axioms A–C in Chapter 16 in Heinonen–Kilpeläinen–Martio [12] are satisfied for p-harmonic functions in complete unbounded metric spaces with a doubling Borel measure and a weak p-Poincaré inequality.

However, to be able to apply the nonlinear axiomatic theory from [12], we also need the following *sheaf property*: If $\Omega_j \subset X$, $j = 1, 2, \ldots$, are open and u is p-harmonic in each Ω_j, then u is p-harmonic in $\bigcup_{j=1}^\infty \Omega_j$. Unfortunately in our setting, it is not known whether the sheaf property holds for p-harmonic functions which are obtained by minimizing the p-energy integral in (2.3).

The situation is more promising for Cheeger p-harmonic functions, i.e. for continuous minimizers of the integral $\int |Du|^p \, d\mu$ in the sense of Definition 2.3 (with g_u replaced by $|Du|$), where Du is the vector-valued Cheeger gradient of u, see Theorems 4.38 and 4.47 in Cheeger [10]. An equivalent definition of Cheeger p-harmonic functions is that $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is continuous and satisfies the integral identity

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, d\mu = 0 \quad \text{for all} \quad \phi \in \mathrm{Lip}_c(\Omega).$$

All the theory of p-harmonic functions goes through for Cheeger p-harmonic functions as well (simply by replacing g_u by $|Du|$ in the proofs). Observe that if u is Cheeger p-harmonic in $\Omega_j \subset X$, $j = 1, 2 \ldots$, $\Omega = \bigcup_{j=1}^\infty \Omega_j$ and $\phi \in \mathrm{Lip}_c(\Omega)$, then

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, d\mu = \sum_{j=1}^\infty \int_{\Omega_j} |Du|^{p-2} Du \cdot D(\phi\eta_j) \, d\mu = 0,$$

where $\{\eta_j\}_{j=1}^\infty$ is a Lipschitz partition of unity subordinate to the sets Ω_j, $j = 1, 2, \ldots$. Hence, u is Cheeger p-harmonic in Ω and the sheaf property holds. This makes it possible to apply the axiomatic potential theory to Cheeger p-harmonic functions. Most of the conclusions in Chapter 16 in Heinonen–Kilpeläinen–Martio [12] have already been proved for Cheeger p-harmonic functions (and also for p-harmonic functions obtained from upper gradients) without the use of the axiomatic potential theory. Nevertheless, the following result seems to be new in the setting of metric measure spaces.

Theorem 5.1 (Theorem 16.24 in [12]). *Let $u:\Omega\to(-\infty,\infty]$ be a lower semicontinuous function which is not identically ∞ in any component of Ω. Then u is Cheeger p-superharmonic if and only if for every regular set $\Omega'\Subset\Omega$ and each $f\in C(\partial\Omega')$, the condition $u\geq f$ on $\partial\Omega'$ implies $u\geq H_{\Omega'}f$ in Ω', where $H_{\Omega'}f$ is the Cheeger p-harmonic function in Ω' with boundary values f.*

In A. Björn [2], other characterizations and equivalent definitions of p-superharmonic functions on metric spaces are given, some of them employing Theorem 1.1.

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(*Received* September 6, 2006, *revised* October 26, 2006)