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Weak-bases and D-spaces

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Abstract. It is shown that certain weak-base structures on a topological space give a D-space. This solves the question by A.V. Arhangel'skii of when quotient images of metric spaces are D-spaces. A related result about symmetrizable spaces also answers a question of Arhangel'skii.

Theorem. *Any symmetrizable space X is a D-space (hereditarily).*

Hence, quotient mappings, with compact fibers, from metric spaces have a D-space image. What about quotient s-mappings? Arhangel'skii and Buzyakova have shown that spaces with a point-countable base are D-spaces so open s-images of metric spaces are already known to be D-spaces.

A collection $\mathcal{W}$ of subsets of a sequential space X is said to be a *w-system* for the topology if whenever $x \in U \subseteq X$, with U open, there exists a subcollection $\mathcal{V} \subseteq \mathcal{W}$ such that $x \in \bigcap \mathcal{V}$, $\bigcup \mathcal{V}$ is a weak-neighborhood of x, and $\bigcup \mathcal{V} \subseteq U$.

Theorem. *A sequential space X with a point-countable w-system is a D-space.*

Corollary. *A space X with a point-countable weak-base is a D-space.*

Corollary. *Any T_2 quotient s-image of a metric space is a D-space.*

Keywords: quotient map, symmetrizable space, weak-base, w-structure, D-space

Classification: Primary 54B15; Secondary 54D70, 54E25, 54E40

1. Introduction

All spaces in this paper are assumed to be at least T_1. A *neighborhood assignment* for a topological space (X, τ) is a function $\phi : X \to \tau$ such that, for all $x \in X$, $x \in \phi(x)$. With a mild abuse of the language we may speak of $\{\phi(x) : x \in X\}$ as the neighborhood assignment. The space X is said to be a *D-space* [vDP] if, whenever ϕ is a neighborhood assignment of X there exists a closed discrete $D \subseteq X$ such that $X = \bigcup_{d \in D} \phi(d)$. The class of D-spaces was introduced by E.K. van Douwen in [vDP]. A remarkable open question asks whether Lindelöf spaces are D-spaces and it is not even known whether subparacompact spaces are always D-spaces. It is clear that compact spaces, in fact σ-compact spaces, are D-spaces. Beyond this it often appears that some base or completeness "structure" is needed in order to prove certain spaces are D-spaces. Results by Borges and Wehrly [BW] show that semi-stratifiable spaces and paracompact p-spaces are D-spaces. Recent papers ([A2], [A3], [ABuz], [Buz1], [Buz2], [D], [FS] and [G]) have contributed a good deal to the study of the class of D-spaces.

A.V. Arhangel'skii has asked whether symmetrizable spaces are D-spaces and (a related question) whether quotient s-images of metric spaces are D-spaces. In this note we show that symmetrizable spaces are, in fact, hereditarily D-spaces. The hereditary result is a little surprising since the property of being symmetrizable is not a hereditary property — symmetrizability is generally only inherited to open or closed subspaces. An immediate corollary is that quotient compact-images of metric spaces are D-spaces. However, the main result of the last section gives the full answer to Arhangel'skii's question about the quotient s-images as a corollary. This result shows that a sequential space with a point-countable w-structure is a D-space. Another corollary of this result is that a space X with a point-countable weak-base is a D-space. This generalizes the result of Arhangel'skii and Buzyakova [ABuz] where it is shown that spaces with a point-countable base are D-spaces.

In the next section we summarize and review some results about closed or open (continuous) images of metric spaces. This may help justify why it was natural to focus on the class of quotient s-images of metric spaces.

2. Closed and open images of metric spaces

The case of closed images of D-spaces is quickly dispatched with the result of Borges and Wehrly.

- *All closed continuous images of D-spaces are D-spaces* [BW].

At first glance it may seem that open continuous images of metric spaces should be D-spaces but this is not the case. Recall that first-countable spaces are exactly the open continuous images of metric spaces so an example here could be any first-countable non-D-space (such as ω_1).

- *Open continuous images of metric spaces need not be D-spaces* .

It is even possible for the domain space to be a σ-discrete metric space — the following specific example may be of interest.

Example 2.1. Let $I(\omega_1)$ denote ω_1 with the discrete topology and let

$$M = \omega_1 \times I(\omega_1) \setminus \{(\alpha, \beta) : \alpha > \beta\}.$$

Then, M is a σ-discrete metric space and M can be mapped onto ω_1 by an open continuous map with discrete fibers.

Proof: Notice that M is metrizable since it is the disjoint union of open metrizable (countable) subspaces. To see that M is σ-discrete, express each countable level $L_\beta = \{(\alpha, \beta) : \alpha \le \beta\}$ as $L_\beta = \{(\alpha_{\beta,n}, \beta) : n \in \mathbb{N}\}$. For $m \in \mathbb{N}$, let $S_m = \{(\alpha_{\beta,m}, \beta) : \beta \in \omega_1\}$ — the selection of exactly one from each level L_β makes S_m a closed discrete set in M and $M = \bigcup_{n\in\mathbb{N}} S_n$. Now, we see that M is an open subspace of $\omega_1 \times I(\omega_1)$; hence the projection $\pi_1|_M : M \to \omega_1$ is an open map from M onto ω_1. $\square$

A function $f : M \to X$ is said to be an *s-mapping* if the fibers (point inverses) of f are separable. The function $\pi_1|_M$ in the previous example is not an s-mapping since the fibers are uncountable and discrete. For continuous open mappings the s-mapping condition turns out to be sufficient for carrying the D-space property from a metric space to its image.

- *All open continuous s-images of metric spaces are D-spaces* [ABuz].

Recall that spaces with a point-countable base are exactly the open continuous s-images of metric spaces so the above follows from the result of Arhangel'skii and Buzyakova that all spaces with a point-countable base are D-spaces [ABuz].

Now it becomes natural to consider the quotient images of metric spaces as the common generalization of closed or open (continuous) images of metric spaces. The following is a corollary to the main result of Section 4.

- *If $f : M \to X$ is a quotient s-map from a metric space M onto a T_2 space X then X is a D-space.*

3. Symmetrizable spaces

Suppose X is a topological space and $d : X \times X \to [0, \infty)$ such that, for all $(x, y) \in X \times X$, $d(x, y) = d(y, x)$ and $d(x, y) = 0 \iff x = y$. The function d is said to be a *symmetric* [A1] for X provided: For all nonempty $A \subseteq X$, A is closed in X if and only if $\inf\{d(x, z) : z \in A\} > 0$ for every $x \in X \setminus A$. In this case, one could say (X, d) (or X) is *symmetrizable* (with symmetric d). For $x \in X$ and $n \in \mathbb{N}$ let $B(x, \frac{1}{n}) = \{z \in X : d(x, z) < \frac{1}{n}\}$. Notice, by their very nature, symmetrizable spaces are always at least T_1. When d is a symmetric for X we see that a subset $W \subseteq X$ is open if and only if

($*$) whenever $x \in W$ there exists $n_x \in \mathbb{N}$ such that $B(x, \frac{1}{n_x}) \subseteq W$.

We remark that, in general, x is not in the interior of $B(x, \frac{1}{n})$. The collection $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ may be thought of as a *weak-base at x* because of the way the topology is determined above. The general notion of a weak-base is defined in Section 4; for now, we can get by with the property given by ($*$). In this setting, by a *weak-neighborhood* of x in X, we will mean a set $V \subseteq X$ such that $B(x, \frac{1}{k}) \subseteq V$, for some $k \in \mathbb{N}$.

Theorem 3.1. *Symmetrizable spaces are hereditarily D-spaces.*

Proof: Suppose (Z, d) is a symmetrizable space and $X \subseteq Z$ with the subspace topology τ. To show X is a D-space, let $U : X \to \tau$ be an open neighborhood assignment for X and let $\mathcal{U} = \{U(x) : x \in X\}$. Since open subspaces of symmetrizable spaces are symmetrizable, we may assume each $U(x)$, $x \in X$, is actually open in Z and $Z = \bigcup\{U(x) : x \in X\}$. We need to find a closed discrete subset D of X such that $X \subseteq \bigcup\{U(x) : x \in D\}$.

For every $x \in X$ let $k_x \in \mathbb{N}$ such that $B(x, \frac{1}{k_x}) \subseteq U(x)$. For every $n \in \mathbb{N}$, let

$$I(n) = \{x \in X : k_x = n\}.$$

Well-order X so that when $y \in I(i)$, $z \in I(j)$ for $i < j$ then $y < z$.

Now, recursively find $J(n) \subseteq I(n)$ as follows:
For $x \in I(n)$,
$x \in J(n) \iff$

$$x = \min((I(n) \setminus \bigcup\{U(y) : i < n, y \in J(i)\}) \setminus \bigcup\{U(y) : y \in J(n), y < x\}).$$

Let $D = \bigcup_{n\in\mathbb{N}} J(n)$.

Claim 0. $D \subseteq X$.

Claim 1. $\mathcal{W} = \{U(x) : x \in D\}$ covers X.

For any $y \in X$ find $m \in \mathbb{N}$ such that $y \in I(m)$. If $y \in J(m)$ then certainly $y \in U(y) \subseteq \bigcup\mathcal{W}$. If $y \notin J(m)$ then

$$y \in \left(\bigcup\{U(x) : i < m, x \in J(i)\}\right) \cup \left(\bigcup\{U(z) : x \in J(m), x < y\}\right) \subseteq \bigcup\mathcal{W}.$$

Claim 2. D is a closed discrete set in $\bigcup\mathcal{W}$ (and hence in X).

This follows if we show that for any $t \in D$, $D \setminus \{t\}$ is closed. To this end we may assume $Z = \bigcup\mathcal{W}$ and let $x \in Z \setminus (D \setminus \{t\})$. It suffices to find a weak neighborhood V of x such that $V \cap (D \setminus \{t\}) = \emptyset$. Let m be the first element of $\mathbb{N}$ such that there is a first element y of $J(m)$ where $x \in U(y)$. Now, for all $z \in D$ with $y < z$ we have $z \notin U(y)$. Also, for all $z \in D$ with $z < y$ we have $B(z, \frac{1}{m}) \subseteq B(z, \frac{1}{k_z}) \subseteq U(z)$ and $x \notin U(z)$ so $z \notin B(x, \frac{1}{m})$. That is, $U(y) \cap B(x, \frac{1}{m})$ is a weak neighborhood of x with $U(y) \cap B(x, \frac{1}{m}) \cap (D \setminus \{y\}) = \emptyset$. If $t = y$ we are done. If $t \neq y$ then since $y \in D$ we know $x \neq y$ and there is $j \in \mathbb{N}$ such that $y \notin B(x, \frac{1}{j})$. This gives $V = B(x, \frac{1}{j}) \cap U(y) \cap B(x, \frac{1}{m})$ as the weak neighborhood of x with $V \cap (D \setminus \{t\}) = \emptyset$. □

The following corollary will be partially superseded in the next section. We state it anyway since this did partially motivate the question about quotient s-images of metric spaces. This also includes the hereditary result.

Corollary 3.2. *The quotient compact-image of a metric space is a D-space (hereditarily).*

Proof: Recall that quotient compact-images of metric spaces are symmetrizable [A1], hence are D-spaces (hereditarily) by the previous theorem. □

A space X is said to be *irreducible* if every open cover $\mathcal{U}$ of X has an irreducible open refinement $\mathcal{V}$; that is, the open refinement $\mathcal{V}$ has no proper subcover. It is easy to show directly that a D-space is irreducible. It is not as easy to show that symmetrizable spaces are irreducible and showing subspaces (of symmetrizable spaces) are irreducible is more difficult. This may not have been previously known, so perhaps it is worthwhile to mention this as another corollary to Theorem 3.1.

Corollary 3.3. *A symmetrizable space is (hereditarily) irreducible.*

4. Quotient images of metric spaces

In this section we answer Arhangel'skii's question about the quotient s-images of metric spaces being D-spaces. Rather than working with the quotient mappings directly we use the notions of a weak-base and a w-system as defined below. This will give a stronger result and the desired theorem about quotient s-images of metric spaces will follow immediately.

Question (Arhangel'skii)**.** Are the quotient s-images of metric spaces D-spaces?

Definition. A *weak-base* [A1] for a space X is a collection of subsets $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ where, for all $x \in X$, $x \in \bigcap \mathcal{B}_x$, $\mathcal{B}_x$ is closed under finite intersections and $\mathcal{B}$ determines the topology on X in the following way: A set $U \subseteq X$ is open in X if and only if for all $z \in U$, there exists $B \in \mathcal{B}_z$ with $B \subseteq U$.

Recall that a space X is said to be *sequential* if and only if for every non-closed subset $A \subseteq X$ there exists a sequence $\langle x_n \rangle_{n\in\omega}$ in A which converges to some $z \in X \setminus A$. We can use the setting of a sequential space to discuss the notion of a weak-neighborhood of an element x without being given an entire weak-base for a topology.

Definition. If X is a sequential space and $x \in W \subseteq X$ we say W is a *weak-neighborhood* of x if whenever $\langle x_n \rangle_{n\in\omega}$ converges to x then $\langle x_n \rangle_{n\in\omega}$ is eventually in W.

The next proposition essentially says that in a sequential space the collection of weak-neighborhoods is a weak-base for X. We leave the proof to the reader.

Proposition 4.1. *If X is a sequential space then a subset $U \subseteq X$ is open if and only if for all $x \in U$ there exists a weak-neighborhood W of x such that $W \subseteq U$.*

Definition. A collection $\mathcal{W}$ of subsets of a sequential space X is said to be a *w-system* for the topology if whenever $x \in U \subseteq X$, with U open, there exists a subcollection $\mathcal{V} \subseteq \mathcal{W}$ such that $x \in \bigcap \mathcal{V}$, $\bigcup \mathcal{V}$ is a weak-neighborhood of x and $\bigcup \mathcal{V} \subseteq U$.

Remark. As noted below in Proposition 4.2, a w-system structure can arise naturally in the context of quotient spaces. A potential advantage in this setting is that the w-system $\mathcal{W}$ is now an internal structure (to the range space) which should contain all of the relevant topological information originally contained in the more cumbersome external structure of the quotient mapping. Besides the obvious topological information it is often desirable for a w-system to retain other attributes of a particular base (e.g., point-countability) — hence the intention that Proposition 4.2 be valid for "any base" on Z. It is clear that the entire topology τ on the domain space Z will induce a w-system on a quotient image but this may not be a very useful structure.

Proposition 4.2. *If $f : Z \to X$ is a quotient map from a space Z onto a T_2 sequential space X and $\mathcal{B}$ is any base for the topology on Z then $\mathcal{W} = \{f(B) : B \in \mathcal{B}\}$ is a w-system for X.*

PROOF: Let $x \in U \subseteq X$, with U open. We need to find a subcollection $\mathcal{V} \subseteq \mathcal{W}$ such that $x \in \bigcap \mathcal{V}$, $\bigcup \mathcal{V}$ is a weak-neighborhood of x, and $\bigcup \mathcal{V} \subseteq U$. In Z, let $\mathcal{C} = \{B \in \mathcal{B} : B \cap f^{-1}(x) \neq \emptyset$ and $B \subseteq f^{-1}(U)\}$ and let $\mathcal{V} = \{f(B) : B \in \mathcal{C}\}$. Clearly $x \in \bigcap \mathcal{V}$ and $\bigcup \mathcal{V} \subseteq U$. To show that $\bigcup \mathcal{V}$ is a weak-neighborhood of x suppose $\langle y_n \rangle_{n\in\omega}$ converges to x; then we need only show that $\langle y_n \rangle_{n\in\omega}$ is eventually in $\bigcup \mathcal{V}$. If this were not the case there would be a subsequence "missing" $\bigcup \mathcal{V}$ completely so without loss of generality we may assume $\{y_n : n \in \omega\} \cap \bigcup \mathcal{V} = \emptyset$. Since X is T_2 we see that $\{y_n : n \in \omega\} \cup \{x\}$ is closed in X and $f^{-1}(\{y_n : n \in \omega\} \cup \{x\})$ is closed in Z. Now, $f^{-1}(x) \subseteq \bigcup \mathcal{C}$ and $\bigcup\{f^{-1}(y_n) : n \in \omega\} \cap \bigcup \mathcal{C} = \emptyset$ implies that $\bigcup\{f^{-1}(y_n) : n \in \omega\}$ is a closed saturated set in Z. This says $\{y_n : n \in \omega\}$ is a closed set in X, a contradiction. $\square$

Unfortunately, the T_2 condition on X in the previous proposition cannot simply be removed. The next example gives a simple illustration of this.

Example 4.3. There is a countable metric space Z, a quotient map $f : Z \to X$ onto a T_1 space X, and a base $\mathcal{B}$ for Z such that $\{f(B) : B \in \mathcal{B}\}$ is not a w-structure for X.

PROOF: Let $Z = \{(k, \frac{1}{n}) \in \mathbb{R}^2 : k \in \omega, n \in \mathbb{N}\} \cup (\omega \times \{0\})$ with the topology inherited from $\mathbb{R}^2$. Notice that Z is the topological sum of countably many convergent sequences. Let X be the quotient map obtained from Z by identifying the set $H_k = \{(n, \frac{1}{k}) : n \in \mathbb{N}\}$ to an element $y_k \in X$, for every $k \in \mathbb{N}$, and by identifying the pair $A_m = \{(0, \frac{1}{m}), (m, 0)\}$ to an element $p_m \in X$, for every $m \in \mathbb{N}$. Let $f : Z \to X$ be the corresponding quotient map and let $\mathcal{B}$ be any base for Z such that whenever $(0,0) \in B \in \mathcal{B}$ then $B \subseteq \{(0,0)\} \cup \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$. If $x = f((0,0))$, notice that $y_n \to x$ but that $\{y_n : n \in \mathbb{N}\} \cap \bigcup\{f(B) : (0,0) \in B\} = \emptyset$. This shows that $\mathcal{W} = \{f(B) : B \in \mathcal{B}\}$ cannot be a w-system for X. $\square$

Theorem 4.4. *A sequential space X with a point-countable w-system is a D-space.*

PROOF: Let $\mathcal{W}$ be a point-countable w-system for X (and for each $x \in X$ let $\mathcal{W}_x$ denote $\{W \in \mathcal{W} : x \in W\}$). Suppose $\mathcal{U} = \{U(x) : x \in X\}$ is an open neighborhood assignment for X. For every $x \in X$ pick a subcollection $\mathcal{V}_x \subseteq \mathcal{W}_x$, with $x \in \bigcap \mathcal{V}_x$, such that $V(x) = \bigcup \mathcal{V}_x$ is a weak-neighborhood of x and $V(x) \subseteq U(x)$. For $t \in X$, let $\mathcal{H}_t$ denote the countable set $\{W \in \mathcal{W} : t \in W \in \bigcup_{x\in X} \mathcal{V}_x\}$. Consider $\mathcal{H}_t$ to be well-ordered with an order-type as a subset of ω. (Keep in mind that the elements of $\mathcal{H}_t$ may not be weak-neighborhoods of t even though these elements all contain t.) Identify the potential "centers" of elements of $H \in \mathcal{H}_t$ by letting $c(H) = \{x \in H : H \in \mathcal{V}_x\}$ and $C(t) = \bigcup\{c(H) : H \in \mathcal{H}_t\}$.

By a recursion process we will identify an ordinal μ, countable sets $A_\alpha \subseteq X$, for $\alpha < \mu$, and open sets $O_\alpha = \bigcup\{U(x) : x \in A_\alpha\}$ so that $\bigcup_{\alpha<\mu} O_\alpha = X$ and $D = \bigcup_{\alpha<\mu} A_\alpha$ is closed and discrete in X. (So $\{U(x) : x \in D\}$ would be the desired subcover of $\mathcal{U}$ witnessing the D-space property.)

For an ordinal β, assuming that A_α (and O_α), for all $\alpha < \beta$, have been defined, continue the process as follows:

If $\bigcup_{\alpha<\beta} O_\alpha = X$, we stop and let $\mu = \beta$.

If $\bigcup_{\alpha<\beta} O_\alpha \neq X$, pick some $z_\beta \in X \setminus \bigcup_{\alpha<\beta} O_\alpha$. Next, we find (by induction on ω) an increasing sequence $\langle F_n^\beta \rangle_{n\in\omega}$ of finite subsets of X, with the initial $F_0^\beta = \{z_\beta\}$, as follows:
Given that F_n^β is defined and $t \in F_n^\beta$, let

$$R(t) = \Big(C(t) \setminus \bigcup_{s\in F_n^\beta} U(s)\Big) \setminus \bigcup_{\alpha<\beta} O_\alpha \quad \text{and} \quad E_n^\beta = \big\{t \in F_n^\beta : R(t) \neq \emptyset\big\}.$$

For $t \in E_n^\beta$, let $k(t,n) = \min\{n, |\{W \in \mathcal{H}_t : R(t) \cap c(W) \neq \emptyset\}|\}$. Now, let $W_{t,i}$, $i = 1, 2, \ldots, k(t,n)$, be the first $k(t,n)$ elements of $\mathcal{H}_t$ such that $R(t) \cap c(W_{t,i}) \neq \emptyset$ and pick $x(t,i) \in R(t) \cap c(W_{t,i})$, for each i. We let

$$F_{n+1}^\beta = F_n^\beta \bigcup \{x(t,i) : t \in E_n^\beta, 1 \leq i \leq k(t,n)\}.$$

If some $E_n^\beta = \emptyset$ then $F_n^\beta = F_{n+1}^\beta = F_{n+2}^\beta = \cdots$. In any case, notice that the resulting F_m^β, $m \in \omega$, form an increasing sequence of finite sets. Now we let $A_\beta = \bigcup_{n\in\omega} F_n^\beta$.

That concludes the recursion process which defines the countable sets $A_\alpha \subseteq X$, for $\alpha < \mu$, and open sets $O_\alpha = \bigcup\{U(x) : x \in A_\alpha\}$. It is clear from the construction that $\bigcup_{\alpha<\mu} O_\alpha = X$. For later use, we state the following two crucial observations which follow from the construction above:

(a) If $\beta \leq \gamma < \mu$ and $0 \leq n < k < \omega$ then $\left(\bigcup_{\alpha<\beta} O_\alpha\right) \cap F_n^\gamma = \emptyset$ and $\left(\bigcup_{s\in F_n^\beta} U(s)\right) \bigcap F_k^\beta \setminus F_n^\beta = \emptyset$.

(b) If $t \in F_m^\beta \subseteq A_\beta$, for some $m \in \omega$, then $\mathcal{W}_t$ has been "revisited often enough" so that $C(t) \subseteq \bigcup_{\alpha\leq\beta} O_\alpha$.

It remains to be shown that $D = \bigcup_{\alpha<\mu} A_\alpha$ is closed and discrete in X. For contradiction, assume otherwise. X is sequential, so D not closed or not discrete implies the existence of an infinite sequence $\langle x_n \rangle_{n\in\omega}$ from D which converges to some $y \in X$. Let γ be the first ordinal where $y \in O_\gamma$ and let m be the smallest integer such that there exists $z \in F_m^\gamma$ where $y \in U(z)$.

Since $x_n \to y$ and $V(y)$ is a weak-neighborhood of y there is $k \in \omega$ such that $x_n \in V(y) \cap U(z)$ for all $n \geq k$. It follows from (a) above that all such x_n, for $n \geq k$, must appear either in F^γ_m (finite) or in $\bigcup_{\alpha<\gamma} A_\alpha$; hence there is some $\beta < \gamma$ and some $p > k$ such that $x_p \in A_\beta$. Now observation (b) above, says that $C(x_p) \subseteq \bigcup_{\alpha\leq\beta} O_\alpha$. However, there must be some $W' \in \mathcal{V}_y \cap \mathcal{H}_{x_p}$ and this puts $y \in C(x_p)$. Recall that $y \notin \bigcup_{\alpha\leq\beta} O_\alpha$ since $\beta < \gamma$ and γ is the first ordinal where $y \in O_\gamma$. This contradiction finishes the proof. □

Theorem 4.4 gives several corollaries. The first improves on the Arhangel'skii-Buzyakova result [ABuz] that a space with a point-countable base is a D-space.

Corollary 4.5. *Any space X with a point-countable weak-base is a D-space.*

Proof: A point-countable weak-base for X makes X a sequential space and this weak-base would certainly be a point-countable w-system. □

Corollary 4.6. *If Z is a metric space and $f : Z \to X$ is a quotient s-map onto a T_2 space X then X is a D-space.*

The above corollary answers the previously mentioned question by Arhangel'skii about whether quotient s-images of metric spaces are D-spaces. In fact, the following more general result holds.

Corollary 4.7. *If Z has a point-countable base and $f : Z \to X$ is a quotient s-map onto a T_2 space X then X is a D-space.*

Proof: The quotient image of a sequential space is sequential, so X is sequential. If $\mathcal{B}$ is a point-countable base for Z and $f^{-1}(x)$, $x \in X$, is any fiber of the map then $f^{-1}(x)$ separable implies that $\{B \in \mathcal{B} : B \cap f^{-1}(x) \neq \emptyset\}$ is countable. Now, Proposition 4.2 gives that $\mathcal{W} = \{f(B) : B \in \mathcal{B}\}$ is a point-countable w-system for X; an application of Theorem 4.4 concludes the argument. □

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