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Singular integral characterization of nonisotropic generalized BMO spaces

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Abstract. We extend a result of Coifman and Dahlberg [*Singular integral characterizations of nonisotropic H^p spaces and the F. and M. Riesz theorem*, Proc. Sympos. Pure Math., Vol. 35, pp. 231–234; Amer. Math. Soc., Providence, 1979] on the characterization of H^p spaces by singular integrals of $\mathbb{R}^n$ with a nonisotropic metric. Then we apply it to produce singular integral versions of generalized BMO spaces. More precisely, if T_λ is the family of dilations in $\mathbb{R}^n$ induced by a matrix with a nonnegative eigenvalue, then there exist $2n$ singular integral operators homogeneous with respect to the dilations T_λ that characterize BMO_φ under a natural condition on φ.

Keywords: singular integral, nonisotropic generalized BMO

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§1. Introduction

We consider in $\mathbb{R}^n$ the translation invariant quasi-distances generated by the nonisotropic dilations given by an $n \times n$ matrix A with some real eigenvalue. Spaces of functions of bounded mean oscillation (BMO), maximal and atomic Hardy spaces (H^p) and Lipschitz spaces (BMO_φ) are all well defined function spaces in this setting with Lebesgue measure. In the usual isotropic case, when A is the identity matrix, a deep characterization of H^1 is given by the Riesz singular integral transforms R_j, $j = 1, \dots, n$. A function f in L^1 belongs to H^1 if and only if $R_j f \in L^1$ for $j = 1, \dots, n$. Even when the atomic and maximal theories of H^p spaces are completely developed in general settings ([MS2]), the existence of enough singular integral operators in order to produce such a characterization in abstract contexts is largely an unsolved problem. A first attempt in this direction is the result by Coifman and Dahlberg [CD].

In this paper we extend the result of Coifman and Dahlberg to more general dilations and we apply it to the characterization of generalized BMO and Lipschitz spaces using our previous result in [C].

In Section 2 we introduce the spaces of homogeneous type and the function spaces, and we state the main result obtained in this article: the characterization

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of BMO_φ through singular integral operators. In Section 3 we present the Hardy space and show that, in the general setting of spaces of homogeneous type, a singular integral characterization of the maximal version of H^1 suffices to show singular integral characterizations of BMO. In Section 4 we prove results that allow us to obtain the nonisotropic version of the Fefferman-Stein Theorem. For a given matrix A with some real eigenvalue, in Section 5 we prove the existence of enough singular integrals in order to get the characterization of the maximal Hardy space.

§2. Statement of the results

Given a set X, a nonnegative symmetric function d defined on $X \times X$ is called a quasi-distance if $d(x, y) = 0$ if and only if $x = y$ and the following generalization of the triangle inequality holds for every x, y and $z \in X$ and some constant K,

$$d(x, z) \leq K\left(d(x, y) + d(y, z)\right).$$

A measure μ defined on the σ-algebra containing the d-balls, $B(x, r) = \{y : d(x, y) < r\}$, is said to satisfy the doubling property if

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)) < \infty,$$

for some constant A, every $x \in X$ and every $r > 0$.

If μ is a doubling measure we say, following [CW], that (X, d, μ) is a space of homogeneous type.

Hardy spaces, on spaces of homogeneous type, in their maximal and atomic approaches have been studied by Macías and Segovia in [MS2]. There, the basic structure is that of normal spaces. We shall say that (X, d, μ) is a normal space if there exist four positive constants A_1, A_2, K_1 and K_2 such that

$$\begin{aligned} A_1 r \leq \mu(B(x, r)) \leq A_2 r \qquad & \text{for } \ K_1\mu(\{x\}) \leq r \leq K_2\mu(X), \\ B(x, r) = X \qquad & \text{if } \ r > K_2\mu(X), \\ B(x, r) = \{x\} \qquad & \text{if } \ r < K_1\mu(\{x\}). \end{aligned}$$

It is clear that we may assume without loosing generality that $K_1 < 1 < K_2$. Moreover, in [MS1] it is proved that every quasi-distance d is equivalent to a quasi-distance d' of order β, i.e., there exist two constants C and $0 < \beta \leq 1$ such that

$$|d'(x, y) - d'(y, z)| \leq C r^{1-\beta} d'(x, z)^\beta,$$

for every x, y, z and r such that $d'(x, y) < r$ and $d'(y, z) < r$.

Let us now introduce the function spaces which concern us in this paper. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function satisfying the Δ_2 Orlicz's condition:

$\varphi(2r) \leq C\varphi(r)$ for some positive constant C and every $r > 0$ (see [KR]). Given a real function f defined on a space of homogeneous type (X, d, μ) we shall say that f satisfies the Lipschitz-φ condition and we shall write $f \in \Lambda_\varphi$ if there exists $C > 0$ such that

$$|f(x) - f(y)| \leq C\varphi(d(x, y)) \text{ for every } x, y \in X.$$

The infimum of those constants C is a semi-norm which added to the L^∞ norm gives a Banach space structure on Λ_φ. When $\varphi(t) = t^\beta$ for $0 < \beta \leq 1$, Λ_φ is the class of Lipschitz-β functions, which under the hypothesis of regularity of the measure μ, is dense in every L^p for $p < \infty$. Sometimes we shall write $\Lambda_\varphi(X, d)$ instead of Λ_φ to emphasize the role of the distance.

Let $f \in L^1_{\mathrm{loc}}$, i.e. $\int_B |f|\, d\mu < \infty$ for every ball B. We say that f is of φ-bounded mean oscillation and write $f \in \mathrm{BMO}_\varphi$ if there exists a constant C such that the inequality

$$\frac{1}{\mu(B)} \int_B |f - f_B|\, d\mu \leq C\varphi(r(B)),$$

holds for every ball B in X, where $r(B)$ is the radius of B and $f_B = \mu(B)^{-1} \int_B f\, d\mu$. If we identify two functions which differ by a constant, BMO_φ becomes a Banach space with the norm

$$\|f\|_{\mathrm{BMO}_\varphi} = \sup_B \frac{1}{\mu(B)\varphi(r(B))} \int_B |f - f_B|\, d\mu,$$

which is equivalent to $\sup_B \inf_{a\in\mathbb{R}} \mu(B)^{-1}\varphi(r(B))^{-1} \int_B |f - a|\, d\mu$. We shall use the notation $\mathrm{BMO}_\varphi(X, d, \mu)$ instead of BMO_φ in order to recall the particular structure of the underlying space of homogeneous type.

We denote by $\mathcal{S}$ the Schwartz class of functions and by $\mathcal{S}'$ the respective distribution space. Throughout this paper C will denote a positive constant, not necessarily the same at each occurrence.

To each $n \times n$ diagonalizable matrix A and each $\lambda > 0$, we associate the nonisotropic dilations whose matrix is given by $T_\lambda = e^{A \log \lambda}$ where $\lambda > 0$. Let us also assume, following [G], that the eigenvalues of A have a real part large enough in order to have a unique solution $\rho = \rho(x)$ of $\|T_{\frac{1}{\rho}}(x)\| = 1$. The function $\rho(x - y)$ becomes a translation invariant distance on $\mathbb{R}^n$. Moreover when μ is the Lebesgue measure on $\mathbb{R}^n$ we have that $(\mathbb{R}^n, \rho, \mu)$ is a space of homogeneous type. Given a ρ-ball $B = B(x_0, r)$ in $\mathbb{R}^n$, we have that $\mu(B) = Cr^\tau$ where $\tau = \sum_i^n a_{ii}$ is the trace of A. The function $d(x, y) = \rho^\tau(x - y)$ is a quasi-distance of order τ^{-1} on $\mathbb{R}^n$ and $(\mathbb{R}^n, d, \mu)$ becomes a normal space of homogeneous type.

A convolution operator R is said to be T_λ-homogeneous of degree m if $R(f \circ T_\lambda)(x) = \lambda^{-m-\tau}(Rf)(T_\lambda x)$.

The main result of this article is given in the next statement.

Theorem 2.1. *Let A be an $n\times n$ matrix with some real eigenvalue. Let T_λ be the nonisotropic family of dilations induced by A. Let φ be a nondecreasing function satisfying Δ_2 and the following growth condition*

$$r^\beta \int_r^\infty \frac{\varphi(t)}{t^{1+\beta}}\,dt \le C\varphi(r),$$

with $\beta=\tau^{-1}$. Then there exist $2n$ singular integral operators $\mathcal{R}_i$, $i=1,\dots,2n$, T_λ-homogeneous of degree $-\tau$ such that

$$\mathrm{BMO}_\varphi(\mathbb{R}^n,d,\mu)\subset \Lambda_\varphi+\sum_{i=1}^{2n}\mathcal{R}_i\Lambda_\varphi.$$

From Theorem II in [C], Theorem 2.1 will follow at once from the characterization of the $\mathrm{BMO}(\mathbb{R}^n,d,\mu)$ space as the next theorem states.

Theorem 2.2. *Let A be an $n\times n$ matrix with some real eigenvalue. Let T_λ be the nonisotropic family of dilations induced by A. Then there exist $2n$ singular integral operators $\mathcal{R}_i$, $i=1,\dots,2n$, T_λ-homogeneous of degree $-\tau$ such that*

$$\mathrm{BMO}(\mathbb{R}^n,d,\mu)=L^\infty+\sum_{i=1}^{2n}\mathcal{R}_iL^\infty.$$

This theorem will be a direct consequence of the characterization of the nonisotropic Hardy space H^1 in terms of $2n$ singular integral operators, that we will present in the next section and prove in the rest of the paper.

§3. Hardy and BMO spaces

Following Macías and Segovia [MS2] let us denote by E^α the space of all functions with bounded supports belonging to Λ_β, for every $0<\beta<\alpha$. We shall say that a linear functional f on E^α is a distribution on E^α if it is continuous when E^α is endowed with the inductive limit topology of the Λ_β with compact support. For a space of homogeneous type (X,d,μ), γ a number such that $0<\gamma<\alpha$, and x in X, we introduce a class $D_\gamma(x)$ which will allow us to define maximal functions of distributions on E^α. We shall say that a function ψ belonging to E^α is in $D_\gamma(x)$ if there exists r such that $r\ge K_1\mu(\{x\})$, the support of ψ is contained in $B(x,r)$, $r\|\psi\|_\infty\le 1$ and $r^{1+\gamma}\|\psi\|_\gamma\le 1$, where $\|\psi\|_\gamma=\inf C$ such that $|\psi(x)-\psi(y)|\le Cd(x,y)^\gamma$. Let f be a distribution on E^α and $0<\gamma<\alpha$. We define the γ-maximal functions $f^*_\gamma(x)$ of f as

$$f^*_\gamma(x)=\sup\{|\langle f,\psi\rangle| : \psi\in D_\gamma(x)\}.$$

In [MS2] Macías and Segovia give the atomic decomposition of the maximal Hardy space H^p: let f be a distribution on E^α such that for some γ, $0 < \gamma < \alpha$, and some p, $(1+\gamma)^{-1} < p \leq 1$, its γ-maximal function $f^*_\gamma(x)$ belongs to L^p. Then there exists a sequence of p-atoms, $\{a_n(x)\}$, and a numerical sequence, $\{\lambda_n\}$, such that $f = \sum_n \lambda_n a_n$ strongly in the dual space of E^α. They actually prove that the atomic and maximal H^p spaces are equivalent. Once we have the atomic decomposition we can prove the duality between the spaces H^1 and BMO in the context of spaces of homogeneous type. This last result allows us to extend the argument in [FS] in order to prove that if for a space of homogeneous type (X, d, μ) we have that there exist $\mathcal{R}_i$, $i = 1, \dots, m$ singular integral anti-hermitian operators such that $H^1(X, d, \mu) = \{f \in L^1(X) : \mathcal{R}_i f \in L^1(X), i = 1, \dots, m\}$ then $\mathrm{BMO}(X, d, \mu) = L^\infty + \sum_{i=1}^m \mathcal{R}_i L^\infty$. Here the expression anti-hermitian means that $\int(\mathcal{R}_i f)\varphi = -\int f(\mathcal{R}_i\varphi)$ for $f \in H_1$ and $\varphi \in L^\infty$. Of course $L^\infty + \sum_{i=1}^m \mathcal{R}_i L^\infty \subset \mathrm{BMO}$, since $\mathcal{R}_i$ $i = 1, \dots, m$ are standard singular integral operators. In order to prove the opposite inclusion let B be the Banach space which consists of the direct sum of $m+1$ copies of $L^1(X)$ with the following norm $\|(f_0, f_1, \dots, f_m)\| = \sum_{j=0}^m \|f_j\|_1$. Let S be the subspace of B with $f_j = \mathcal{R}_j(f_0)$, $j = 1, \dots, m$. S is a closed subspace of B and the mapping $f_0 \to (f_0, \mathcal{R}_1 f, \dots, \mathcal{R}_n f_0)$ is a Banach space isometry of H^1 to S. Then any continuous linear functional on H^1 can be identified with a corresponding functional defined on S, and hence by the Hahn-Banach theorem, it extends to a continuous linear functional on B. Now $B = L^1 \oplus L^1 \dots \oplus L^1$ and thus the dual of B is equivalent to $L^\infty \oplus L^\infty \dots \oplus Ł^\infty$. Restricting our attention to S (and hence H^1) we get the following conclusion. Any $g \in \mathrm{BMO}(X, d, \mu)$ defines a linear and continuous functional on H^1 by $l(f) = \int_X gf\, d\mu$. Then there exists $\varphi_0, \varphi_1, \dots, \varphi_m \in L^\infty$, such that

$$l(f) = \sum_{j=0}^m \int_X f_j \varphi_j \, dx, \quad \text{where} \quad f = f_0, \quad \text{and} \quad f_j = \mathcal{R}_j(f), \quad j = 1, \dots, m.$$

Now the anti-hermitian character of the singular integral operators gives us the desired result since

$$l(f) = \int_X fg = \int_X f \left\{ \varphi_0 - \sum_{j=1}^n \mathcal{R}_j(\varphi_j) \right\} dx,$$

for every $f \in H^1$. Thus Theorem 2.2 and hence Theorem 2.1 will be a consequence of the next result.

Theorem 3.1. *Let A be an $n \times n$ matrix with some real eigenvalue. Let T_λ be the nonisotropic family of dilations induced by A. Then there exist $2n$ singular integral operators $\mathcal{R}_i$, $i = 1, \dots 2n$, T_λ-homogeneous of degree $-\tau$ such that a function f belongs to the maximal version of the Hardy space $H^1(\mathbb{R}^n, d, \mu)$ if and only if $\mathcal{R}_i f \in L^1$ for every $i = 1, \dots, 2n$.*

§4. A characterization of H^1

In this section we consider $A = (a_{ij})$, an $n \times n$ matrix in Jordan canonical form with $a_{11} = 1$. Let τ be the trace of A and ρ the nonisotropic metric associated to A.

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. For $S \in S^{n-1}$ we consider the subset $\Gamma = \{T_\lambda \xi \ / \ \xi \in S \subset S^{n-1}, \lambda > 0\}$ of $\mathbb{R}^n$. Let φ be a function belonging to the Schwartz class $\mathcal{S}(\mathbb{R})$ such that, $\widehat{\varphi}(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\widehat{\varphi}(x) = 0$ if $|x| > 1$. It is easy to see, for $\varphi_\mu(x) = \mu\varphi(\mu x)$, that $\widehat{\varphi}_\mu(t) = \widehat{\varphi}(\frac{t}{\mu})$.

Let ζ be a C^∞ function defined on S^{n-1} such that $\zeta \equiv 1$ on $S \subset S^{n-1}$ and $\zeta \equiv 0$ outside a given open neighborhood of S. Let us now consider the function Θ T_λ-homogeneous of degree zero such that $\Theta|_{S^{n-1}} = \zeta$. In other words $\Theta(\xi) = \zeta(T_{\frac{1}{\rho(\xi)}}\xi)$ with $\xi \in \mathbb{R}^n - \{0\}$.

For each $\lambda > 0$ let us define a function ϕ_λ such that

$$\widehat{\phi}_\lambda(\xi) = \widehat{\varphi}(\frac{\rho(\xi)}{\lambda})\Theta(\xi).$$

Notice that the right hand side of the above expression belongs to $L^1(\mathbb{R}^n)$, since it is bounded and supported on a bounded set. Now we obtain the following result.

Lemma 4.1. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that* $\operatorname{supp} \widehat{f} \subset \Gamma$. *Then for any $\lambda > 0$ and $F_\lambda(\xi) = \phi_\lambda * f(\xi)$ we have $\widehat{F}_\lambda(\xi) = (\widehat{\varphi}_\lambda \circ \rho)\widehat{f}$. Moreover if μ is a positive number with $2\mu < \lambda$ then $F_\mu = \phi_\mu * F_\lambda$.*

Proof: The first statement follows using the fact that $\Theta \equiv 1$ on Γ and $\widehat{f} \equiv 0$ out of Γ. For the other statement it is enough to see that if ξ is such that $\widehat{F_\mu}(\xi)$ is not zero then $\widehat{\phi_\lambda}(\xi) = 1$, in fact let $\xi \in \Gamma$ (this implies $\widehat{f}(\xi) \neq 0$) be such that $\frac{\rho(\xi)}{\mu} \leq 1$ (this implies that $\widehat{\phi_\mu}(\xi) \neq 0$); then we have that $\Theta(\xi) = 1$ and, for our choice of μ, we have that $\frac{\rho(\xi)}{\lambda} \leq \frac{\rho(\xi)}{2\mu} \leq \frac{1}{2}$ hence $\widehat{\varphi}(\frac{\rho(\xi)}{\lambda}) = 1$ and the result follows. □

In this form F_μ is a convolution operator with kernel $K_\mu(\xi) = (\widehat{\varphi_\mu} \circ \rho)^\vee(\xi)$ belonging to $\mathcal{S}(\mathbb{R}^n)$.

Associated with the metric ρ and the dilations T_λ, $\lambda > 0$ one can define, following [G], in a natural way a system of polar coordinates through the following mapping $x \in \mathbb{R}^n - \{0\} \mapsto (x', \rho(x)) \in S^{n-1} \times (0, \infty)$ where $x' = T_{\frac{1}{\rho(x)}}x$. It is not difficult to see that the integral in $\mathbb{R}^n$ can be expressed in this system of polar coordinates in the following way

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_{\rho=0}^{\infty} \int_{x' \in S^{n-1}} f(T_\rho x')\ |\langle Ax', x'\rangle|\,dx'\rho^{\tau-1}d\rho.$$

Here dx' means the ordinary Lebesgue measure on the unit sphere. For simplicity we use the notation $|\langle Ax', x'\rangle| dx' = d\sigma(x')$. This change of variable and the fact that $T_{st}(x) = T_s \circ T_t(x)$ allow us to prove, in an easy way, that $K_\mu(\xi) = \mu^\tau K(T^*_\mu \xi)$, where T^* is the adjoint operator and $K(\xi) = (\widehat{\varphi} \circ \rho)^\vee(\xi)$.

For $y' \in S^{n-1}$, $\mu > 0$ and $r > 0$ we denote

$$\mathcal{K}^{y'}_\mu(r) = K_\mu(T_r y') r^{\tau-1} = r^{\tau-1}\mu^\tau K \circ T^*_\mu(T_r y')$$

and

$$\Psi^{y'}_\mu(r) = r^{\tau-1} K(T^*_\mu T_{\frac{1}{\mu}} T_r y') = r^{\tau-1}(K \circ T^*_\mu)(T_{\frac{r}{\mu}} y'),$$

therefore $\mathcal{K}^{y'}_\mu(r) = \mu \Psi^{y'}_\mu(\mu r)$.

For $\lambda > 0$, $y' \in S^{n-1}$ and $\xi_2 \in \mathbb{R}^{n-1}$ we consider the real function

$$F^{y',\xi_2}_\lambda(r) = F_\lambda((T_r y')_1, \xi_2),$$

where $(T_r y')_1$ denotes the first coordinate of $T_r y'$ and F_λ is defined in the same way that the Lemma 4.1 with $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $\operatorname{supp} \widehat{f} \subset \Gamma$. For these functions we obtain the following result.

Lemma 4.2. (i) *The function $F^{y',\xi_2}_\lambda(r)$ has an analytic extension to the upper half plane, i.e. $F^{y',\xi_2}_\lambda(z)$ is analytic in $\operatorname{Im} z > 0$.*

(ii) *Let s, r, s_0 be such that $|s_0 - r| \geq 2|s_0 - s|$. Then there exists a positive constant ϵ such that for $C = C(\mu, \tau)$ we have*

$$|\Psi^{y'}_\mu(\mu(s - r)) - \Psi^{y'}_\mu(\mu(s_0 - r))| \leq C \frac{|s_0 - s|^\epsilon}{(1 + |s_0 - r|)^{n+\epsilon}}.$$

Proof: For the first statement, we observe that $\widehat{\phi_\lambda}\widehat{f}$ is a smooth function with compact support in $\Gamma \cap \{\rho(\xi) \leq \lambda\}$; hence from Paley-Wiener theory we have that $F_\lambda(\xi)$ is an analytic function in all variables. Therefore we must prove that $F^{y',\xi_2}_\lambda(z) = F_\lambda((T_z y')_1, \xi_2)$ is an analytic function in $\operatorname{Im} z > 0$. For each $\lambda > 0$, $y' \in S^{n-1}$ and $\xi_2 \in \mathbb{R}^{n-1}$, using that $\widehat{\phi_\lambda}\widehat{f}$ is smooth enough, we can see that $F^{y',\xi_2}_\lambda(z)$ is a continuous function. Let γ be a triangular path in $\mathbb{R}^{n+1}_+$ and such that the first eigenvalue of A is one. We have that $(T_z y')_1 = z y'_1$; this one and the fact that the exponential is a analytic function allow us to obtain that

$$\begin{aligned}\int_\gamma F^{y',\xi_2}_\lambda(z)\, dz &= \int_\gamma \int e^{iu((T_z y')_1, \xi_2)} \widehat{\phi_\lambda}(u) \widehat{f}(u)\, du\, dz \\ &= \int \widehat{\phi_\lambda}(u)\widehat{f}(u) \int_\gamma e^{i(z y'_1 u_1 + \xi_2 u_2)}\, dz\, du = 0,\end{aligned}$$

therefore $F_\lambda^{y',\xi_2}(z)$ is an analytic function.

In order to prove (ii), it is enough to see that there exists a positive constant ϵ such that

$$|\Psi_\mu^{y'}(\mu(r-s)) - \Psi_\mu^{y'}(\mu r)| \leq C\frac{s^\epsilon}{(1+r)^{n+\epsilon}},$$

for $2s < r$ and $C = C(\mu,\tau)$. If we denote by η the function in $\mathcal{S}(\mathbb{R}^n)$ given by $\eta = K \circ T_\mu^*$ then

$$\begin{aligned}|\Psi_\mu^{y'}(\mu(r-s)) - \Psi_\mu^{y'}(\mu r)| &= |(\mu(r-s))^{\tau-1}\eta(T_{r-s}y') - (\mu r)^{\tau-1}\eta(T_r y')| \\ &\leq \mu^{\tau-1}\left[(r-s)^{\tau-1}|\eta(T_{r-s}y') - \eta(T_r y')|\right] \\ &\quad + \mu^{\tau-1}|\eta(T_r y')||(r-s)^{\tau-1} - r^{\tau-1}| \\ &= \mu^{\tau-1}[(I) + (II)].\end{aligned}$$

We analyze each term separately. As $\eta \in \mathcal{S}$ for every $k \in \mathbb{N}$ there exists a constant C such that $|\eta(T_r y')| \leq \frac{C}{(1+\|T_r y'\|)^k}$. The fact that $\|T_r y\| \geq \|y\| r$, taken $k = [\tau + n]$, where $[x]$ denote the integer part function, gives us

$$\begin{aligned}(II) &\leq C\frac{r^{\tau-2}s}{(1+r)^k} \\ &\leq C\frac{s}{(1+r)^{n+1}}.\end{aligned}$$

For (I) we observe that

$$|\eta(T_{r-s}y') - \eta(T_r y')| = s\ \|\frac{d}{dr}(\eta(T_r(y')))(r_0)\|,$$

for r_0 such that $r - s < r_0 < r$, therefore $\frac{r}{2} < r_0 < r$. Using again the fact that η is a function in the Schwartz class we have that for every $k \in \mathbb{N}$ there exists a constant C such that $|\frac{\partial\eta}{\partial x_i}(T_R y')| \leq \frac{C}{(1+\|T_R y'\|)^k}$, for each $i = 1, \ldots, n$, and therefore $\|\nabla\eta(T_R y')\| \leq \frac{C}{(1+\|T_R y'\|)^k}$. This estimate, the fact that T_r is a nondecreasing function of r and the equivalence between r_0 and r allow us to obtain that

$$\begin{aligned}\|\frac{d}{dr}(\eta(T_r(y')))(r_0)\| &\leq \|\nabla\eta(T_{r_0}y')\|\frac{1}{r_0}\|A\|\ \|e^{A\log r_0}\| \\ &\leq \|\nabla\eta(T_{r_0}y')\|\frac{1}{r_0}\|A\|\ \|T_r\| \\ &\leq \frac{C\|A\|\ \|T_r\|}{r}\frac{1}{(1+r)^k}.\end{aligned}$$

Finally, choosing $k = [\tau + n + \|A\|]$, we obtain that

$$\begin{aligned}(I) &\le Cr^{\tau-2}s\|A\|\ \|T_r\|\frac{1}{(1+r)^k}\\ &\le Cr^{\tau-2}s\frac{\|A\|\ r^{\|A\|}}{(1+r)^k}\\ &\le Cs\frac{r^{\tau-2+\|A\|}}{(1+r)^k}\\ &= C\frac{s}{(1+r)^{n+1}},\end{aligned}$$

and the result follows. □

Lemma 4.3 (Fefferman-Stein, nonisotropic version) [FS]**.** *If $\xi_2 \in \mathbb{R}^{n-1}$, $y' \in S^{n-1}$, and λ and μ are real numbers then*

$$\|\mathcal{K}_\mu^{y'} * F_\lambda^{y',\xi_2}\|_{L^1} \le c(\mu, y')\|F_\lambda^{y',\xi_2}\|_{H_1}.$$

Proof: From the duality between H_1 and BMO in spaces of homogeneous type, it is enough to prove, for $g \in L^\infty(\mathbb{R}^n)$, that the function $\phi_\mu^{y',\xi_2} = \mathcal{K}_\mu^{y'} * g^{y',\xi_2}$ belongs to BMO, where $g^{y',\xi_2}(r) = g((T_r y')_1, \xi_2)$. Let $I = I_h = [s_0 - \frac{h}{2}, s_0 + \frac{h}{2}]$ and let I_{2h} be an concentric interval with length $2h$. Decompose $g^{y',\xi_2} = g^1 + g^2$, with $g^1 = g^{y',\xi_2}\chi_{I_{2h}}$. This decomposition induces two functions ϕ_1 and ϕ_2 where $\phi_i = g^i * \mathcal{K}_\mu^{y'}$, $i = 1, 2$. We will prove that each one belongs to BMO, in fact

$$\begin{aligned}\int_I |\phi_1(s)|ds &\le \int_{\mathbb{R}} |\phi_1(s)|\,ds\\ &\le \int_{\mathbb{R}}\int_{\mathbb{R}} \mu(\mu r)^{\tau-1}|K(T_\mu^* T_r y')||g^1(s-r)|\,dr\,ds\\ &= \int_{\mathbb{R}} \mu(\mu r)^{\tau-1}|K(T_\mu^* T_r y')|\left(\int_{\mathbb{R}} |g^1(s-r)|\,ds\right)dr\\ &\le \|g^{y',\xi_2}\|_\infty |I_{2h}| \int_{\mathbb{R}} \mu(\mu r)^{\tau-1}|K(T_\mu^* T_r y')|\,dr.\end{aligned}$$

If we denote $G(\mu, y') = \int_{\mathbb{R}} \mu(\mu r)^{\tau-1}|K(T_\mu^* T_r y')|\,dr$, this function is finite for almost every $y' \in S^{n-1}$; in fact, using again the change of variable $y = T_r y'$ and observing that the kernel $K \in L^1(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{S^{n-1}}\int_{r\in\mathbb{R}} \mu(\mu r)^{\tau-1}|KT_\mu^* T_r y'|\,dr\,d\sigma(y') &= \int_{\mathbb{R}^n} \mu^\tau |K(T_\mu^* x)|\,dx\\ &= C(\mu)\|K\|_1.\end{aligned}$$

Therefore we have

$$\frac{1}{|I|}\int_I |\phi_1(s)|\,ds \le c\|g^{y',\xi_2}\|_\infty G(\mu,y').$$

For the other function we consider

$$a_I = \int_{I^c_{2h}} \mathcal{K}^{y'}_\mu(s_0-r)g^2(r)\,dr = \int_{I^c_{2h}} \mu\Psi^{y'}_\mu(\mu(s_0-r))g^2(r)\,dr.$$

Taking $s \in I_h$ and using Lemma 4.2 we obtain that

$$\begin{aligned}
|\phi_2(s)-a_I| &\le \int_{I^c_{2h}} \mu|g^2(r)||\Psi^{y'}_\mu(\mu(s-r))-\Psi^{y'}_\mu(\mu(s_0-r))|\,dr\\
&\le \mu\|g^{y',\xi_2}\|_\infty \int_{|s_0-r|>2h} |\Psi^{y'}_\mu(\mu(s-r))-\Psi^{y'}_\mu(\mu(s_0-r))|\,dr\\
&\le C\mu\|g^{y',\xi_2}\|_\infty \int_{|s_0-r|>2h} \frac{|s_0-s|^\epsilon}{(1+|s_0-r|)^{n+\epsilon}}\,dr\\
&\le C\mu\|g^{y',\xi_2}\|_\infty |s_0-s|^\epsilon \int_{|s_0-r|>2h} \frac{1}{|s_0-r|^{n+\epsilon}}\,dr\\
&\le C\mu\|g^{y',\xi_2}\|_\infty \frac{h^\epsilon}{\epsilon(2h)^\epsilon}\\
&= C\|g^{y',\xi_2}\|_\infty.
\end{aligned}$$

Then we have

$$\frac{1}{|I|}\int_I |\phi_2(s)-a_I| \le C\|g^{y',\xi_2}\|_\infty.$$

This statement together with that one obtained for ϕ_1 give us

$$\frac{1}{|I|}\int_I |\phi^{y',\xi_2}_\mu - a_I| \le C\|g^{y',\xi_2}\|_\infty(1+G(\mu,y')),$$

and the proof follows. □

Now, let us consider an appropriate change of variable, which together with the last result allow us to arrive at the principal result of this section.

For $y' \in S^{n-1}$, $r \in \mathbb{R}$ and $x=(x_1,x_2)\in\mathbb{R}^n$ with $x_1\in\mathbb{R}$ and $x_2\in\mathbb{R}^{n-1}$, let, as before, $(T_r y)_1$ be the first coordinate of $T_r y$ and $(T_r y)_2$ the others. Consider the following change of variables,

$$\begin{cases} x_1 = (T_{\xi_1-r}y')_1 + (T_r y')_1\\ x_2 = \xi_2 + (T_r y')_2.\end{cases}$$

Using the fact that $(T_\lambda y)_1 = \lambda y_1$, we obtain

$$x - T_r y' = ((T_{\xi_1-r}y')_1,\xi_2) = ((\xi_1-r)y'_1,\xi_2).$$

The determinant of the jacobian matrix of this change of variable is y'_1.

Theorem 4.4. *Let A be an $n \times n$ matrix with some real eigenvalue. Let f be a distribution in $\mathcal{S}'(\mathbb{R}^n)$ such that* $\operatorname{supp} \widehat{f} \subset \Gamma$. *Then there exists a constant C such that*

$$\| \sup_{\lambda>0} |\phi_\lambda * f| \|_{L^1} \leq C \lim_{\lambda\to\infty} \|\phi_\lambda * f\|_{L^1}.$$

Proof: For a fixed number N, we consider λ such that $\lambda > 2N$ (this implies that $2\mu < \lambda$ for all $\mu \leq N$). Then from Lemma 4.1 and 4.2 and using the previous two changes of variables we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{\mu\leq N} |F_\mu(x)|\, dx = \int_{\mathbb{R}^n} \sup_{\mu\leq N} |K_\mu * F_\lambda(x)|\, dx \\
&= \int_{\mathbb{R}^n} \sup_{\mu\leq N} \left| \int_{\mathbb{R}^n} K_\mu(y) F_\lambda(x-y)\, dy \right| dx \\
&\leq \int_{\mathbb{R}^n} \int_{S^{n-1}} \sup_{\mu\leq N} \left| \int_{\mathbb{R}} K_\mu(T_r y') F_\lambda(x - T_r y') r^{\tau-1}\, dr \right| d\sigma(y')\, dx \\
&= \int_{S^{n-1}} \int_{\xi_2\in\mathbb{R}^{n-1}} \int_{\xi_1\in\mathbb{R}} |y_1'| \sup_{\mu\leq N} \left| \int_{\mathbb{R}} \mathcal{K}_\mu^{y'}(r) F_\lambda((T_{\xi_1-r}y')_1, \xi_2)\, dr \right| d\xi_1\, d\xi_2\, d\sigma(y') \\
&= \int_{S^{n-1}} |y_1'| \int_{\xi_2\in\mathbb{R}^{n-1}} \int_{\xi_1\in\mathbb{R}} \sup_{\mu\leq N} \left| \int \mathcal{K}_\mu^{y'}(r) F_\lambda^{y',\xi_2}(\xi_1 - r)\, dr \right| d\xi_1\, d\xi_2\, d\sigma(y') \\
&= \int_{S^{n-1}} |y_1'| \int_{\xi_2\in\mathbb{R}^{n-1}} \| \sup_{\mu\leq N} |\mathcal{K}_\mu^{y'} * F_\lambda^{y',\xi_2}| \|_1\, d\xi_2\, d\sigma(y').
\end{aligned}$$

Therefore, using Lemma 4.3, we obtain

$$\begin{aligned}
\| \sup_{\mu\leq N} \mathcal{K}_\mu^{y'} * F_\lambda^{y',\xi_2} \|_1 &\leq C(1 + G(\mu, y')) \|F_\lambda^{y',\xi_2}\|_{H_1} \\
&= C(1 + G(\mu, y')) \sup_{y>0} \int_{\mathbb{R}} |F_\lambda^{y',\xi_2}(x + iy)|\, dx \\
&\leq C(1 + G(\mu, y')) \int_{\mathbb{R}} |F_\lambda^{y',\xi_2}(x)|\, dx,
\end{aligned}$$

and then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sup_{\mu\leq N} |F_\mu(x)|\, dx \leq C \int_{S^{n-1}} |y_1'| \int_{\xi_2\in\mathbb{R}^{n-1}} (1 + G(\mu, y')) \|F_\lambda^{y',\xi_2}\|_1\, d\xi_2\, d\sigma(y') \\
&= C \int_{S^{n-1}} (1 + G(\mu, y')) \int_{\xi_2\in\mathbb{R}^{n-1}} |y_1'| \int_{\xi_1} |F_\lambda((T_{\xi_1}y')_1, \xi_2)|\, d\xi_1\, d\xi_2\, d\sigma(y') \\
&= C \int_{S^{n-1}} (1 + G(\mu, y')) \int_{\mathbb{R}^n} |F_\lambda(x)|\, dx\, d\sigma(y').
\end{aligned}$$

In the proof of Lemma 4.3, we have obtained that $\int_{S^{n-1}}(1+G(\mu,y')) < \infty$, hence $\int_{\mathbb{R}^n} \sup_{\mu\le N} |F_\mu(x)|\,dx \le C\int_{\mathbb{R}^n} |F_\lambda(x)|\,dx$, for $\lambda > 2N$ and then

$$\int_{\mathbb{R}^n} \sup_{\mu\le N} |F_\mu(x)|\,dx \le C \lim_{\lambda\to\infty} \|\phi_\lambda * f\|_1.$$

Taking the limit as N goes to infinity we obtain the desired inequality. □

§5. Proof of Theorem 3.1

Let us first reduce the problem to a matrix in a Jordan form.

Lemma 5.1. *Let A be an $n\times n$ matrix and $J = PAP^{-1}$ its Jordan canonical matrix. If we denote by BMO_J and BMO_A the BMO related to the metrics ρ_J and ρ_A, and by T^A_λ and T^J_λ the nonisotropic dilations associated to the matrix A and J, respectively, then*

(i) *$f\in \mathrm{BMO}_J$ if and only if $f\circ P \in \mathrm{BMO}_A$;*
(ii) *if for $f\in\mathrm{BMO}_J$ there exist m singular integral operators $\mathcal{R}_i$, $i=1,\dots,m$ T^J_λ-homogeneous of degree $-\tau$ and functions g_i, $i=1,\dots,m$ belonging to L^∞ such that $f=\sum_{i=1}^m \mathcal{R}_i g_i$ then $f\circ P$ have a similar decomposition.*

Proof: Let B_A and B_J be the balls for the metrics associated to A and J respectively. We observe that $PB_A = \{Pz/\rho_A(z) < r\} = \{u/\rho_A(P^{-1}u) < r\}$, with r the B_A ratios. For $x\in\mathbb{R}^n$ we consider the norm $|||x||| = \|Px\|$, and for a matrix A we take the metric $\bar\rho_A$ such that $|||T^A_{\frac{1}{\bar\rho_A(x)}}x||| = 1$. Then

$$\begin{aligned}
1 &= |||T^A_{\frac{1}{\bar\rho_A(P^{-1}u)}} P^{-1}u|||\\
&= \|Pe^{A\log\frac{1}{\bar\rho_A(P^{-1})}}P^{-1}u\|\\
&= \|e^{J\log\frac{1}{\bar\rho_A(P^{-1}u)}}u\|\\
&= \|T^J_{\frac{1}{\bar\rho_A(P^{-1}u)}}u\|.
\end{aligned}$$

By uniqueness we have that $\rho_J u = \bar\rho_A(P^{-1}u)$. The equivalence between the norms in $\mathbb{R}^n$ implies an equivalence between the metrics $\bar\rho_A$ and ρ_A and then we have that PB_A is equivalent to $\{u/\bar\rho_A(P^{-1}u) < r\} = \{u/\rho_J(u) < r\} = B_J$, and therefore the families PB_A and B_J are equivalent. Using the change of variable $Px = y$ we obtain (i). For the other statement, let f be a function in BMO_J such that $f=\sum_{i=1}^m \mathcal{R}_i g_i$ with $\mathcal{R}_i$ singular integral operators and $g_i\in L^\infty$. Then we have that $f \circ P(x) = \sum_i \mathcal{R}_i g_i(Px)$. Each term in the sum satisfies that

$$\begin{aligned}\mathcal{R}g(Px) &= \int k(Px-y)g(y)\,dy\\ &= \int k\circ P(x-z)g(Pz)\frac{dz}{|\det P|^{-1}}\\ &= \tilde{\mathcal{R}}\tilde{g}(x),\end{aligned}$$

with $\tilde{\mathcal{R}}$ the singular integral operator with kernel $k\circ P$ and $\tilde{g} = g\circ P$. It is clear that $g\circ P \in L^\infty$ if $g\in L^\infty$. Now we analyze the homogeneity of $\tilde{k}$.

$$\begin{aligned}\tilde{k}(T_\lambda^A(x)) = k(PT_\lambda^A x) &= k(T_\lambda^J Px)\\ &= \lambda^\tau k\circ P(x)\\ &= \lambda^\tau \tilde{k}(x),\end{aligned}$$

and the result follows. □

Proof of Theorem 3.1: The proof follows the lines of that one by Coifman and Dahlberg [CD]. We can find a C^∞ partition of unity on S^{n-1}, consisting of $2n$ functions $w_i(\xi)$ having the following properties

$$\begin{aligned} e_i &\in \operatorname{sop} w_i \subset \{\xi\in S^{n-1}, \xi_i>0\} &&\text{for}\quad i=1,\dots,n,\\ -e_i &\in \operatorname{sop} w_{n+j} \subset \{\xi\in S^{n-1}, \xi_i<0\} &&\text{for}\quad i=n+j,\quad j=1,\dots,n;\end{aligned}$$

here e_i denotes the standard orthonormal basis of $\mathbb{R}^n$. If we now extend w_i by homogeneity, i.e. $w_i(\xi) = w_i(T_{\frac{1}{\rho(\xi)}}\xi)$, we can define the operator

$$\mathcal{R}_i f = \left(w_i(\xi)\widehat{f}(\xi)\right)^\vee.$$

By construction, $w_i(\xi)\widehat{f}(\xi)$ is supported in a region of the type Γ considered in Theorem 4.4, hence

$$\|\sup_\lambda |\phi_\lambda * \mathcal{R}_i f|\,\|_1 \le C \lim_{\lambda\to\infty}\|\phi_\lambda * \mathcal{R}_i f\|_1 \le \|\mathcal{R}_i f\|_1.$$

Therefore, we obtain one part of the equivalence using the fact that $f = \sum_i \mathcal{R}_i f$. The other part follows using a more general result on singular integral operator in spaces of homogeneous type, as a consequence of the result in [MS2]. □

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