Comment.Math.Univ.Carolin. 48,2 (2007)291–301

Eberlein spaces of finite metrizability number

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Abstract. Yakovlev [*On bicompacta in Σ-products and related spaces*, Comment. Math. Univ. Carolin. **21.2** (1980), 263–283] showed that any Eberlein compactum is hereditarily σ-metacompact. We show that this property actually characterizes Eberlein compacta among compact spaces of finite metrizability number. Uniformly Eberlein compacta and Corson compacta of finite metrizability number can be characterized in an analogous way.

Keywords: metrizability number, Eberlein compactum, separating family

Classification: Primary 54A38, 54A15; Secondary A15

A compact space is called an *Eberlein compactum* if it is homeomorphic to a weakly compact subset of some Banach space. Answering a question of Arhangel'skii, E. Michael and M.E. Rudin showed that any compact space which is the union of two metrizable subspaces is an Eberlein compactum (cf. [10]). Upon noticing there are easy examples of compact spaces that can be represented as the union of three metrizable subspaces which are not even Corson compacta, neither them nor anybody else (to our best knowledge) have pursued this line of research. The aim of this paper is to determine which compact spaces of finite metrizability number, i.e. representable as unions of finitely many metrizable subspaces, are Eberlein.

The paper is organized as follows. In Section 1, we characterize internally open subspaces of Eberlein compacta as well as other Eberlein type compacta, like uniform Eberlein or Corson compacta. In Section 2, we give proofs showing that these Eberlein type spaces enjoy some hereditary properties (along the lines Yakovlev did in [12]). Then we utilize a representation theorem for locally compact spaces of finite metrizability number (Ismail and Szymanski cf. [6]) to show that the appropriate hereditary properties characterize the spaces in question, provided that they have finite metrizability number. Concerning our characterization, it should be mentioned that in 1984 G. Gruenhage obtained a characterization of Corson and Eberlein compacta in a similar vein. Namely, he showed (cf. [5,

(a) Research supported by OTKA grant no. 37758.

(b) Research done while the author was a visiting professor at P. Erdös Research Center of Mathematics. The author thanks the Center for support for this research.

Theorem 2.2]) that a compact space X is a Corson (resp. Eberlein) compactum if and only if X^2 is hereditarily metalindelöf (resp. σ-metacompact).

All undefined terms and notions can be found in [4] or in [7].

1. Open subspaces of Eberlein and Eberlein like compacta

Let κ be a cardinal and let $\mathcal{P}$ be a family of subsets of a set X.

$\mathcal{P}$ is a *point-κ family* if $|\{U \in \mathcal{P} : x \in U\}| \leq \kappa$ for each $x \in X$. $\mathcal{P}$ is a *point-finite family* in X if $|\{U \in \mathcal{P} : x \in U\}| < \omega$ for each $x \in X$. $\mathcal{P}$ is a *σ-point finite family* in X if $\mathcal{P}$ is the union of countably many point-finite subfamilies.

$\mathcal{P}$ is a T_0 *separating family* in X if for any two distinct points x, y in X there is $U \in \mathcal{P}$ such that U contains exactly one of x, y. (If it is so, we use the phrase: U separates x and y). $\mathcal{P}$ is an *F-separating family* in X if for any two distinct points x, y in X there is $U \in \mathcal{P}$ such that $x \in U$ and $y \notin \operatorname{cl} U$ or vice versa. Let us point out that any separating family in X covers all but, perhaps, a single point of X.

For the purposes of our paper, we need the following topological characterizations of Eberlein compacta.

Theorem 1 (H.P. Rosenthal [11]). *A compact space X is an* Eberlein compactum *if and only if there is a collection of non-empty open F_σ subsets of X which is T_0 separating and σ-point finite in X.*

Theorem 2 (E. Michael and M.E. Rudin [9]). *A compact Hausdorff space is an Eberlein compact if and only if it has a collection of open subsets of X which is σ-point finite and F-separating in X.*

We recommend [1] and [8] as basic sources for, other than topological, contexts related to Eberlein compacts.

One of the goals of this section is to study open subspaces of Eberlein compacta. The following general lemma comes very handy in this respect.

Lemma 1. *Let $\mathcal{C}$ be a class of compact spaces closed under continuous images. A non-compact space Y is homeomorphic to an open subspace of a space $X \in \mathcal{C}$ if and only if Y is locally compact and the one-point compactification of Y belongs to $\mathcal{C}$.*

Proof: The sufficiency is obvious. To prove the necessity, suppose that Y is homeomorphic to the open subspace G of a space $X \in \mathcal{C}$. Then Y is locally compact and $F = X - G$ is a non-empty closed subset of X. Since the quotient map $q : X \to X/F$ (obtained by collapsing the set F to a single point) is continuous, the quotient space $X/F \in \mathcal{C}$. Clearly, Y is homeomorphic to $X/F - \{F\}$. □

Theorem 3. *For a locally compact space Y, the following conditions are equivalent:*

(i) *Y is homeomorphic to an open subspace of an Eberlein compact;*

(ii) *Y has a σ-point finite, T_0 separating collection of non-empty open F_σ sets with compact closures;*
(iii) *Y has a σ-point finite, F-separating collection of non-empty open F_σ sets with compact closures;*
(iv) *Y has a σ-point finite, F-separating collection of non-empty open sets with compact closures.*

Proof: The theorem is non-trivial only in the case when Y is not compact.

The following argument proves both implications (i) $\Rightarrow$(ii) and (ii) $\Rightarrow$(iii).

By the theorem of Y. Benyamini, M.E. Rudin, and M. Wage [2], the class of Eberlein compacta is closed under continuous images. By Lemma 1, $Y = X - \{p\}$ for some Eberlein compactum X and $p \in X$. Let $\mathcal{P}$ be a σ-point finite, T_0 separating collection of non-empty open F_σ subsets of X. For each $U \in \mathcal{P}$ fix open F_σ sets $U(n)$, $n \in \omega$, such that:

(a) $\operatorname{cl} U(n) \subseteq U(n+1)$ for each $n \in \omega$;
(b) $\bigcup\{U(n) : n \in \omega\} = U$.

For each $U \in \mathcal{P}$ and for each $n \in \omega$, fix a closed G_δ set $\widetilde{U}(n)$ such that $\operatorname{cl} U(n) \subseteq \widetilde{U}(n) \subseteq U(n+1)$. We set

$$\mathcal{Q} = \big\{U(n) : p \notin U \in \mathcal{P},\ n \in \omega\big\} \cup \big\{X - \widetilde{U}(n) : p \in U(n), U \in \mathcal{P},\ n \in \omega\big\}.$$

To see that $\mathcal{Q}$ is σ-point finite, notice that $\{U(n) : p \notin U \in \mathcal{P},\ n \in \omega\}$ is the union of countably many σ-point finite families $\mathcal{Q}_n = \{U(n) : p \notin U \in \mathcal{P}\}$ and that $\{X - \widetilde{U}(n) : p \in U \in \mathcal{P},\ n \in \omega\}$ is a countable family.

To see that $\mathcal{Q}$ is F-separating, take two distinct points x, y of $X - \{p\}$. Suppose $U \in \mathcal{P}$ separates x and y, say $x \in U$ and $y \notin U$. If $p \notin U$, then take an $n \in \omega$ such that $x \in U(n)$. Thus $y \notin \operatorname{cl} U(n)$. Clearly $U(n) \in \mathcal{Q}$. If $p \in U$, then $p, x \in U$ and $y \notin U$. Take an $n \in \omega$ such that $p, x \in U(n)$. Hence $X - \widetilde{U}(n)$ F-separates x and y. Clearly, $Y - \widetilde{U}(n) \in \mathcal{Q}$.

From the definition of sets $U(n)$ and $\widetilde{U}(n)$, it follows immediately that every member of $\mathcal{Q}$ is F_σ as well as that the closure of every member of $\mathcal{Q}$ is a compact subset of the space $X - \{p\}$.

Since the implication (iii) $\Rightarrow$(iv) is obvious, it remains to prove the implication (iv) $\Rightarrow$(i). Assume that Y is a locally compact non-compact space, and let $\mathcal{Q}$ be a σ-point finite, F-separating collection of non-empty open sets with compact closures in Y. We may assume that $\bigcup \mathcal{Q} = Y$. If X is the one-point compactification of Y, then X is a compact space in which $\mathcal{Q}$ is a σ-point-finite, F-separating collection of non-empty open sets. Thus X is an Eberlein compactum. $\square$

A similar line of arguments can be used to characterize open subspaces of uniform Eberlein compacta or of Corson compacta. We need pertaining definitions and facts first.

A compact Hausdorff space X is called a *uniform Eberlein compactum* if there are a separating collection $\mathcal{P}$ of non-empty open F_σ subsets of X, a function $\varphi : \omega \to \omega$, and a decomposition $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \omega\}$ such that each point $x \in X$ belongs to at most $\varphi(n)$ many members of $\mathcal{P}_n$.

A compact Hausdorff space X is called a *Corson compactum* if there is a collection of non-empty open F_σ subsets of X which is separating and point countable in X.

Since the class of uniform Eberlein compacta and the class of Corson compacta are both closed under continuous images (see Y. Benyamini, M.E. Rudin, and M. Wage [2] and E. Michael and M.E. Rudin [2]???, respectively), the next two theorems can be proved virtually the same way as Theorem 3.

Theorem 4. *For a locally compact Hausdorff space Y, the following conditions are equivalent:*

(j) *Y is homeomorphic to an open subspace of a uniform Eberlein compactum;*
(jj) *Y has a separating collection $\mathcal{P}$ of non-empty open F_σ sets with compact closures for which there is a function $\varphi \in {}^\omega\omega$ and a decomposition, $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \omega\}$, such that each $x \in Y$ belongs to at most $\varphi(n)$ sets in $\mathcal{P}_n$;*
(jjj) *Y has an F-separating collection $\mathcal{P}$ of non-empty open F_σ sets with compact closures for which there is a function $\varphi \in {}^\omega\omega$ and a decomposition, $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \omega\}$, such that each $x \in Y$ belongs to at most $\varphi(n)$ sets in $\mathcal{P}_n$.*

Theorem 5. *For a locally compact Hausdorff space Y, the following conditions are equivalent:*

($\diamond$) *Y is homeomorphic to an open subspace of a Corson compactum;*
($\diamond\diamond$) *Y has a point countable, separating collection of non-empty open F_σ sets with compact closures;*
($\diamond\diamond\diamond$) *Y has a point countable, F-separating collection of non-empty open F_σ sets with compact closures;*
($\diamond v$) *Y has a point countable, F-separating collection of non-empty open sets with compact closures.*

A space homeomorphic to an open subspace of an Eberlein compactum is going to be called an *Eberlein space*. Similarly, a space homeomorphic to an open subspace of a uniform Eberlein (or Corson) compactum is going to be called a *uniform Eberlein* (*resp. Corson*) *space*.

Proposition 1. *Let X be a locally compact Hausdorff space that can be represented as $X = G \cup H$, where G is an open Eberlein subspace and $H \cap G = \emptyset$. Suppose further that there is a σ-point finite collection $\mathcal{P}$ of open sets in X that covers H and such that the family $\mathcal{P} \restriction H = \{U \cap H : U \in \mathcal{P}\}$ consists of σ-compact sets and is separating in H. Then X is an Eberlein space.*

In particular, if X is compact Hausdorff and $X = G \cup H$, where G is an open Eberlein subspace and H is compact and metric, then X is an Eberlein compact.

PROOF: We shall show that X has a σ-point finite, F-separating collection $\mathcal{Q}$ of non-empty open sets with compact closures. Towards this goal, take a family $\mathcal{Q}_1$ that is a σ-point finite, F-separating collection of non-empty open sets with compact closures in G. We may assume that this family covers G and thus $\mathcal{Q}_1$ F-separates any pair of distinct points of X if at least one of them is in G. It remains to construct another σ-point finite collection of open subsets of X that F-separates any pair of distinct points from H. For $U \in \mathcal{P}$, let $\{E(U,n) : n \in \omega\}$ be a sequence of compact sets such that $\bigcup\{E(U,n) : n \in \omega\} = U \cap H$. For each set $E(U,n)$ pick an open set $V(U,n) \subseteq X$ such that $E(U,n) \subseteq V(U,n) \subseteq \operatorname{cl} V(U,n) \subseteq U$ and $\operatorname{cl} V(U,n)$ is compact. Then $\mathcal{P}_n = \{V(U,n) : U \in \mathcal{P}\}$ is σ-point finite collection of open sets in X, hence so is $\mathcal{Q}_2 = \bigcup\{\mathcal{P}_n : n \in \omega\}$. Notice that $\mathcal{Q}_2$ F-separates any pair of distinct points from H. Thus $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ is as required. □

Remark. There exists a compact Hausdorff space X such that $X = G \cup H$, where G is an open Eberlein subspace and H is an Eberlein compactum but the space X itself is not an Eberlein compactum. Take, for example, X to be the one-point compactification of a Ψ-space. Thus $X = \omega \cup \mathcal{M} \cup \{\infty\}$, where $\mathcal{M}$ is a MAD on ω. It is well known that X is not an Eberlein compactum, however ω is its open Eberlein subspace and $\mathcal{M} \cup \{\infty\}$ is an Eberlein compactum.

2. Spaces of finite metrizability number and Eberlein like spaces

A topological space X is said to be *σ-metacompact* if every open cover of X has a σ-point finite refinement; X is hereditarily *σ-metacompact* if every (open) subspace of X is σ-metacompact.

A topological space X is said to be *metalindelöf* if every open cover of X has a point countable open refinement; X is *hereditarily metalindelöf* if every (open) subspace of X is metalindelöf. In [12], N. Yakovlev proved the following remarkable facts.

Theorem 6. (a) *If X is an Eberlein compactum, then X is hereditarily σ-metacompact.*
(b) *If X is a Corson compactum, then X is hereditarily metalindelöf.*

We shall augment Yakovlev's results by exhibiting a similar hereditary property of uniform Eberlein compacta.

For a given collection $\mathcal{P}$ of non-empty subsets of a set X and for any point $x \in X$ let $\mathcal{P}^+(x) = \{U \in \mathcal{P} : x \in U\}$ and $\mathcal{P}^-(x) = \{U \in \mathcal{P} : x \notin U\}$. Clearly, for each $x \in X$, $\mathcal{P}^+(x) \cap \mathcal{P}^-(x) = \emptyset$ and $\mathcal{P}^+(x) \cup \mathcal{P}^-(x) = \mathcal{P}$. Notice that $\mathcal{P}$ is separating iff $\bigcap\{U : U \in \mathcal{P}^+(x)\} \cap \bigcap\{X - U : U \in \mathcal{P}^-(x)\} = \{x\}$ for each $x \in X$. If $\mathcal{P}$ is F-separating, then $\bigcap\{U : \operatorname{cl} U \in \mathcal{P}^+(x)\} \cap \bigcap\{X - U : U \in \mathcal{P}^-(x)\} = \{x\}$ for each $x \in X$.

Now assume that $\mathcal{P}$ is a family of non-empty open F_σ subsets of a compact space X. For each $U \in \mathcal{P}$, fix a sequence $\{U(n) : n \in \omega\}$ of open sets such that $\operatorname{cl} U(n) \subseteq U(n+1)$, for each $n \in \omega$, and $\bigcup\{U(n) : n \in \omega\} = U$. For $A, B \in [\mathcal{P}]^{<\omega}$ and $n, m \in \omega$, let

$$V(A,B,n,m) = \bigcap\{U(n) : U \in A\} \cap \bigcap\{X - \operatorname{cl} U(m) : U \in B\}.$$

The set $V(A,B,n,m)$ is called a *$\mathcal{P}$-canonical neighborhood* of a point $x \in X$ if $A \in [\mathcal{P}^+]^{<\omega}$, $B \in [\mathcal{P}^-]^{<\omega}$, and $x \in U(n)$ for each $U \in A$. Under the assumptions on $\mathcal{P}$ and X, as above, we have the following lemma.

Lemma 2. *If $\mathcal{P}$ is an F-separating collection in X and $x \in X$, then the family of all $\mathcal{P}$-canonical neighborhoods of x is a base at x.*

PROOF: Let W be an open neighborhood of a point x. Since $\bigcap\{\operatorname{cl} U : U \in \mathcal{P}^+(x)\} \cap \bigcap\{X - U(m) : U \in \mathcal{P}^-(x),\ m \in \omega\} = \{x\}$, there exists a finite subset A of $\mathcal{P}^+(x)$, a finite subset B of $\mathcal{P}^-(x)$, and $m \in \omega$ such that $\bigcap\{\operatorname{cl} U : U \in A\} \cap \bigcap\{X - U(m) : U \in B\} \subseteq W$. Pick an $n \in \omega$ such that $x \in U(n)$ for each $U \in A$. Then $V(A,B,n,m)$ is a $\mathcal{P}$-canonical neighborhood of x contained in W. $\square$

$\mathcal{P}$ is said to be *uniformly σ-point-finite* if $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \omega\}$, where each $\mathcal{P}_n$ is a point-$\varphi(n)$ family for some function $\varphi : \omega \to \omega$.

A topological space X is called *uniformly σ-metacompact* if every open cover of X has a uniformly σ-point finite refinement; X is *hereditarily uniformly σ-metacompact* if every (open) subspace of X is uniformly σ-metacompact.

Theorem 7. *If X is a uniform Eberlein compactum, then X is hereditarily uniformly σ-metacompact.*

PROOF: By Theorem 4(jjj), X has an F-separating collection $\mathcal{P}$ of non-empty open F_σ sets for which there are a function $\varphi : \omega \to \omega$ and a decomposition $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \omega\}$ such that each $x \in X$ belongs to at most $\varphi(n)$ sets in $\mathcal{P}_n$. For each $U \in \mathcal{P}$, fix a sequence $\{U(n) : n \in \omega\}$ of open sets such that $\operatorname{cl} U(n) \subseteq U(n+1)$, for each $n \in \omega$, and $\bigcup\{U(n) : n \in \omega\} = U$. Without loss of generality we may assume that $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \cdots \subseteq \mathcal{P}_n \subseteq \dots$

Let $\mathcal{Q}$ be any collection of non-empty open subsets of X. By transfinite induction, it is easy to construct a collection of points $\{x_\xi : \xi < \beta\}$, and a collection $\{V_\xi : \xi < \beta\}$ of $\mathcal{P}$-canonical sets that satisfy the following conditions:

(1) for each $\alpha < \beta$, $V_\alpha = V(A_\alpha, B_\alpha, n_\alpha, m_\alpha)$ is a $\mathcal{P}$-canonical neighborhood of x_α such that, for some $n \in \omega$, $A_\alpha \cup B_\alpha \subseteq \mathcal{P}_n$ and $A_\alpha = \mathcal{P}_n^+(x_\alpha)$;

(2) for each $\alpha < \beta$, $x_\alpha \notin \bigcup\{\ V_\xi : \xi < \alpha\}$; (3) $\{V_\xi : \xi < \alpha\}$ is a refinement of $\mathcal{Q}$.

Claim. For each $k, m, n \in \omega$ and $A \in [\mathcal{P}]^{<\omega}$ there is at most one $\xi < \beta$ such that: $A_\xi = A$ and $n_\xi = n$ and $m_\xi = m$ and $A_\xi \cup B_\xi \subseteq \mathcal{P}_k$ and $A_\xi = \mathcal{P}_k^+(x_\xi)$.

For otherwise, there would exist $\xi < \zeta < \beta$ such that $\mathcal{P}_k^+(x_\xi) = \mathcal{P}_k^+(x_\varsigma)$. Thus $B_\xi \in \mathcal{P}^-(x_\zeta)$ which would imply that $x_\varsigma \in V_\xi$, contradicting (2).

For each $k, m, n \in \omega$, let

$$\mathcal{Q}(k,n,m) = \left\{V_\xi : n_\xi = n,\, m_\xi = m,\; A_\xi \cup B_\xi \subseteq \mathcal{P}_k, \;\text{ and }\; A_\xi = \mathcal{P}_k^+\left(x_\xi\right)\right\}.$$

Clearly, each V_ξ belongs to at least one of the families $\mathcal{Q}(k,n,m)$. To end the proof we show that $\mathcal{Q}(k,n,m)$ is point-$2^{\varphi(k)}$. To see this, let x be in the intersection of a subfamily $\mathcal{R}$ of $\mathcal{Q}(k,n,m)$. By the claim, there will be at most $|2^{\mathcal{P}_k^+(x)}| \le 2^{\varphi(k)}$ members of $\mathcal{R}$. The proof of the theorem is finished. $\square$

Let Π be a property of collections of subsets of a given set (like being (σ-)disjoint, (σ-)point finite, point-countable, etc.). A collection $\mathcal{D}$ of subsets of a space X is said to be Π-*good in* X if there exists a collection $\mathcal{P}$ with property Π of open subsets of X such that $\bigcup\mathcal{D} \subseteq \bigcup\mathcal{P}$ and every element of $\mathcal{P}$ intersects exactly one element of $\mathcal{D}$. The space X is said to be a Π-*witness* if each discrete (in X) collection $\mathcal{D}$ of closed 2nd countable subsets of X is Π-good in X. The following statements are then obvious from the definitions. (In fact, 2nd countability of the members of $\mathcal{D}$ is not used in them.)

Proposition 2. (a) *If X is collectionwise normal, then X is a Π-witness for $\Pi =$ disjoint.*

(b) *If X is uniformly σ-metacompact, then X is a Π-witness for $\Pi =$ uniformly σ-point finite.*

(c) *If X is σ-metacompact, then X is a Π-witness for $\Pi = \sigma$-point finite.*

(d) *If X is metalindelöf, then X is Π-witness for $\Pi =$ point countable.*

The following lemma is trivial but also indispensable for our forthcoming arguments.

Lemma 3. *Let X be a locally compact Hausdorff space. Suppose that F is a compact subset of X and a subset of an open set U. There exists an open F_σ set V such that $F \subseteq V \subseteq \operatorname{cl} V \subseteq U$ and $\operatorname{cl} V$ is compact.*

Theorem 8. *Let X be a locally compact Hausdorff space and let $\mathcal{D}$ be a collection of disjoint, closed, 2nd countable subsets of X. Let Π be one of the properties:*

σ-disjoint;

σ-point k; $k \in \omega$;

uniformly σ-point finite;

σ-point finite;

point countable;

point-κ; $\kappa > \omega$.

If $\mathcal{D}$ is Π-good in X, then there is a collection $\mathcal{P}$ with property Π of open F_σ subsets of X with compact closures such that:

($\blacktriangle$) *If $x \in F \in \mathcal{D}$ and V is an open neighborhood of x in F then there exists $U \in \mathcal{P}$ such that $x \in U$ and $\operatorname{cl} U \cap (\bigcup \mathcal{D}) \subseteq V$.*

Proof: We give a proof only in the case $\Pi = \sigma$-disjoint, the other cases being quite analogous.

Let $\mathcal{Q}$ be a σ-disjoint family of open subsets of X that covers $\bigcup \mathcal{D}$ and every $U \in \mathcal{Q}$ intersects exactly one $F_U \in \mathcal{D}$. We have $\mathcal{Q} = \bigcup\{\mathcal{Q}_n : n \in \omega\}$, where each $\mathcal{Q}_n$ is a disjoint open family of X. Consider one of those families, say $\mathcal{Q}_0$. Fix a countable open base $\mathcal{B}_F$ in F for every member F of the collection $\mathcal{D}$. Let $\{F_U(n) : n \in \omega\}$ be all the elements of the base $\mathcal{B}_{F_U}$ that are contained in $U \cap F_U$. Let $\{F_U(n,m) : m \in \omega\}$ be open sets in F such that $\bigcup\{F_U(n,m) : m \in \omega\} = F_U(n)$, $\operatorname{cl} F_U(n,m) \subseteq F_U(n)$, and $\operatorname{cl} F_U(n,m)$ is compact.

Now, for each $U \in \mathcal{Q}_0$ and $n \in \omega$, let $\widetilde{F}_U(n)$ be an open subset in X such that $\widetilde{F}_U(n) \subseteq U$ and $\widetilde{F}_U(n) \cap (\bigcup \mathcal{D}) = F_U(n)$. Applying Lemma 3 to the open set $\widetilde{F}_U(n)$ and its compact subset $\operatorname{cl} F_U(n,m)$ we get an open F_σ subset $V_U(n,m)$ of X such that

$$\operatorname{cl} F_U(n,m) \subseteq V_U(n,m) \subseteq \operatorname{cl} V_U(n,m) \subseteq \widetilde{F}_U(n) \tag{2.1}$$

and $\operatorname{cl} V_U(n,m)$ is compact. Let $\mathcal{Q}^0(n,m) = \{V_U(n,m) : U \in \mathcal{Q}_0\}$. From construction, $\mathcal{Q}^0(n,m)$ is a disjoint family consisting of open F_σ sets with compact closures. Analogously, we construct families $\mathcal{Q}^k(n,m)$ for each $k \in \omega$. We set: $\mathcal{P} = \bigcup\{\mathcal{Q}^k(n,m) : k,m,n \in \omega\}$. It remains to show that $\mathcal{P}$ satisfies condition ($\blacktriangle$).

Let $x \in F \in \mathcal{D}$ and let $x \in U \in \mathcal{Q}_k$ for some $k \in \omega$. If V is an open neighborhood of x in F, then $x \in F_U(n,m) \subseteq F_U(n) \subseteq V$ for some $m,n \in \omega$. By 2.1, the set $V_U(n,m)$, that belongs to $\mathcal{Q}^k(n,m) \subseteq \mathcal{P}$, satisfies the condition $\operatorname{cl} V_U(n,m) \cap (\bigcup \mathcal{D}) \subseteq V$. □

The *metrizability number* $m(X)$ of a space X is the smallest cardinal number κ such that X can be represented as the union of κ many metrizable subspaces. Locally compact Hausdorff spaces with finite metrizability number are studied e.g., in [6]. Typical examples include the one-point compactification of an uncountable discrete space, the Alexandroff duplicate of the unit segment, and the space Ψ. We need the following representation theorem (cf. [6, Corollaries 7 and 8]):

Theorem 9. *Let X be a locally compact Hausdorff space with $m(X) = n$, $2 \le n < \omega$. Then*

(a) *X can be represented as $X = G \cup F$, where G is open dense metrizable subspace of X, $G \cap F = \emptyset$, and $m(F) = n - 1$;*

(b) *X can be represented as $X = H \cup E$, where H is open dense subspace of X with $m(H) = n - 1$, $G \cap E = \emptyset$, and E is metrizable.*

We can state and prove our main results.

Theorem 10. *Let X be a locally compact Hausdorff space of finite metrizability number. Let Π be the property of being σ-point finite. The following conditions are equivalent:*

(1) *X is hereditarily σ-metacompact;*
(2) *every dense open subspace of X is a Π-witness;*
(3) *X is an Eberlein space.*

Proof: The implication (1)$\Rightarrow$(2) is obvious. The implication (3)$\Rightarrow$(1) follows from Theorem 6(a). It remains to prove (2)$\Rightarrow$(3). We will use induction on the metrizability number $m(X)$.

The case $m(X) = 1$ is obvious.

Assume that any locally compact Hausdorff space with metrizability number $\leq n$ and whose every dense open subspace is a Π-witness (for $\Pi = \sigma$-point finite) is an Eberlein space. Let X be a locally compact Hausdorff space with every dense open subspace a Π-witness and $m(X) = n + 1$. By Theorem 9(b), $X = H \cup E$, where H is open dense subspace of X with $m(H) = n$, $H \cap E = \emptyset$, and E is metrizable. By the induction assumption, H is an Eberlein space. So H has a σ-point finite, (F-)separating collection $\mathcal{Q}$ of non-empty open F_σ sets with compact closures. Since E is a locally compact metric space, E can be represented as the topological sum, $E = \bigoplus\{E_\xi : \xi < \kappa\}$, of its second countable subspaces E_ξ. Since X is a Π-witness, by Theorem 8 applied to $\mathcal{D} = \{E_\xi : \xi < \kappa\}$, there is a collection $\mathcal{P}$ with the property Π of non-empty open F_σ subsets of X with compact closures such that ($\blacktriangle$) holds for $\mathcal{P}$. Then $\mathcal{P} \cup \mathcal{Q}$ is a required family for X to be an Eberlein space. $\square$

Proofs of our other two main theorems can be done virtually the same way as the proof above. We have to make only suitable references to Theorem 6(b), in case of Corson spaces, or to Theorem 7, in case of uniform Eberlein spaces. Therefore we do not include the proofs of these two theorems.

Theorem 11. *Let X be a locally compact Hausdorff space of finite metrizability number. Let Π be the property of being uniformly σ-metacompact. The following conditions are equivalent:*

(1) *X is hereditarily uniformly σ-metacompact;*
(2) *every dense open subspace of X is a Π-witness;*
(3) *X is a uniform Eberlein space.*

Theorem 12. *Let X be a locally compact Hausdorff space with a finite metrizability number. Let Π be the property of being point countable. The following conditions are equivalent:*

(1) *X is hereditarily metalindelöf;*
(2) *every dense open subspace of X is Π-witness;*
(3) *X is a Corson space.*

Next, for each natural number $n \geq 2$, we will construct a hereditarily screenable uniform Eberlein compactum of metrizability number exactly n. We first state a proposition. Since its proof follows the same line of argument as the previous ones, we again omit it here.

Proposition 3. *Let X be a locally compact Hausdorff space of finite metrizability number. Let Π be the property of being σ-disjoint. If X is a Π-witness, then X has a σ-disjoint, F-separating collection of non-empty F_σ open sets with compact closures. In particular, X is a uniform Eberlein compactum.*

We may now present the spaces promised above.

Example 1. Let Y_0 denote the one-point compactification of the discrete space ω_1. Notice that Y_0 is hereditarily screenable compact and $m(Y_0) = 2$.
By induction, let Y_{n+1} be the one-point compactification of $\omega_1 \times Y_n$. Clearly, Y_{n+1} is hereditarily screenable and compact. It remains to show that $m(Y_{n+1}) = n+3$ provided that $m(Y_n) = n + 2$.
It is clear (by induction) that $m(Y_{n+1}) \leq n + 3$. Assuming to the contrary that $m(Y_{n+1}) \leq n + 2$, from Theorem 9(b), we would get that $Y_{n+1} = G \cup F$, where G is an open dense subspace of Y_{n+1} with $m(G) \leq n + 1$, Since F can have at most countably many pairwise disjoint open sets, G must contain one of subspaces $\{\alpha\} \times Y_n$, which is of metrizability number $n + 2$; a contradiction.

Remark 2. Let X be a compact Hausdorff space with $m(X) = 2$.

(a) From the representation theorem, it follows that X has a σ-disjoint, separating collection of non-empty open F_σ sets. Thus X is a uniform Eberlein compact (cf. [10]).

(b) Let Y be the one-point compactification of a discrete space of cardinality ω_1, say $Y = \omega_1 \cup \{\infty\}$. Let Z be the one-point compactification of a discrete space of cardinality ω, say $Z = \omega \cup \{\infty\}$. Then $X = Y \times Z$ is a compact Hausdorff space and $m(X) = 2$. However $X - \{(\infty, \infty)\}$ is not normal: $\omega_1 \times \{\infty\}$ and $\{\infty\} \times \omega$ are closed in $X - \{(\infty, \infty)\}$ and cannot be separated. This example gives a negative answer to a question by J. Gerlits (who found, independently, another such example).

Let us conclude by noting that hereditary metalindelöfness (resp. hereditarily σ-metacompactness) alone does not characterize Corson (resp. Eberlein) compacta among compact spaces. Indeed, it is well known that Alexandrov's "double-arrow" space is not a Corson compactum although it is hereditarily lindelöf, and thus hereditarily paracompact. But the metrizability number of Alexandrov's double-arrow space is equal to continuum, hence to show that our results are sharp, we would need a compact space of countable metrizability number that is hereditarily metalindelöf or (hereditarily σ-metacompact) but not Corson (or Eberlein). At present, we do not have such an example.

References

[1] Amir D., Lindenstrauss J., *The structure of weakly compact sets in Banach spaces*, Ann. of Math. **88** (1968), 35–46.

[2] Benyamini Y., Rudin M.E., Wage M., *Continuous images of weakly compact subsets of Banach spaces*, Pacific J. Math. **70.2** (1977), 309–324.

[3] Benyamini Y., Starbird T., *Embedding weakly compact sets into Hilbert space*, Israel J. Math. **23** (1976), 137–141.

[4] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1988.

[5] Gruenhage G., *Covering properties on $X^2\backslash\Delta$, W-sets, and compact subsets of Σ-products*, Topology Appl. **17** (1984), 287–304.

[6] Ismail M., Szymanski A., *On locally compact Hausdorff spaces with finite metrizability number*, Topology Appl. **114.3** (2001), 285–293.

[7] Juhász I., *Cardinal Functions in Topology — Ten Years After*, Mathematical Centre Tracts 123, Amsterdam, 1983.

[8] Lindenstrauss J., *Weakly compact sets, their topological properties and the Banach spaces they generate*, Annals of Math. Studies 69, Princeton University Press, Princeton, 1972, pp. 235–273.

[9] Michael E., Rudin M.E., *A note on Eberlein compacts*, Pacific J. Math. **72.2** (1977), 487–495.

[10] Michael E., Rudin M.E., *Another note on Eberlein compacts*, Pacific J. Math. **72.2** (1977), 497–499.

[11] Rosenthal H.P., *The heredity problem for weakly compactly generated Banach spaces*, Compositio Math. **28** (1974), 83–111.

[12] Yakovlev N.N., *On bicompacta in Σ-products and related spaces*, Comment. Math. Univ. Carolin. **21.2** (1980), 263–283.

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(*Received* August 24, 2006)