Strong boundedness and algebraically closed groups

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Abstract. Let G be a non-trivial algebraically closed group and X be a subset of G generating G in infinitely many steps. We give a construction of a binary tree associated with (G, X). Using this we show that if G is ω_1-existentially closed then it is strongly bounded.

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Introduction

Definition ([1], [2])**.** A group G is *Cayley bounded* if for every generating subset $U \subset G$ there exists $n \in \omega$ such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$. A group is *strongly bounded* if it is Cayley bounded and cannot be presented as the union of a strictly increasing chain $\{H_n : n \in \omega\}$ of proper subgroups (has cofinality $> \omega$).

It is shown in [2], that strongly bounded groups have property (FH) that every affine isometric action of G on a Hilbert space has a fixed point [3]. Recent papers [1], [2], [6] and [7] contain a number of uncountable examples of strongly bounded groups. Most of them can be presented as closed subgroups of the group $\mathrm{Sym}(\omega)$ of all permutations of the set of natural numbers ω. The paper [7] contains some other examples (for example, the group of Lipschitz homeomorphisms of the Baire space), but the proof of their strong boundedness uses some reductions to automorphism groups of countable structures.

ω_1-existentially closed groups provide a different construction of uncountable strongly bounded groups. In this case the property of strong boundedness has been obtained in [2] (where the author of [2] mentions that A. Khelif has also proved this).

We study non-trivial algebraically closed groups. Our main result (Proposition 2) associates an infinite binary tree with any set generating G in infinitely many steps. Using this we give another proof that ω_1-existentially closed groups are strongly bounded. Our proof follows the approach of [6] and [7].

When these results were obtained the author did not know that Y. de Cornulier extended the first version of [2] by the material about algebraically closed groups.

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Algebraically closed groups

A group G is *algebraically closed* if any finite system of equations $\Sigma(\bar{x}, \bar{a})$ with parameters $\bar{a}$ from G and having a solution in some group extending G, already has a solution in G. It is well-known that a non-trivial algebraically closed group is *existentially closed*, i.e. any quantifier free formula $\phi(\bar{x}, \bar{a})$ (where we may assume that $\phi(\bar{x}, \bar{a})$ is a conjunction of equalities and inequalities of the form $w(\bar{x}) = (\neq)1$) with parameters from G and having a solution in some group extending G, already has a solution in G (see [8]).

A group G is ω_1-*existentially closed* if for every set $\Sigma(\bar{x})$ of equalities and inequalities of the form $w(\bar{x}, \bar{a}) = (\neq)1$ depending on variables $\bar{x}$ and at most countably many parameters from G, if $\Sigma(\bar{x})$ has a solution in some group extending G, then it is satisfied already in G. Every group embeds into an ω_1-existentially closed group [9]. The following theorem has been already proved in [2].

Theorem 1. *Every ω_1-existentially closed group G is strongly bounded.*

We have found that Theorem 1 can be proved by methods resembling those of [6] and [7] (which in turn is based on the proof of Theorem 6.1 of [5]). We will use the following statement from [1, Lemma 10] and [2, Proposition 2.7].

> *A group G is strongly bounded if and only if for every presentation of G as $G = \bigcup X_n$ for an increasing sequence X_n, $n \in \omega$, with $\{1\} \cup X_n^{-1} \cup X_n \cdot X_n \subseteq X_{n+1}$ there is a number n such that $X_n = G$.*

Our proof of Theorem 1 is based on the following proposition.

Proposition 2. *Let G be an existentially closed group. Let a sequence $\{X_n\}$ define a presentation of G as above and $G \neq X_n$ for all n. Then there is a binary tree $\{g_s : s \in 2^{<\omega}\} \subset G$ such that all g_s generate in G the free product of groups $\langle g_s \rangle$ where $|g_s| = \infty$. After possibly replacing X_n, $n \in \omega$, by a subsequence, the tree satisfies the following property: for every $s \in 2^{<\omega}$ with $|s| = n$, $g_{s0} \in X_n$ and $g_{s1} \in G \setminus X_{n+2}$.*

Before the proof we note that in the formulation G may be countable. In this case a sequence X_n, $n \in \omega$, as above (and the corresponding tree) can be easily found.

The proof below uses a fundamental theorem of Ziegler [10] (or see Theorem 3.3.7 of [4]). We give it in a form appropriate for our applications:

> *Let $\Phi(\bar{x})$ be a recursively enumerable set of quantifier-free Horn formulas of the following form (strict Horn formulas):*
>
> $$\bigwedge_{i=1}^{k} w_i(\bar{x}) = 1 \rightarrow w_0(\bar{x}) = 1,$$

where w_i are group words depending on variables $\bar{x}$. Then there is a formula $\phi(\bar{x})$ of the form $\exists\bar{y}(\bigwedge_{j=1}^{l} w'_j(\bar{x},\bar{y})=1)$, such that a tuple $\bar{c}\in G$ satisfies $\Phi(\bar{x})$ if and only if $\phi(\bar{c})$ is satisfied in some group extending G. If G is existentially closed then $\Phi(\bar{x})$ is equivalent to $\phi(\bar{x})$ in G.

PROOF: Assume that all g_s with $|s|\le n$ are already defined. To define all g_{s0} and g_{s1} with $|s|=n$ we firstly find a tuple $h'_1,\dots,h'_{2^n}\in X_n$ such that the set $\{h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ freely generates (as a basis) in G a free subgroup. To see that such a tuple exists take the set $\Phi(x'_1,\dots,x'_{2^n},z,\bar{z})$ of all strict Horn formulas of the form

$$w(x'_1,\dots,x'_{2^n},\bar{z})=1\to z=1,$$

where the tuple $\bar{z}$ consists of all z_r with $r\in 2^{\le n}$, and $w(\bar{x}',\bar{z})$ is a non-trivial reduced word. By Ziegler's theorem there is a formula $\phi(\bar{x}',z,\bar{z})$ of the form above, which is equivalent to $\Phi(\bar{x}',z,\bar{z})$ in G. Note that $\phi(\bar{x}',h,\bar{g})$ is realized in some group extending $G*\langle x_1,\dots,x_{2^n}\rangle$, where $h\in G\setminus\{1\}$ is arbitrary. Since G is existentially closed we can find a required tuple $h'_1,\dots,h'_{2^n}$ in G (using ϕ and parameters from $\{h\}\cup\{g_r : |r|\le n\}$).

As $\bigcup X_i=G$, after possible changing of the enumeration $\{X_n\}$ we can arrange that $h'_1,\dots,h'_{2^n}\in X_n$.

Now consider the complement of X_{n+2}.

Lemma 3. *The set $G\setminus X_{n+2}$ contains a tuple $h_1,\dots,h_{2^n}$ such that the elements $\{h_1,\dots,h_{2^n},h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ freely generate in G a free subgroup.*

PROOF: Assuming the contrary we find the maximal i such that there are $h_1,\dots,h_i\in G\setminus X_{n+2}$ such that $\{h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ freely generate in G a free subgroup. Thus $i<2^n$. We claim that

For any $g\in G\setminus\langle\{h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}\rangle$ there are h_{i+1} and h_{i+2} satisfying the following conditions: $g=h_{i+1}h_{i+2}$ and each of the sets $\{h_1,\dots,h_i,h_{i+1},h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ and $\{h_1,\dots,h_i,h_{i+2},h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ freely generates in G a free subgroup.

Indeed, let

$$H=\langle y_1\rangle*\langle\{g,h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}\rangle.$$

Then the subgroup of H generated by $\{y_1,h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ is the free product $P=\langle y_1\rangle*\langle\{h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}\rangle$. Let $y_2=y_1^{-1}\cdot g$. Then the subgroup of H generated by $\{y_2,h_1,\dots,h_i,h'_1,\dots,h'_{2^n}\}\cup\{g_r : |r|\le n\}$ is the free product $\langle y_2\rangle*\langle\{h_1,\dots,h_i\}\cup\{g_r : |r|\le n\}\rangle$. To verify this take any non-trivial reduced word $w(y_2,\bar{h},\bar{g})$ and replace all occurrences of y_2 by $y_1^{-1}\cdot g$. It is easy to see that no occurrence of y_1 can be reduced in this word. Thus $w(y_2,\bar{h},\bar{g})$ cannot be equal to 1 in H.

Let $\Phi(y, z, x_1, \ldots, x_i, x'_1, \ldots, x'_{2^n}, \ldots, z_r, \ldots)$, $r \in 2^{\leq n}$, be a set of strict quantifier-free Horn formulas describing the property that $\{y, z, x_1, \ldots, x_i, x'_1, \ldots, x'_{2^n}, \ldots, z_r, \ldots\}$ generates the free product $\langle y \rangle * \langle z, x_1, \ldots, x_i, x'_1, \ldots, x'_{2^n}, \ldots, z_r, \ldots \rangle$. Applying Ziegler's theorem and the fact that G is existentially closed to Φ and an appropriate extension of $G * \langle y_1 \rangle$, we find h_{i+1} and $h_{i+2} \in G$ satisfying the same equations with y_1 and y_2 over $\{g, h_1, \ldots, h_i, h'_1, \ldots, h'_{2^n}\} \cup \{g_r : |r| \leq n\}$. These elements satisfy the statement of the claim.

By the assumptions on $h_i, \ldots, h_i$ the claim implies that the set

$$G \setminus \langle \{h_1, \ldots, h_i, h'_1, \ldots, h'_{2^n}\} \cup \{g_r : |r| \leq n\} \rangle$$

is a subset of $X_{n+2} \cdot X_{n+2} \subseteq X_{n+3}$. Since this set is non-trivial, we see that $G \subseteq X_{n+3} \cdot X_{n+3} \subseteq X_{n+4}$. This is a contradiction with the assumptions of the proposition. □

We now finish the proof of Proposition 2 as follows. Define $g_{s0} \in X_n$, $|s| = n$, to be h'_i, where $i - 1$ is $\{0, 1\}$-presented by s. Define $g_{s1} \in G \setminus X_{n+2}$ to be the corresponding h_i. The statement of the proposition is obvious. □

We now notice that the assumptions of Proposition 2 also imply *existence of a binary tree $\gamma_s \in G$, $s \in 2^{<\omega}$, so that for all s, $g_{s1}^{\gamma_{s1}} = g_{s0}^{\gamma_{s0}} = g_{s0}^{\gamma_s}$.*

Indeed, assume that all γ_s with $|s| = n$ are defined. Let $f_s \in G$ conjugate g_{s1} to g_{s0} and commute with each g_r where r is an initial segment of s. The existence of such f_s follows by an obvious argument involving HNN-extensions and existential closure. Let $\gamma_{s0} = \gamma_s$ and $\gamma_{s1} = f_s \cdot \gamma_s$.

Proof of Theorem 1: Find trees $\{g_s : s \in 2^{<\omega}\}$ (as in Proposition 2) and $\{\gamma_s : s \in 2^{<\omega}\}$ as above. Since G is ω_1-existentially closed there are $\gamma_\sigma \in G$, $\sigma \in 2^\omega$, satisfying $g_s^{\gamma_\sigma} = g_s^{\gamma_s}$ for all s of the form $\sigma|n$. For $\sigma \neq \tau$ with $\sigma|n = s0$ and $\tau|n = s1$ we have $g_{s0}^{\gamma_\sigma} = g_{s0}^{\gamma_{s0}} = g_{s0}^{\gamma_s}$ and $g_{s1}^{\gamma_\tau} = g_{s1}^{\gamma_{s1}} = g_{s0}^{\gamma_s}$. Since $g_{s0} \in X_n$ and $g_{s1} \notin X_{n+2}$, we see that $\gamma_\sigma \gamma_\tau^{-1} \notin X_n$. On the other hand, as $G = \bigcup X_n$, we may assume that X_{n-2} contains uncountably many elements γ_δ. Thus we may assume that γ_σ and γ_τ as above are in X_{n-2}. This gives a contradiction with the condition $\gamma_\sigma \gamma_\tau^{-1} \notin X_n$. □

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