Embedding into discretely absolutely star-Lindelöf spaces

Yan-Kui Song

Abstract. A space X is *discretely absolutely star-Lindelöf* if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a countable subset F of D such that F is discrete closed in X and $\mathrm{St}(F,\mathcal{U}) = X$, where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$. We show that every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace.

Keywords: normal, star-Lindelöf, centered-Lindelöf

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1. Introduction

By a space, we mean a topological space. A space X is *absolutely star-Lindelöf* (see [1]) (*discretely absolutely star-Lindelöf*)(see [10], [11]) if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a countable subset F of D such that $\mathrm{St}(F,\mathcal{U}) = X$ (F is discrete and closed in X and $\mathrm{St}(F,\mathcal{U}) = X$, respectively), where $\mathrm{St}(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$.

A space X is *star-Lindelöf* (see [4], [7] under different names) (*discretely star-Lindelöf*) (see [9], [15]) if for every open cover $\mathcal{U}$ of X, there exists a countable subset (a countable discrete closed subset, respectively) F of X such that $\mathrm{St}(F,\mathcal{U}) = X$. It is clear that every separable space and every discretely star-Lindelöf space are star-Lindelöf as well as every space of countable extent (in particular, every countably compact space or every Lindelöf space).

A family of subsets is *centered* (*linked*) provided every finite subfamily (every two elements, respectively) has nonempty intersection and a family is called *σ-centered* (*σ-linked*) if it is the union of countably many centered subfamilies (linked subfamilies, respectively). A space X is *centered-Lindelöf* (*linked-Lindelöf*) (see [2], [3]) if every open cover $\mathcal{U}$ of X has a σ-centered (σ-linked) subcover.

From the above definitions, it is not difficult to see that every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, every absolutely star-Lindelöf space is star-Lindelöf, every star-Lindelöf space is centered-Lindelöf, and every centered-Lindelöf space is linked-Lindelöf.

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Bonanzinga and Matveev [2] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed subspace in a Hausdorff (regular, Tychonoff, respectively)star-Lindelöf space. They asked if every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed G_δ-subspace in a Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. The author [10] gave a positive answer to their question. The author [11] showed that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed G_δ-subspace in a Hausdorff (regular, Tychonoff, respectively) absolutely star-Lindelöf space. The author [12] showed that every separable Hausdorff (regular, Tychonoff, normal) star-Lindelöf space can be represented in a Hausdorff (regular, Tychonoff, normal, respectively) discretely absolutely star-Lindelöf space as a closed G_δ-subspace. Thus, it is natural for us to consider the following question:

Question. *Is it true that every Hausdorff* (*regular, Tychonoff*) *linked-Lindelöf space can be represented a closed subspace in a Hausdorff* (*regular, Tychonoff, respectively*) *discretely absolutely star-Lindelöf space? And can it be embedded as a closed G_δ-subspace?*

The purpose of this note is to give a construction showing that every Hausdorff linked-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace, which gives a positive answer to the question in the class of Hausdorff spaces.

Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$ and $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$. Other terms and symbols that we do not define will be used as in [5].

2. Embedding into discretely absolutely star-Lindelöf spaces

First, we show that every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace.

Recall the definition of the Alexandorff duplicate $A(X)$ of a space X. The underlying set of $A(X)$ is $X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is of the from $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x in X. It is well-known that $A(X)$ is Hausdorff (regular, Tychonoff, normal) iff X is, $A(X)$ is compact iff X is and $A(X)$ is Lindelöf iff X is.

Recall from [6] that a space X is *absolutely countably compact* (= acc) if for every open cover $\mathcal{U}$ of X and every dense subset D of X, there exists a finite subset F of D such that $\mathrm{St}(F, \mathcal{U}) = X$. It is not difficult to show that every

Hausdorff absolutely countably compact space is countably compact (see [6]). In our construction, we use the following lemma.

Lemma 2.1 ([8], [14]). *If X is countably compact, then $A(X)$ is acc. Moreover, for any open cover $\mathcal{U}$ of $A(X)$, there exists a finite subset F of $X \times \{1\}$ such that $A(X) \setminus \mathrm{St}(F,\mathcal{U}) \subseteq X \times \{0\})$ is a finite subset consisting of isolated points of $X \times \{0\}$.*

Theorem 2.2. *Every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace.*

Proof: If $|X| \leq \omega$, then X is separable; the author [12] showed that every separable Hausdorff (regular, Tychonoff, normal) space can be represented in Hausdorff (regular, Tychonoff, normal respectively) discretely absolutely star-Lindelöf space as a closed G_δ-subspace.

Let X be a star-Lindelöf space with $|X| > \omega$, let T be X with the discrete topology and let

$$Y = T \cup \{\infty\}, \text{ where } \infty \notin T$$

be the one-point Lindelöfication of T. Pick a cardinal κ with $\kappa \geq |X|$. Define

$$S(X,\kappa) = X \cup (Y \times \kappa^+).$$

We topologize $S(X,\kappa)$ as follows: $Y \times \kappa^+$ has the usual product topology and is an open subspace of $S(X,\kappa)$, and a basic neighborhood of a point x of X takes the form

$$G(U,\alpha) = U \cup (U \times (\alpha,\kappa^+)),$$

where U is a neighborhood of x in X and $\alpha < \kappa^+$. Then, it is easy to see that X is a closed subset of $S(X,\kappa)$ and $S(X,\kappa)$ is Hausdorff if X is Hausdorff.

Let

$$\mathcal{R}(X) = A(S(X,\kappa)) \setminus (X \times \{1\}).$$

Then $\mathcal{R}(X)$ is Hausdorff if X is Hausdorff.

We show that $\mathcal{R}(X)$ is discretely absolutely star-Lindelöf. To this end, let $\mathcal{U}$ be an open cover of $\mathcal{R}(X)$. Without loss of generality, we assume that $\mathcal{U}$ consists of basic open sets of $\mathcal{R}(X)$. Let S be the set of all isolated points of κ^+ and let

$$D_1 = ((T \times S) \times \{0\}) \cup ((T \times \kappa^+) \times \{1\}) \text{ and } D_2 = (\{\infty\} \times \kappa^+) \times \{1\}.$$

Set $D = D_1 \cup D_2$. Then, every element of D is isolated in $\mathcal{R}(X)$, and so every dense subset of $\mathcal{R}(X)$ contains D. Thus, it is sufficient to show that there exists a countable subset F of D such that F is discrete closed in $\mathcal{R}(X)$ and $\mathrm{St}(F,\mathcal{U}) = \mathcal{R}(X)$.

For each $x \in X$, there exists a $U_x \in \mathcal{U}$ such that $\langle x, 0\rangle \in U_x$, Hence there exist $\alpha_x < \kappa^+$ and an open neighborhood V_x of x in X such that

$$(V_x \times \{0\}) \cup A(V_x \times (\alpha_x, \kappa^+)) \subseteq U_x.$$

If we put $\mathcal{V} = \{V_x : x \in X\}$, then $\mathcal{V}$ is an open cover of X, hence there exists a countable subset F_1' of X such that $X = \mathrm{St}(F_1', \mathcal{V})$, since X is star-Lindelöf. We pick $\alpha_0 > \sup\{\alpha_x : x \in X\}$. Let

$$X_1 = (X \times \{0\}) \cup A(T \times [\alpha_0, \kappa^+));$$
$$X_2 = A(T \times [0, \alpha_0]) \ \text{ and } \ X_2 = A(\{\infty\} \times \kappa^+).$$

Then,

$$X = X_1 \cup X_2 \cup X_3.$$

Let

$$F_1 = (F_1' \times \{\alpha_0\}) \times \{1\}.$$

Then, F_1 is a countable subset of D_1 and

$$X_1 \subseteq \mathrm{St}(F_1, \mathcal{U}),$$

since $U_x \cap F_1 \neq \emptyset$ for each $x \in X$. Since $F_1 \subseteq D_1$ and F_1 is countable, F_1 is closed in $\mathcal{R}(X)$ by the construction of the topology of $\mathcal{R}(X)$.

On the other hand, since Y is Lindelöf and $[0, \alpha_0]$ is compact, $Y \times [0, \alpha_0]$ is Lindelöf, hence $X_1 = A(Y \times [0, \alpha_0])$ is Lindelöf. For each $\alpha \leq \alpha_0$ there exists a $U_\alpha \in \mathcal{U}$ such that $\langle\langle\infty, \alpha\rangle, 0\rangle \in U_\alpha$, hence there exists an open neighborhood V_α of α in κ^+ and an open neighborhood V_α' of ∞ in Y such that

$$A(V_\alpha' \times V_\alpha) \setminus (\langle\langle\infty, \alpha\rangle, 1\rangle) \subseteq U_\alpha.$$

Let $\mathcal{V}' = \{V_\alpha : \alpha \leq \alpha_0\}$. Then, $\mathcal{V}'$ is an open cover of $[0, \alpha_0]$. Hence, there exists a finite subcover $V_{\alpha_1}, V_{\alpha_2}, \ldots V_{\alpha_n}$, since $[0, \alpha_0]$ is compact. Let

$$T_1 = \bigcup\{T \setminus V_{\alpha_i}' : i \leq n\}.$$

Then, T_1 is a countable subset of T. For each $i \leq n$, we pick $x_i \in D_1 \cap U_{\alpha_i}$. Let $F_2' = \{x_i : i \leq n\}$. Then, F_2' is a finite subset of D_1 and

$$((\{\infty\} \times [0, \alpha_0]) \times \{0\}) \cup A((T \setminus T_1) \times [0, \alpha_0]) \subseteq \mathrm{St}(F_2', \mathcal{U}).$$

For each $t \in T_1$, since $\{t\} \times [0, \alpha_0]$ is compact, $A(\{t\} \times [0, \alpha_0])$ is compact as well, hence there exists a finite subset F_t of D_1 such that

$$A(\{t\} \times [0, \alpha_0]) \subseteq \mathrm{St}(F_t, \mathcal{U}).$$

Let $F_2'' = \bigcup\{F_t : t \in T_1\}$. Then, F_2'' is countable, since T_1 is countable. Since $F_2'' \cap A(\{\alpha\} \times Y)$ is countable for each $\alpha < \kappa^+$ and $F_2'' \cap A(\{\kappa^+\} \times \{t\})$ is finite for each $t \in T$, F_2'' is closed in $\mathcal{R}(X)$ by the construction of the topology of $\mathcal{R}(X)$. By the definition of F_2'', we have

$$A(T_1 \times [0, \alpha_0]) \subseteq \mathrm{St}(F_2'', \mathcal{U}).$$

Then, $F_2 = F_2' \cup F_2''$ is a countable closed subset of D_2, since F_1' and F_2'' are closed in $\mathcal{R}(X)$, and

$$X_2 \cup ((\{\infty\} \times [0, \alpha_0]) \times \{0\}) \subseteq \mathrm{St}(F_2, \mathcal{U}).$$

Finally, we show that there exists a finite subset F_3 of D such that $X_3 \subseteq \mathrm{St}(F_3, \mathcal{U})$. Since $\{\infty\} \times \kappa^+$ is countably compact, then, by Lemma 2.1, there exists a finite subset $F_3' \subseteq (\{\infty\} \times \kappa^+) \times \{1\}$ such that

$$E = X_3 \setminus \mathrm{St}(F_3', \mathcal{U}) \subseteq (\{\infty\} \times \kappa^+) \times \{0\} \text{ is a finite subset}$$

and each point of E is an isolated point of $(\{\infty\} \times \kappa^+) \times \{0\}$. For each point $x \in E$, there exists a $U_x \in \mathcal{U}$ such that $x \in U_x$. For each point $x \in E$, pick $d_x \in D \cap U_x$. Let $F_3'' = \{d_x : x \in E\}$; then F_3'' is a finite subset of D and $E \subseteq \mathrm{St}(F_3'', \mathcal{U})$. If we put $F_3 = F_3' \cup F_3''$, then F_3 is a finite subset of D and

$$X_3 \subseteq \mathrm{St}(F_3, \mathcal{U}).$$

If we put $F = F_1 \cup F_2 \cup F_3$, then F is a countable subset of D such that $\mathrm{St}(F, \mathcal{U}) = \mathcal{R}(X)$. Since F_1 and F_2 are closed in $\mathcal{R}(X)$, F_3 is finite, and each point of F is isolated, F is discrete closed in X, which completes the proof. □

Since every discretely absolutely star-Lindelöf space is discretely star-Lindelöf, the next corollary follows from Theorem 2.2.

Corollary 2.3. *Every Hausdorff star-Lindelöf space can be represented in a Hausdorff discretely star-Lindelöf space as a closed subspace.*

Since every discretely absolutely star-Lindelöf space is absolutely star-Lindelöf, the next corollary follows from Theorem 2.2.

Corollary 2.4. *Every Hausdorff star-Lindelöf space can be represented in a Hausdorff absolutely star-Lindelöf space as a closed subspace.*

Bonanzinga and Matveev [2] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be represented a closed subspace in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space. Thus, we have the following corollary.

Corollary 2.5. *Every Hausdorff linked-Lindelöf space can be represented in a Hausdorff discretely absolutely star-Lindelöf space as a closed subspace.*

We have the following two propositions on the separation of Theorem 2.2.

Proposition 2.6. *If X is locally countable (i.e., each point of X has a neighborhood U with $|U| \leq \omega$) and Tychonoff, then $S(X,\kappa)$ is Tychonoff (hence, $\mathcal{R}(X)$ is Tychonoff).*

Proof: Assume that X is locally countable, Tychonoff and let $x \in X$. Since a locally countable, Tychonoff space is zero-dimensional, x has a neighborhood base $\mathcal{U}(x)$ in X consisting countable, open-closed sets in X. If $U \in \mathcal{U}(x)$, then the set

$$G(U,\alpha) = U \cup (U \times (\alpha, \kappa^+))$$

is open-closed in $S(X,\kappa)$ for each $\alpha < \kappa^+$. Hence, x has a neighborhood base in $S(X,\kappa)$ consisting of open-closed sets, which implies $S(X,\kappa)$ is Tychonoff. □

Proposition 2.7. *If X is not locally countable, then $S(X,\kappa)$ is not regular (hence, $\mathcal{R}(X)$ is not regular).*

Proof: If X is not locally countable, then there exists a point $x \in X$ which has no countable neighborhood in X. Let $\mathcal{U}(x)$ be a neighborhood base of x in X. If $U \in \mathcal{U}$ and $\alpha < \kappa^+$, then the closure of $G(U,\alpha)$ in $S(X,\kappa)$ contains $\langle\infty,\gamma\rangle$ for each $\gamma > \alpha$ by construction of Y. This means that for any $U, V \in \mathcal{U}(x)$ and any $\alpha, \beta < \kappa^+$,

$$\langle\infty,\gamma\rangle \in \mathrm{cl}_{S(X,\kappa)}\, G(U,\alpha) \setminus G(V,\beta)$$

for each $\gamma > \alpha$. Hence, $S(X,\kappa)$ is not regular. □

By Theorem 2.2 and Proposition 2.6, we have the following theorem:

Theorem 2.8. *Every locally-countable, star-Lindelöf Tychonoff space can be represented in a discretely absolutely star-Lindelöf Tychonoff space as a closed subspace.*

Remark 1. In Theorem 2.2, even if X is locally-countable normal, $\mathcal{R}(X)$ need not be normal. Indeed, $X \times \{0\}$ and $A(\{\infty\} \times \kappa^+)$ are disjoint closed subsets of $\mathcal{R}(X)$ that cannot be separated by disjoint open subsets of $\mathcal{R}(X)$. Thus, the author does not know if every normal star-Lindelöf space can be represented in a normal discretely absolutely star-Lindelöf space as a closed subspace.

Remark 2. From Theorem 2.2, it is not difficult to see that $X \times \{0\}$ is not a closed G_δ subset of $\mathcal{R}(X)$. Thus, the author does not know if every Hausdorff (regular, Tychonoff) star-Lindelöf space can be represented in a Hausdorff (regular, Tychonoff, respectively) discretely absolutely star-Lindelöf space as a closed G_δ-subspace.

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Institute of Mathematics, School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing, 210097, P.R. China

E-mail: songyankui@njnu.edu.cn

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