Banach space valued mappings of the first Baire class contained in usco mappings

Jiří Spurný

Abstract. We prove that any Baire-one usco-bounded function from a metric space to a closed convex subset of a Banach space is the pointwise limit of a usco-bounded sequence of continuous functions.

Keywords: Baire-one functions, usco map, usco-bounded sequence of continuous functions

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1. Introduction

O. Kalenda studied in [2] the following question:

Let X be a metric space, Y a convex subset of a normed linear space and $f : X \to Y$ a Baire-one function whose graph is contained in the graph of a usco mapping $\varphi : X \to Y$. Does there exist a sequence $\{f_n\}$ of continuous functions $f_n : X \to Y$ such that $f_n \to f$ and the graphs of all f_n's are contained in a usco map $\psi : X \to Y$?

(We refer the reader to the next section and [2] for terminology.) He answered the question affirmatively in case Y is a closed convex subset of the Euclidean space $\mathbb{R}^d$ ([2, Theorem 3.3]). The aim of this note is a positive answer to [2, Question 4.1] given by the following theorem.

Theorem 1.1. *Let (X, ρ), (Y, σ) be metric spaces and $f : X \to Y$ be a usco-bounded Baire-one mapping. Then for each $\varepsilon > 0$ there exists a usco-bounded simple function $g : X \to Y$ such that $\sup_{x\in X} \sigma(f(x), g(x)) < \varepsilon$.*

Using [2, Theorem 3.2] we get from Theorem 1.1 the following strengthening of [2, Theorem 3.3].

Theorem 1.2. *Let X be a metric space, Y a closed convex subset of a Banach space and $f : X \to Y$ a Baire-one usco-bounded function. Then there exists a usco-bounded sequence $\{f_n\}$ of continuous functions from X to Y such that $f_n \to f$.*

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2. Proofs

We recall that a nonempty-valued mapping $\varphi : X \to Y$ between topological spaces X and Y is called *upper semi-continuous compact-valued* (briefly *usco*) if $\varphi(x)$ is a nonempty compact subset of Y for each $x \in X$ and $\{x \in X : \varphi(x) \subset U\}$ is open in X for every open $U \subset Y$. A function $f : X \to Y$ is termed *Baire-one* if f is the pointwise limit of a sequence of continuous functions. A family of functions defined on X with values in Y is called *usco-bounded* if there is a usco map $\varphi : X \to Y$ whose graph contains the graph of every function from the family.

A family $\mathcal{A}$ of subsets of a topological space X is *discrete* if each point of X has a neighbourhood intersecting at most one element of the family, $\mathcal{A}$ is *σ-discrete* if $\mathcal{A}$ is a countable union of discrete families. The family $\mathcal{A}$ is *locally finite* if each point of X has a neighbourhood meeting at most finitely many elements of $\mathcal{A}$. A family $\mathcal{B}$ is a *refinement* of $\mathcal{A}$ if $\bigcup \mathcal{A} = \bigcup \mathcal{B}$ and for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subset A$.

A function $f : X \to Y$ is called *simple* if there is a σ-discrete partition of X consisting of F_σ-sets such that f is constant on each element of the partition.

Lemma 2.1. *Let X and Y be metric spaces and let $\varphi : X \to Y$ be a set-valued mapping with nonempty values. Then the following assertions are equivalent:*

(i) *there exists a usco map $\psi : X \to Y$ such that $\varphi \subset \psi$ (i.e., the graph of φ is contained in the graph of ψ),*
(ii) *if $\{x_n\} \subset X$ converges to $x \in X$ and $y_n \in \varphi(x_n)$, then the sequence $\{y_n\}$ has a convergent subsequence.*

Proof: See [2, Lemma 2.1]. □

Lemma 2.2. *Let X be a metric space and $\varepsilon > 0$. Then there exists a σ-discrete locally finite partition of X consisting of F_σ-sets of diameter smaller than ε.*

Proof: Given $\varepsilon > 0$, let $\mathcal{U}$ be an open cover of X consisting of sets of diameter smaller than ε. By [1, Theorem 4.4.1] we can find an open σ-discrete locally finite refinement $\mathcal{V}$ of $\mathcal{U}$. We pick a well-ordering $\leq$ of $\mathcal{V}$ and set

$$P_V = V \setminus \bigcup \{W : W \in \mathcal{V}, W < V\}, \quad V \in \mathcal{V}.$$

Then $\mathcal{P} = \{P_V : V \in \mathcal{V}\}$, as a shrinking of $\mathcal{V}$ (see [1, p. 386]), is also σ-discrete and locally finite. Obviously, $\mathcal{P}$ consists of F_σ-sets of diameter smaller than ε. This finishes the proof. □

Proof of Theorem 1.1: Let f be as in the premise and $\varepsilon > 0$. We select $\eta \in (0, \frac{\varepsilon}{4})$. According to [2, Lemma 2.2], there exists a simple function $g_1 : X \to Y$ such that $\sup_{x \in X} \sigma(f(x), g_1(x)) < \eta$. By the definition of simple functions, there is a σ-discrete partition $\mathcal{A}$ of X consisting of F_σ-sets such that g_1 is constant on each element of $\mathcal{A}$.

For each $A \in \mathcal{A}$ we find a point $x_A \in A$ and set

$$g_2(x) = f(x_A), \quad x \in A \in \mathcal{A}.$$

Then g_2 is also a simple function and $\sup_{x\in X} \sigma(f(x), g_2(x)) \leq 2\eta$. Indeed, for $x \in A \in \mathcal{A}$ we have

$$\begin{aligned}\sigma(f(x), g_2(x)) &= \sigma(f(x), f(x_A))\\ &\leq \sigma(f(x), g_1(x_A)) + \sigma(g_1(x_A), f(x_A))\\ &= \sigma(f(x), g_1(x)) + \sigma(g_1(x_A), f(x_A))\\ &< 2\eta.\end{aligned}$$

Let $\mathcal{A} = \bigcup_n \mathcal{A}_n$ where each $\mathcal{A}_n$ is discrete. Using Lemma 2.2 we find σ-discrete locally finite partitions $\mathcal{P}_n$, $n \in \mathbb{N}$, of X such that each element of $\mathcal{P}_n$ is an F_σ-set of diameter smaller than $\frac{1}{n}$. For each $n \in \mathbb{N}$ we set $\mathcal{B}_n = \mathcal{A}_n \wedge \mathcal{P}_n$, i.e.,

$$\mathcal{B}_n = \{A \cap P : A \in \mathcal{A}_n, P \in \mathcal{P}_n\}.$$

A routine verification yields that each $\mathcal{B}_n$ is a σ-discrete locally finite family of pairwise disjoint sets. Then $\mathcal{B} = \bigcup_n \mathcal{B}_n$ is a σ-discrete partition of X consisting of F_σ-sets.

For each $B \in \mathcal{B}$ we pick a point $x_B \in B$ and define

$$g(x) = f(x_B), \quad x \in B \in \mathcal{B}.$$

Then g is a simple function and $\sup_{x\in X} \sigma(f(x), g(x)) \leq 4\eta$. Indeed, given $x \in B \in \mathcal{B}$, let A be the unique set in $\mathcal{A}$ such that $B \subset A$. Then $g_2(x_B) = g_2(x_A) = g_2(x)$ and

$$\begin{aligned}\sigma(f(x), g(x)) &= \sigma(f(x), f(x_B))\\ &\leq \sigma(f(x), g_2(x_B)) + \sigma(g_2(x_B), f(x_B))\\ &= \sigma(f(x), g_2(x)) + \sigma(g_2(x_B), f(x_B))\\ &< 2\eta + 2\eta.\end{aligned}$$

To finish the proof we have to verify that g is usco-bounded. To this end, let $\{x_k\}$ be a sequence of points of X converging to x. Our aim is to find a convergent subsequence of $\{g(x_k)\}$.

For each $k \in \mathbb{N}$ we find $n_k \in \mathbb{N}$ such that $x_k \in \bigcup \mathcal{B}_{n_k}$. Assume first that $\{n_k\}$ is a bounded sequence. Then there is an integer $n \in \mathbb{N}$ such that for infinitely many k's we have $x_k \in \bigcup \mathcal{B}_n$. Since $\mathcal{B}_n$ is a locally finite family and $x_k \to x$, there is a set $B \in \mathcal{B}_n$ such that $x_k \in B$ for infinitely many k's. Since g is constant on B, $\{g(x_k)\}$ has a convergent subsequence.

If $\{n_k\}$ is not bounded, we may assume that $\{n_k\}$ is increasing. For each $k \in \mathbb{N}$ we find $B_k \in \mathcal{B}_{n_k}$ such that $x_k \in B_k$. As diameter of B_k is smaller than $\frac{1}{n_k}$ and $x_k \to x$, $x_{B_k} \to x$ as well. Since $g(x_k) = f(x_{B_k})$, we can use the hypothesis on f to conclude that $\{g(x_k)\}$ has a convergent subsequence. This finishes the proof. $\square$

Proof of Theorem 1.2: Let $f : X \to Y$ be a Baire-one usco-bounded function. Using Theorem 1.1 we construct a sequence $\{f_n\}$ of functions $f_n : X \to Y$, $n \in \mathbb{N}$, such that each f_n is usco-bounded and $\{f_n\}$ converges to f uniformly. By [2, Theorem 3.1], each f_n is a pointwise limit of a usco-bounded sequence of continuous functions from X to Y. According to [2, Theorem 3.2], the same holds true for the function f. This concludes the proof. $\square$

References

[1] Engelking R., *General Topology*, Heldermann Verlag, Berlin, 1989.

[2] Kalenda O.F.K., *Baire-one mappings contained in a usco map*, Comment. Math. Univ. Carolin. **48** (2007), 135–145.

Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis, Sokolovská 83, 186 75 Prague 8, Czech Republic

E-mail: spurny@karlin.mff.cuni.cz

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