Representation of bilinear forms in non-Archimedean Hilbert space by linear operators II

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Abstract. The paper considers the representation of non-degenerate bilinear forms on the non-Archimedean Hilbert space $\mathbb{E}_\omega \times \mathbb{E}_\omega$ by linear operators. More precisely, upon making some suitable assumptions we prove that if φ is a non-degenerate bilinear form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then φ is representable by a unique linear operator A whose adjoint operator A^* exists.

Keywords: non-Archimedean Hilbert space, bilinear form, continuous linear functionals, non-Archimedean Riesz theorem, bounded bilinear form, stable unbounded bilinear form, unstable unbounded bilinear form

Classification: 47S10, 46S10

1. Introduction

Let $\mathbb{K}$ be a field, which is complete with respect to a non-Archimedean valuation denoted $|\cdot|$. Classical examples of such a field include $\mathbb{Q}_p$, the field of p-adic numbers, where $p \geq 2$ is a prime, $\mathbb{C}_p$, the field of p-adic complex numbers, and the field of Laurent series. Fix once and for all a sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$ of nonzero elements of $\mathbb{K}$ and define $\mathbb{E}_\omega$ as the set of all sequences $u = (u_i)_{i\in\mathbb{N}}$ of elements of $\mathbb{K}$ such that the series $\sum_{i\in\mathbb{N}} \omega_i u_i^2$ converges in $\mathbb{K}$, equivalently, $\lim_{i\to\infty}(|u_i| \,.\, |\omega_i|^{1/2}) = 0$. A natural norm is defined on $\mathbb{E}_\omega$ as follows:

$$u = (u_i)_{i\in\mathbb{N}}, \quad \|u\| = \sup_{i\in\mathbb{N}} \left(|u_i| \,.\, |\omega_i|^{1/2}\right).$$

This norm is non-Archimedean in the sense that, for any $u, v \in \mathbb{E}_\omega$,

$$\|u + v\| \leq \max\left(\|u\|, \|v\|\right)$$

with the equality holding if $\|u\| \neq \|v\|$. An inner product (symmetric, bilinear, non-degenerate form) is defined on $\mathbb{E}_\omega$ as follows: for all $u = (u_i)_{i\in\mathbb{N}}$, $v = (v_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$,

$$\langle u, v\rangle := \sum_{i=0}^{\infty} \omega_i \, u_i \, v_i.$$

The vector space $\mathbb{E}_\omega$ has a special base, denoted $(e_i)_{i\in\mathbb{N}}$ where e_i is the sequence whose j-th term is 0 if $i \neq j$, and the i-th term is 1. This base satisfies the following: (i) for every i, $\|e_i\| = |\omega_i|^{1/2}$; (ii) $\langle e_i, e_j\rangle = 0$ if $i \neq j$; (iii) for every i, $\langle e_i, e_i\rangle = \omega_i$; and (iv) every $u \in \mathbb{E}_\omega$ can be written uniquely as

$$u = \sum_{i=0}^{\infty} u_i e_i, \quad \text{where} \quad u_i \in \mathbb{K}, \quad \text{and} \quad \lim_{i\to\infty} \left(|u_i| \,.\, |\omega_i|^{1/2}\right) = 0.$$

The base $(e_i)_{i\in\mathbb{N}}$ is called the canonical *orthogonal* base of $\mathbb{E}_\omega$.

In the literature, the space $\mathbb{E}_\omega$ endowed with its norm and inner product given above, is called a p-adic or non-Archimedean Hilbert space. However, one should point out that the norm on $\mathbb{E}_\omega$ does not stem from the inner product. In addition to that the space $\mathbb{E}_\omega$ contains isotropic vectors, that is, $\langle u, u\rangle = 0$ while $0 \neq u \in \mathbb{E}_\omega$.

A bilinear form $\varphi : D(\varphi) \times D(\varphi) \mapsto \mathbb{K}$ with domain $D(\varphi)$ is said to be *representable* (Definition 3.7) whenever there exists a (possibly unbounded) linear operator $A : D(A) \mapsto \mathbb{E}_\omega$ ($D(A)$ being the domain of A) such that

$$\varphi(u, v) = \langle Au, v\rangle, \quad \forall\, u \in D(A), v \in D(\varphi).$$

An unbounded bilinear form $\varphi : D(\varphi) \times D(\varphi) \mapsto \mathbb{K}$ whose domain $D(\varphi)$ contains all elements of the canonical base $(e_i)_{i\in\mathbb{N}}$ is called *stable*. The subclass of all these stable unbounded bilinear forms is denoted $\Sigma_S(\mathbb{E}_\omega \times \mathbb{E}_\omega)$. Similarly, the subclass of all bilinear forms whose domains do not contain the above-mentioned canonical base is called *unstable* and denoted $\Sigma_U(\mathbb{E}_\omega \times \mathbb{E}_\omega)$.

In Diagana [3], it was shown that if φ is a non-degenerate, symmetric bilinear form satisfying

$$\lim_{i\to\infty} \left(\frac{|\varphi(e_i, e_j)|}{\|e_i\|}\right) = 0, \quad \forall\, j \in \mathbb{N}, \tag{1.1}$$

then φ is uniquely representable. Moreover, if A denotes the (possibly unbounded) linear operator associated with φ, then its adjoint A^* does exist with $A = A^*$.

In this paper we are interested in studying representation theorems for bounded and stable unbounded bilinear forms not necessarily symmetric. Namely, it is shown that a non-degenerate (stable) bilinear form φ on $\mathbb{E}_\omega \times \mathbb{E}_\omega$ is representable whenever

$$\lim_{i\to\infty} \left(\frac{|\varphi(e_i, e_j)|}{\|e_i\|}\right) = \lim_{i\to\infty} \left(\frac{|\varphi(e_j, e_i)|}{\|e_i\|}\right) = 0, \quad \forall\, j \in \mathbb{N}. \tag{1.2}$$

Moreover, if A denotes the linear operator on $\mathbb{E}_\omega$ associated with the form φ, then the adjoint A^* of A does exist.

Beside of the above-mentioned representation results for bilinear forms, we also establish a non-Archimedean version of the Riesz's representation theorem for a subclass of linear functionals on $\mathbb{E}_\omega$. Namely, it is shown that if $F : \mathbb{E}_\omega \mapsto \mathbb{K}$ is a linear functional such that

$$\lim_{i\to\infty} \frac{|F(e_i)|}{\|e_i\|} = 0,$$

then there exists a unique vector $u_0 \in \mathbb{E}_\omega$ such that $F(u) = \langle u, u_0\rangle$ for each $u \in \mathbb{E}_\omega$. Furthermore, $|||F||| = \|u_0\|$, where $||| \cdot |||$ is the natural norm on $\mathbb{E}^*_\omega$, the (topological) dual of $\mathbb{E}_\omega$.

Representing (un)bounded bilinear forms by linear operators in the classical setting is a topic that arises in several fields such as quantum mechanics through the study of form sums associated with the Hamiltonian, mathematical physics, symplectic geometry, and the study of weak solutions to some linear partial differential equations, see, e.g., [2], [7], [11], [12]. In the non-Archimedean realm, one may expect some related applications in: (i) the study of weak solutions to some p-adic partial differential equations; and (ii) the study of a non-Archimedean version of the square root problem of Kato, which is of a great interest to the second named author.

To deal with the above-mentioned issues we shall make extensive use of the formalism of unbounded linear operators on non-Archimedean Hilbert spaces $\mathbb{E}_\omega$ [4], [5], [8] and that of (un)bounded bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$, recently introduced in [6] while studying non-Archimedean counterparts of the convergence in the sense of quadratic forms of bilinear forms defined on a Hilbert space.

3. Preliminary results

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $\omega = (\omega_i)_{i\in\mathbb{N}}$ be a sequence of nonzero elements in $\mathbb{K}$. Throughout the rest of the paper, $\mathbb{E}_\omega$ denotes the non-Archimedean Hilbert space associated with the sequence $\omega = (\omega_i)_{i\in\mathbb{N}}$, and $(e_i)_{i\in\mathbb{N}}$ stands for the canonical orthogonal base associated with $\mathbb{E}_\omega$. Define $e'_j \in \mathbb{E}^*_\omega$ for each $j \in \mathbb{N}$ by

$$x = \sum_{i\in\mathbb{N}} x_i e_i \in \mathbb{E}_\omega, \quad e'_j(x) = x_j.$$

Definition 2.1 ([4], [5], [8])**.** A stable unbounded linear operator A from $\mathbb{E}_\omega$ into $\mathbb{E}_\omega$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathbb{E}_\omega$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$. Namely, the domain $D(A)$ contains the basis $(e_i)_{i\in\mathbb{N}}$ and consists of all $u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$ such $Au = \sum_{i\in\mathbb{N}} u_i Ae_i$ converges in $\mathbb{E}_\omega$, that is,

$$\begin{cases} D(A) := \big\{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\big\}, \\[2mm] \quad Au = \displaystyle\sum_{i,j\in\mathbb{N}} a_{ij}\, e'_j(u)\, e_i \ \text{ for each } \ u \in D(A). \end{cases}$$

Definition 2.2 ([4], [5], [8]). A stable linear operator

$$\begin{cases} D(A) := \big\{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \quad \lim\limits_{i\to\infty} |u_i| \, \|Ae_i\| = 0\big\}, \\ Au = \displaystyle\sum_{i,j\in\mathbb{N}} a_{ij}\, e'_j(u)\, e_i \quad \text{for each} \quad u \in D(A), \end{cases}$$

is said to have an adjoint A^* if and only if

$$\lim_{j\to\infty} \left(\frac{|a_{ij}|}{|\omega_j|^{1/2}}\right) = 0, \quad \forall\, i \in \mathbb{N}. \tag{2.1}$$

In this case the adjoint A^* of A is uniquely expressed by

$$\begin{cases} D(A^*) := \big\{v = (v_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \quad \lim\limits_{i\to\infty} |v_i| \, \|A^*e_i\| = 0\big\}, \\ A^*u = \displaystyle\sum_{i,j\in\mathbb{N}} a^*_{ij}\; e'_j(u) e_i \quad \text{for each} \quad u \in D(A^*), \end{cases}$$

where $a^*_{ij} = \frac{\omega_j a_{ji}}{\omega_i}$.

Remark 2.3. (i) In contrast with the classical context, in the non-Archimedean setting, there are linear operators, which do not have adjoint operators.

(ii) In the classical setting, if A is a closable unbounded linear operator on a Hilbert space, then $A^{**} = \overline{A}$, where $\overline{A}$ is the closure of A. However, in the non-Archimedean setting, if the adjoint A^* of a stable unbounded linear operator A exists, then $A^{**} = A$.

2.1 Continuous linear functionals on $\mathbb{E}_\omega$

Definition 2.4. A linear functional $F : \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be continuous if there exists $K \geq 0$ such that

$$|F(u)| \leq K \,.\, \|u\| \quad \text{for each} \quad u \in \mathbb{E}_\omega.$$

The smallest constant K such that the previous inequality holds is called the norm of the continuous linear functional F and is defined by

$$|||F||| = \sup_{u\neq 0} \left(\frac{|F(u)|}{\|u\|}\right).$$

Let us remind that the space of all continuous linear functionals on $\mathbb{E}_\omega$ is denoted by $\mathbb{E}^*_\omega$ and called the (topological) dual of $\mathbb{E}_\omega$. The space $(\mathbb{E}^*_\omega, |||\cdot|||)$ is a Banach space over $\mathbb{K}$.

Proposition 2.5. *Let $F \in \mathbb{E}_\omega^*$. Then its norm $|||F|||$ can be explicitly expressed as*

$$|||F||| = \sup_{i\in\mathbb{N}} \left(\frac{|F(e_i)|}{\|e_i\|}\right).$$

The next theorem constitutes a non-Archimedean version of the well-known Riesz representation theorem [12].

Theorem 2.6. *Let $F : \mathbb{E}_\omega \mapsto \mathbb{K}$ be a linear functional such that*

$$\lim_{i\to\infty} \left(\frac{|F(e_i)|}{\|e_i\|}\right) = 0. \tag{2.2}$$

Then there exists a unique $u_0 \in \mathbb{E}_\omega$ such that

$$F(u) = \langle u, u_0\rangle, \quad \text{for all} \;\; u \in \mathbb{E}_\omega.$$

Moreover, $|||F||| = \|u_0\|$.

Proof: Obviously, (2.2) yields F is continuous, as $|||F||| = \sup_{i\in\mathbb{N}} \frac{|F(e_i)|}{\|e_i\|} < \infty$.

Let $u = \sum_{i\in\mathbb{N}} u_i e_i \in \mathbb{E}_\omega$. Now, $F(u) = \sum_{i\in\mathbb{N}} u_i F(e_i)$ is well-defined. Indeed, since $u \in \mathbb{E}_\omega$, that is, $\lim_{i\to\infty} |u_i|\|e_i\| = 0$, we have

$$\lim_{i\to\infty} |u_i F(e_i)| \le \|u\| \,.\, \lim_{i\to\infty} \left(\frac{|F(e_i)|}{\|e_i\|}\right) = 0,$$

by using assumption (2.2).

Now, set $u_0 = \sum_{i\in\mathbb{N}} (\frac{F(e_i)}{\omega_i}) e_i$. Again using assumption (2.2), one can easily see that $u_0 \in \mathbb{E}_\omega$. Moreover, $F(u) = \langle u, u_0\rangle$ for each $u \in \mathbb{E}_\omega$.

Suppose that there exists another $v_0 \in \mathbb{E}_\omega$ such that $F(u) = \langle u, v_0\rangle$ for each $u \in \mathbb{E}_\omega$. Then, $\langle u_0 - v_0, u\rangle = 0$ for each $u \in \mathbb{E}_\omega$, that is, $u_0 - v_0 \perp \mathbb{E}_\omega$. In particular, $\langle u_0 - v_0, e_i\rangle = 0$ for each $i \in \mathbb{N}$, that is, all coordinates of $u_0 - v_0$ in the canonical base $(e_i)_{i\in\mathbb{N}}$ of $\mathbb{E}_\omega$ are zero, and hence $u_0 = v_0$.

Now

$$\|u_0\| := \sup_{i\in\mathbb{N}} \left\| \frac{F(e_i)}{\omega_i} e_i \right\| = \sup_{i\in\mathbb{N}} \frac{|F(e_i)|}{\|e_i\|} = |||F|||.$$

□

3. Bilinear forms on $\mathbb{E}_\omega \times \mathbb{E}_\omega$

Definition 3.1. A mapping $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be a bilinear form whenever $u \mapsto \varphi(u, v)$ is linear for each $v \in \mathbb{E}_\omega$ and $v \mapsto \varphi(u, v)$ linear for each $u \in \mathbb{E}_\omega$.

Note that if $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is a bilinear form over $\mathbb{E}_\omega \times \mathbb{E}_\omega$, then the sum

$$\varphi(u,v) = \sum_{i,j=0}^{\infty} \Omega_{ij}\, u_i\, v_j \tag{3.1}$$

may or may not be convergent. However if both $u = (u_i)_{i\in\mathbb{N}}$ and $v = (v_i)_{i\in\mathbb{N}}$ are taken in $\mathbb{E}_\omega$ with

$$\lim_{i,j\to\infty}\left(|u_i|\,.\,|\Omega_{ij}|^{1/2}\right) = 0 \quad\text{and}\quad \lim_{i,j\to\infty}\left(|v_j|\,.\,|\Omega_{ij}|^{1/2}\right) = 0,$$

where $\Omega_{ij} = \varphi(e_i, e_j)$ for all $i, j \in \mathbb{N}$, then the sum in (3.1) converges.

3.1 Bounded bilinear forms

Definition 3.2. A non-Archimedean bilinear form $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be bounded if there exists $M \geq 0$ such that

$$|\varphi(u,v)| \leq M\,.\,\|u\|\,.\,\|v\| \quad \text{for all} \quad u, v \in \mathbb{E}_\omega. \tag{3.2}$$

The smallest M such that (3.2) holds is called the norm of the bilinear form φ and is defined by

$$\|\varphi\| = \sup_{u,v\neq 0}\left(\frac{|\varphi(u,v)|}{\|u\|\,.\,\|v\|}\right).$$

Proposition 3.3. *Let $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a bounded bilinear form. Then its norm $\|\varphi\|$ can be explicitly expressed as*

$$\|\varphi\| = \sup_{i,j\in\mathbb{N}}\left(\frac{|\varphi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right).$$

Proof: The inequality, $\|\varphi\| \geq \sup_{i,j\in\mathbb{N}}(\frac{|\varphi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|})$, is a straightforward consequence of the definition of the norm $\|\varphi\|$ of φ.

Now suppose $u, v \neq 0$. In view of the above, one has

$$\begin{aligned}
|\varphi(u,v)| &= \left|\sum_{i,j=0}^{\infty} \varphi(e_i,e_j)\; u_i v_j\right| \\
&\leq \sup_{i,j\in\mathbb{N}}\left(|\varphi(e_i,e_j)|\,.\,|u_i|\,.\,|v_j|\right) \\
&= \sup_{i,j\in\mathbb{N}}\left(\frac{|\varphi(e_i,e_j)|(|u_i|\,.\,\|e_i\|)\,(|v_j|\,.\,\|e_j\|)}{\|e_i\|\,.\,\|e_j\|}\right) \\
&\leq \|u\|\,.\,\|v\|\,.\,\sup_{i,j\in\mathbb{N}}\left(\frac{|\varphi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right)
\end{aligned}$$

and hence

$$\|\varphi\| \le \sup_{i,j\in\mathbb{N}} \left(\frac{|\varphi(e_i,e_j)|}{\|e_i\|\,.\,\|e_j\|}\right).$$

One completes the proof by combining the first and the last inequalities. □

Definition 3.4. A bounded bilinear form $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be representable whether there exists a bounded linear operator $A : \mathbb{E}_\omega \mapsto \mathbb{E}_\omega$ such that

$$\varphi(u,v) = \langle Au, v\rangle, \quad \forall\, u,v \in \mathbb{E}_\omega.$$

Theorem 3.5. *Let $\varphi : \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ be a non-degenerate bounded bilinear form on $\mathbb{E}_\omega \times \mathbb{E}_\omega$. Then φ is representable whenever* (1.2) *holds. In this case, if A denotes the linear operator associated with φ, then the adjoint A^* of A exists.*

Proof: Define the linear operator A on $\mathbb{E}_\omega$ associated with φ as follows:

$$Au := \sum_{i,j\in\mathbb{N}} \left[\frac{\varphi(e_j,e_i)}{\omega_i}\right] e'_j(u)\, e_i$$

for each $u \in \mathbb{E}_\omega$.

We first check that the linear operator A given above is well-defined on $\mathbb{E}_\omega$. For that, it suffices to see that, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left|\frac{\varphi(e_j,e_i)}{\omega_i}\right| \|e_i\| = \lim_{i\to\infty} \frac{|\varphi(e_i,e_j)|}{\|e_i\|} = 0,$$

by using assumption (1.2). Furthermore, it is routine to see that $\varphi(u,v) = \langle Au, v\rangle$ for all $u,v \in \mathbb{E}_\omega$. Of course, the linear operator A given above is unique since ϕ is non-degenerate.

Now

$$\|A\| := \sup_{i,j\in\mathbb{N}} \left(\frac{\left|\frac{\varphi(e_j,e_i)}{\omega_i}\right| \|e_i\|}{\|e_j\|}\right) = \sup_{i,j\in\mathbb{N}} \left(\frac{|\varphi(e_j,e_i)|}{\|e_j\|\,.\,\|e_i\|}\right) = \|\varphi\|,$$

and hence A is bounded.

It remains to show that A^*, the adjoint of A exists. Indeed,

$$\lim_{j\to\infty} \left(\frac{\left|\frac{\varphi(e_j,e_i)}{\omega_i}\right|}{\|e_j\|}\right) = \frac{1}{|\omega_i|}\cdot \lim_{j\to\infty} \left(\frac{|\varphi(e_j,e_i)|}{\|e_j\|}\right) = 0, \quad \forall\, i \in \mathbb{N},$$

by using assumption (1.2), and hence the adjoint A^* of A exists. □

Example 3.6. Let $(\mathbb{K}, |\cdot|) = (\mathbb{Q}_p, |\cdot|)$ equipped with the p-adic absolute value and let $\omega_i = p^{-i}$ for each $i \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ with $N_0 \geq 1$ (fixed) and set

$$\pi_{ij}^{N_0} = 1 + \frac{1}{\omega_j} + \frac{1}{\omega_i^2 \omega_j^2} + \cdots + \frac{1}{\omega_i^{N_0} \omega_j^{N_0}}$$

for all $i, j \in \mathbb{N}$.

Now, $\forall\, j \in \mathbb{N}$, $\lim_{i\to\infty} \frac{|\pi_{ij}^{N_0}|}{\|e_i\|} = \lim_{i\to\infty} \frac{|\pi_{ji}^{N_0}|}{\|e_i\|} = 0$, since $|\pi_{ij}^{N_0}| = |\pi_{ji}^{N_0}| = 1$ and $\|e_i\| = p^{i/2}$ for all $i \in \mathbb{N}$. For all $u = (u_i)_{i\in\mathbb{N}}, v = (v_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega$, define the bilinear form as follows:

$$\varphi(u, v) = \sum_{i,j=0}^{\infty} \pi_{ij}^{N_0}\ u_i v_j.$$

Obviously, φ is well-defined since, $\forall\, j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left(|u_i| \,.\, |\pi_{ij}^{N_0}|^{1/2} \right) \leq \|u\| \,.\, \lim_{i\to\infty} \frac{1}{\|e_i\|} = 0.$$

Moreover φ is non-degenerate and its norm $\|\varphi\| = 1$. Therefore, the only bounded linear operator on $\mathbb{E}_\omega$ associated with φ is the one defined by

$$Au = \sum_{i,j\in\mathbb{N}} \left[\frac{\pi_{ji}^{N_0}}{\omega_i} \right] e'_j(u) e_i$$

for each $u \in \mathbb{E}_\omega$ with $\|A\| = \sup_{i,j\in\mathbb{N}} \left(\frac{|\pi_{ij}^{N_0}|}{\|e_i\| \,.\, \|e_j\|} \right) = 1$.

It is also clear that A^*, the adjoint of A exists.

3.2 Stable unbounded bilinear forms

In this subsection we present with a representation theorem for some unbounded bilinear forms. More precisely, we consider those unbounded bilinear forms whose domains contain all elements of the canonical base $(e_i)_{i\in\mathbb{N}}$ of $\mathbb{E}_\omega$, as such a base plays a key role in the present setting. The subclass of all those types of unbounded bilinear forms will be called stable and denoted by $\Sigma_S(\mathbb{E}_\omega \times \mathbb{E}_\omega)$.

Similarly, the subclass of all unbounded bilinear forms whose domains do not contain elements of the above-mentioned canonical base will be called unstable and denoted by $\Sigma_U(\mathbb{E}_\omega \times \mathbb{E}_\omega)$. Note that a representation theorem similar to Theorem 3.9 for elements of $\Sigma_U(\mathbb{E}_\omega \times \mathbb{E}_\omega)$ will be left as an open question.

Definition 3.7. A mapping $\varphi : D(\varphi) \times D(\varphi) \subset \mathbb{E}_\omega \times \mathbb{E}_\omega \mapsto \mathbb{K}$ is said to be a stable unbounded bilinear form if $u \mapsto \varphi(u, v)$ is linear for each $v \in D(\varphi)$,

$v \mapsto \varphi(u,v)$ is linear for each $u \in D(\varphi)$, where $D(\varphi)$ is a vector subspace of $\mathbb{E}_\omega$ that contains the base $(e_i)_{i\in\mathbb{N}}$, and

$$\begin{cases} D(\varphi) := \big\{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim\limits_{i,j\to\infty} \Big(|u_i|\ |\Omega_{ij}|^{1/2}\Big) \\ \qquad\qquad = \lim\limits_{i,j\to\infty} \Big(|u_i|\ |\Omega_{ji}|^{1/2}\Big) = 0\big\}, \\ \varphi(u,v) = \sum\limits_{i,j=0}^{\infty} \Omega_{ij}\ u_i v_j, \quad \text{for all} \ \ u,v \in D(\varphi), \end{cases}$$

where $\Omega_{ij} = \varphi(e_i, e_j)$.

The space $D(\varphi)$ defined above is called the *domain* of the bilinear form φ.

Definition 3.8. A bilinear form $\varphi : D(\varphi) \times D(\varphi) \mapsto \mathbb{K}$ ($D(\varphi)$ being its domain) is said to be *representable* whenever there exists a (possibly unbounded) linear operator $A : D(A) \mapsto \mathbb{E}_\omega$ ($D(A)$ being the domain of A) such that

$$\varphi(u,v) = \langle Au, v\rangle, \quad \forall\, u \in D(A), v \in D(\varphi).$$

Theorem 3.9. *Let $\varphi : D(\varphi) \times D(\varphi) \mapsto \mathbb{K}$ be a non-degenerate stable unbounded bilinear form. Then φ is representable whenever assumption* (1.2) *holds. In this case, if A denotes the linear operator associated with φ, then the adjoint A^* of A exists.*

Proof: For all $u = (u_i)_{i\in\mathbb{N}}$, $v = (v_j)_{j\in\mathbb{N}} \in D(\varphi)$, write $\varphi(u,v) = \sum_{i,j=0}^{\infty} \Omega_{ij}\ u_i v_j$ and define the linear operator A on $\mathbb{E}_\omega$ associated to it as follows:

$$\begin{cases} D(A) := \big\{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \ \lim\limits_{i\to\infty} |u_i|\ \|Ae_i\| = 0\big\}, \\ Au = \sum\limits_{i,j\in\mathbb{N}} \left[\dfrac{\varphi(e_j,e_i)}{\omega_i}\right] e_j'(u) e_i \ \text{ for each } \ u = (u_i)_{i\in\mathbb{N}} \in D(A). \end{cases}$$

Obviously, A is well-defined. Indeed, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \left|\frac{\varphi(e_j,e_i)}{\omega_i}\right| \|e_i\| = \lim_{i\to\infty} \frac{|\varphi(e_j,e_i)|}{\|e_i\|} = \lim_{i\to\infty} \frac{|\varphi(e_i,e_j)|}{\|e_i\|} = 0,$$

by using assumption (1.2).

Now

$$Au = \sum_{i\in\mathbb{N}} \frac{1}{\omega_i} \Big(\sum_{j\in\mathbb{N}} u_j \varphi(e_j,e_i)\Big) e_i \ \text{ for each } \ u = (u_i)_{i\in\mathbb{N}} \in D(A),$$

and hence $\langle Ae_i, e_j \rangle = \varphi(e_j, e_i)$ for all $i, j \in \mathbb{N}$.

Moreover, $D(A) \subset D(\varphi)$. Indeed, if $u = (u_i)_{i \in \mathbb{N}} \in D(A)$, then using the Cauchy-Schwartz inequality it follows that, $\forall\, i, j \in \mathbb{N}$,

$$\begin{aligned} |u_i| \,.\, |u_j| \,.\, |\varphi(e_i, e_j)| &= |u_i| \,.\, |u_j| \,.\, |\langle Ae_j, e_i \rangle| \\ &\le (|u_j| \,.\, \|Ae_j\|) \,.\, (\|e_i\| \,.\, |u_i|), \end{aligned}$$

and hence

$$\begin{aligned} \left(\lim_{i,j \to \infty} |u_i| \,.\, |\varphi(e_i, e_j)|^{1/2} \right)^2 &\le \lim_{i,j \to \infty} (|u_j| \,.\, \|Ae_j\|) \,.\, (\|e_i\| \,.\, |u_i|) \\ &= 0, \end{aligned}$$

that is, $u \in D(\varphi)$.

Note that $u_i v_k \varphi(e_i, e_k) \to 0$ as $i, k \to \infty$, by using the fact that ($u \in D(A) \subset D(\varphi)$ and $v \in D(\varphi)$):

$$|u_i v_k \varphi(e_i, e_k)| = \left(|u_i| |\varphi(e_i, e_k)|^{1/2} \right) \,.\, \left(|\varphi(e_i, e_k)|^{1/2} |v_k| \right) \to 0, \quad \text{as} \quad i, k \to \infty,$$

and hence

$$\sum_{k \in \mathbb{N}} \sum_{i \in \mathbb{N}} u_i v_k \varphi(e_i, e_k) = \sum_{i \in \mathbb{N}} \sum_{k \in \mathbb{N}} u_i v_k \varphi(e_i, e_k).$$

Consequently, the following successive equalities are justified:

$$\begin{aligned} \langle Au, v \rangle &= \sum_{k \in \mathbb{N}} \omega_k v_k \frac{1}{\omega_k} \Big(\sum_{i \in \mathbb{N}} u_i \varphi(e_i, e_k) \Big) \\ &= \sum_{k \in \mathbb{N}} v_k \Big(\sum_{i \in \mathbb{N}} u_i \varphi(e_i, e_k) \Big) \\ &= \sum_{i,k \in \mathbb{N}} \varphi(e_i, e_k) u_i v_k \\ &= \varphi(u, v) \end{aligned}$$

for all $u = (u_i)_{i \in \mathbb{N}} \in D(A)$ and $v = (v_i)_{i \in \mathbb{N}} \in D(\varphi)$.

Furthermore, the uniqueness of A is guaranteed by the fact that φ is non-degenerate. It remains to show that A^*, the adjoint of A exists; however, this can be done as in the bounded case. $\square$

Example 3.10. This example is a generalization of Example 3.6. Consider the bilinear form defined by

$$\varphi(u,v) = \sum_{i,j\in\mathbb{N}} \pi_{ij} \,.\, u_i v_j, \quad \forall\, u = (u_i)_{i\in\mathbb{N}}, v = (v_i)_{i\in\mathbb{N}} \in D(\varphi),$$

where $(\pi_{ij})_{i,j\in\mathbb{N}} \subset \mathbb{K}$ is an arbitrary sequence, and the domain $D(\varphi)$ of φ is defined by

$$D(\varphi) = \Big\{ u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i,j\to\infty} \Big(|u_i| \,.\, |\pi_{ij}|^{1/2}\Big) = \lim_{i,j\to\infty} \Big(|u_i| \,.\, |\pi_{ji}|^{1/2}\Big) = 0 \Big\}.$$

Note that $\varphi(e_i, e_j) = \pi_{ij}$ for all $i, j \in \mathbb{N}$ and hence an equivalent of assumption (1.2) is:

$$\lim_{i\to\infty} \frac{|\pi_{ij}|}{\|e_i\|} = \lim_{i\to\infty} \frac{|\pi_{ji}|}{\|e_i\|} = 0. \tag{3.3}$$

Upon making assumption (3.3), the unique (possibly unbounded) linear operator associated with φ is given by

$$Au = \sum_{i,j\in\mathbb{N}} \frac{\pi_{ji}}{\omega_i}\, e'_j(u)\, e_i, \quad \forall\, u = (u_i)_{i\in\mathbb{N}} \in D(A),$$

where $D(A) = \{u = (u_i)_{i\in\mathbb{N}} \in \mathbb{E}_\omega : \lim_{i\to\infty}(\|Ae_i\| \,.\, |u_i|) = 0\}$.

In addition to the above, the adjoint A^* of A does exist under assumption (3.3).

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