A homogeneous space of point-countable but not of countable type

Désirée Basile, Jan van Mill

Abstract. We construct an example of a homogeneous space which is of point-countable but not of countable type. This shows that a result of Pasynkov cannot be generalized from topological groups to homogeneous spaces.

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1. Introduction

All spaces under discussion are Tychonoff. A space X is *of (point-) countable type* if every (point) compact subspace of X is contained in a compact subspace of X with countable character in X. It is a classical result of Henriksen and Isbell [5] that a space X is of countable type if and only if for every (equivalently: some) compactification γX of X the remainder $\gamma X \backslash X$ is Lindelöf. Note that every first countable space is of point-countable type.

By a result of Pasynkov [9], every topological group of point-countable type is a paracompact p-space and hence is of countable type. Hence in topological groups, point-countable type and countable type are equivalent notions. So, it is a natural question if the same is true in homogeneous spaces (including coset spaces of topological groups). We will construct several examples of homogeneous spaces that are even coset spaces of topological groups that are of point-countable type but not of countable type.

2. The examples

Our first example is σ-compact and hence Lindelöf.

Example 2.1. There is a homogeneous σ-compact space Y such that

(1) Y is first countable, hence of point-countable type,
(2) Y is not of countable type,
(3) Y has a compactification γX such that $\gamma X \setminus X$ is completely metrizable.

Proof: Let 2^ω denote the Cantor set, and let $\underline{0}$ denote the point in 2^ω all coordinates of which are 0. Topologize $X = 2^\omega \times 2^\omega$ in the following way:

(a) a basic neighborhood of a point $\langle x, \underline{0}\rangle$ has the form
$$U(x,A,B) = (A \times 2^\omega) \setminus (\{x\} \times B),$$
where $A, B \subseteq 2^\omega$ are clopen, $\underline{0} \notin B$ and $x \in A$;

(b) a basic neighborhood of a point $\langle x, b\rangle$, where $b \in 2^\omega \setminus \{\underline{0}\}$, has the form
$$V(x,B) = \{x\} \times B,$$
where B is a clopen subset of 2^ω such that $b \in B$ and $\underline{0} \notin B$.

It is not hard to prove that X is compact (Hausdorff) and first countable (see van Mill [7, Lemma 2.2]).

Claim 1. Let $x \in 2^\omega$, and let $A, B \subseteq 2^\omega$ be clopen such that $x \in A$ and $\underline{0} \notin B$. Then

(1) $U(x,A,B)$ is homeomorphic to X by a homeomorphism that sends $A \times \{\underline{0}\}$ onto $2^\omega \times \{\underline{0}\}$,

(2) $V(x,B)$ is homeomorphic to 2^ω.

Proof: This is clear since all nonempty clopen subsets of 2^ω are homeomorphic to 2^ω. ◊

Now put $D = (2^\omega \times 2^\omega) \setminus (2^\omega \times \{\underline{0}\})$, and $Y = (X^\omega \setminus D^\omega) \times 2^\omega$, respectively. Then Y is a dense F_σ-subset of $X^\omega \times 2^\omega$. Observe that D is not Lindelöf, being a topological sum of continuum many copies of $2^\omega \setminus \{\underline{0}\}$. Hence $Z = X^\omega \times 2^\omega$ is a compactification of Y whose remainder $D' = D^\omega \times 2^\omega$ is not Lindelöf but is completely metrizable.

The Homogeneity Lemma in van Mill [8, Lemma 2.1] states that a separable metrizable zero-dimensional space is homogeneous if and only if all points $x, y \in X$ have arbitrarily small homeomorphic clopen neighborhoods. An inspection of the proof shows that it in fact works for the broader class of all first countable spaces. We use this to prove that Y is homogeneous.

Claim 2. Any point in Z has arbitrarily small clopen neighborhoods W that are homeomorphic to Z by a homeomorphism that maps $W \cap Y$ onto Y.

Proof: Let $z \in Z$, and let $W = \prod_{n<\omega} V_n \times C$ be a basic clopen neighborhood of z in Z. Each V_n is either of the form $U(p,A,B)$ or of the form $V(q,B)$ for certain A and B (see (a) and (b) above). There is by Claim 1 a (possibly empty) finite subset $F \subseteq \omega$ such that $V_n \approx 2^\omega$ if and only if $n \in F$. For each $n \notin F$ let $f_n\colon V_n \to X$ be a homeomorphism such as in Claim 1(1). In addition, let $g\colon \prod_{n\in F} V_n \times C \to 2^\omega$ be a homeomorphism (Claim 1(2)). Let $\tau\colon \omega \to \omega \setminus F$ be an arbitrary bijection. Now define a function $F\colon \prod_{n<\omega} V_n \times C \to X^\omega \times 2^\omega$ by
$$F(\langle v, c\rangle) = \langle w, c'\rangle,$$

where for each $n < \omega$,

$$w_n = f_{\tau(n)}(v_{\tau(n)}),$$

and

$$c' = g(\langle \langle v_n \rangle_{n \in F}, c \rangle).$$

It is clear that F is a homeomorphism, and we claim that $F(W \cap Y) = Y$. To prove this, take an arbitrary $y \in W$. Then for every $n < \omega$ we have $x_{\tau(n)} \in D$ if and only if $f_{\tau(n)}(x_{\tau(n)}) \in D$ by the properties of the homeomorphism $f_{\tau(n)}$. This evidently means that $F(y) \in Y$ if and only if $y \in Y$. So we conclude that the basic neighborhood $W \cap Y$ of y in Y is homeomorphic to Y. ◊

Since Y is first countable, it is of point-countable type. So the question arises whether it is of countable type. It clearly is not since it has a compactification whose remainder is $D^\omega \times 2^\omega$ and hence is not Lindelöf. So we are done by the Henriksen-Isbell Theorem from [5]. □

Example 2.1 raises the naive question whether every σ-compact homogeneous space is of point-countable type. This can be answered rather easily. Any countable dense subgroup Z of 2^{ω_1} is a counterexample to this question. Simply observe that every compact subspace of Z is countable, hence has an isolated point. Hence if there were a compact subspace of Z with countable character, then Z would be first countable at some (equivalently: at all) points. Since this is not the case, we are done.

It is clear and well-known that a Čech complete space is of countable type. Hence it is not by accident that our example Y is far from being Čech complete. So it is quite natural to ask whether every homogeneous Baire space of point-countable type is of countable type. It is not, as is shown in our next example which was motivated by an example of van Douwen (cf. [2]).

Example 2.2. There is a homogeneous Baire space T such that

(1) T is first countable, hence of point-countable type,
(2) T is not of countable type.

Proof: We adopt the same notation as in Example 2.1. Let T be the following subspace of $Z \times Z$:

$$T = (Y \times Y) \cup (D' \times D').$$

We claim that again we can use the Homogeneity Lemma in van Mill [8, Lemma 2.1] to conclude that T is homogeneous. To this end, let $U \times V$ be a nonempty basic clopen subset of $Z \times Z$. Hence both U and V are nonempty clopen subsets of Z. There are homeomorphisms $\alpha: U \to Z$ and $\beta: V \to Z$, such that

$$\alpha(U \cap Y) = Y, \quad \beta(V \cap Y) = Y.$$

Let $\gamma = \alpha \times \beta$. Then $\gamma: U \times V \to Z \times Z$ is a homeomorphism, and, clearly, $\gamma(T \cap (U \times V)) = T$. Hence all points in T have arbitrarily small homeomorphic clopen neighborhoods in T. Hence T is homogeneous.

Observe that $D' \times D'$ is a dense completely metrizable subspace of $Z \times Z$. Hence $D' \times D'$ is a Baire space, and so is T.

Since T is first countable, it is of point-countable type. To check that it is not of countable type, simply observe that its remainder $R = (Z \times Z) \setminus (T \times T)$ contains many (relatively) closed copies of D'. Hence R is not Lindelöf, and so we are again done by the Henriksen-Isbell Theorem from [5]. $\square$

Observe that the space T in Example 2.2 is not Lindelöf since it contains a closed copy of D'. In the light of Example 2.1, this motivates the following problem.

Question 2.3. Let X be a homogeneous Lindelöf Baire space of point-countable type. Is X of countable type?

Remark 2.4. If G is a topological group acting on a space X then for every $x \in X$ we let $\gamma_x: G \to X$ be defined by $\gamma_x(g) = gx$. We also let $G_x = \{g \in G : gx = x\}$ denote the *stabilizer* of $x \in X$. Then G_x is evidently a closed subgroup of G.

A space X is a *coset space* provided that there is a topological group G with *closed* subgroup H such that X and $G/H = \{xH : x \in G\}$ are homeomorphic. Observe that G acts transitively on G/H and that the natural quotient map $\pi: G \to G/H$ is open. It is well-known, and easy to prove, that G/G_x is homeomorphic to X if γ_x is open. Observe that $H \subseteq G$ is the stabilizer of $H \in G/H$. So for a space X to be a coset space it is necessary and sufficient that there is a topological group G acting transitively on X such that for some $x \in X$ (equivalently: for all $x \in X$) the function $\gamma_x: G \to X$ is open.

It is known that many homogeneous spaces are coset spaces. Ford [4] proved that all strongly locally homogeneous spaces are coset spaces. As a consequence, all zero-dimensional homogeneous spaces are coset spaces. Ungar [10] proved that if X is homogeneous, separable metrizable, and locally compact then X is a coset space. This is a consequence of the Effros Theorem on transitive actions of Polish groups on Polish spaces (Effros [1]).

Ford [4] gave also an example of a homogeneous space that is not a coset space, hence the class of coset spaces is a proper subclass of the class of all homogeneous spaces.

Since our examples are zero-dimensional, they are even coset spaces. So, as our results show, Pasynkov's result from [9] that was mentioned in §1, cannot be generalized to coset spaces.

Remark 2.5. Let G be a topological group acting transitively on X. We saw in Remark 2.4, that if for every $x \in X$, $\gamma_x: G \to X$ is open, then X need not be of countable type provided it is of point-countable type. This motivates the question whether something of interest can be concluded if for example the maps γ_x are all closed. Let us call the action it perfect if for some $x \in X$ (equivalently: for all $x \in X$), $\gamma_x: G \to X$ is a perfect map. Then if X is of point-countable type, it is of countable type. This follows trivially from known results. Indeed, G is clearly of point-countable type, hence a paracompact p-space by Pasynkov's result quoted in §1. But then, X is a paracompact p-space as well by Filippov [3] and Ishii [6]. We do not know whether it can be shown that X is of countable type if $\gamma_x: G \to X$ is merely assumed to be closed. Unfortunately, we do not have natural examples of group actions that have this property, so pursuing this question does not seem to be very interesting.

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Faculteit Exacte Wetenschappen, Afdeling Wiskunde, Vrije Universiteit,
De Boelelaan 1081A, 1081 HV Amsterdam, The Netherlands

E-mail: basile@dmi.unict.it

Faculteit Exacte Wetenschappen, Afdeling Wiskunde, Vrije Universiteit,
De Boelelaan 1081A, 1081 HV Amsterdam, The Netherlands

E-mail: vanmill@few.vu.nl

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