Comment.Math.Univ.Carolin. 48,3 (2007)465–485

Ultrafilter-limit points in metric dynamical systems

S. García-Ferreira, M. Sanchis

Abstract. Given a free ultrafilter p on $\mathbb{N}$ and a space X, we say that $x \in X$ is the *p-limit* point of a sequence $(x_n)_{n\in\mathbb{N}}$ in X (in symbols, $x = p\text{-}\lim_{n\to\infty} x_n$) if for every neighborhood V of x, $\{n \in \mathbb{N} : x_n \in V\} \in p$. By using p-limit points from a suitable metric space, we characterize the selective ultrafilters on $\mathbb{N}$ and the P-points of $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$. In this paper, we only consider dynamical systems (X, f), where X is a compact metric space. For a free ultrafilter p on $\mathbb{N}^*$, the function $f^p : X \to X$ is defined by $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$ for each $x \in X$. These functions are not continuous in general. For a dynamical system (X, f), where X is a compact metric space, the following statements are shown:

1. If X is countable, $p \in \mathbb{N}^*$ is a P-point and f^p is continuous at $x \in X$, then there is $A \in p$ such that f^q is continuous at x, for every $q \in A^*$.

2. Let $p \in \mathbb{N}^*$. If the family $\{f^{p+n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at $x \in X$, then f^{p+q} is continuous at x, for all $q \in \beta(\mathbb{N})$.

3. Let us consider the function $F : \mathbb{N}^* \times X \to X$ given by $F(p, x) = f^p(x)$, for every $(p, x) \in \mathbb{N}^* \times X$. Then, the following conditions are equivalent.

(1) f^p is continuous on X, for every $p \in \mathbb{N}^*$.
(2) There is a dense G_δ-subset D of $\mathbb{N}^*$ such that $F|_{D\times X}$ is continuous.
(3) There is a dense subset D of $\mathbb{N}^*$ such that $F|_{D\times X}$ is continuous.

Keywords: ultrafilter, P-limit point, dynamical system, selective ultrafilter, P-point, compact metric

Classification: Primary 54G20, 54D80, 22A99: secondary 54H11

1. Preliminaries and notation

All the spaces are assumed to be Tychonoff (= completely regular and Hausdorff). If $f : X \to Y$ is a continuous function, then $\overline{f} : \beta(X) \to \beta(Y)$ will stand for the Stone extension of f. For a metric space X and $\epsilon > 0$, $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. For short, $x_n \to x$ means that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x. The Stone-Čech compactification $\beta(\mathbb{N})$ of the natural numbers $\mathbb{N}$ with the discrete topology will be identified with the set of all ultrafilters on $\mathbb{N}$, and its remainder $\mathbb{N}^* = \beta(\mathbb{N}) \setminus \mathbb{N}$ with the set of all free ultrafilters on $\mathbb{N}$. If $A \subseteq \mathbb{N}$,

Research of the second-named author was supported by Generalitat Valenciana, grant number CTESIN/2005/015, CONACYT grant no. 40057-F and PAPIIT grant no. IN-106103. Hospitality and financial support received from the *Department of Mathematics of Jaume I University* (*Spain*) where this research was essentially performed are gratefully acknowledged.

then $\hat{A} = \mathrm{cl}_{\beta(\mathbb{N})} A = \{p \in \beta(\mathbb{N}) : A \in p\}$ is a basic clopen subset of $\beta(\mathbb{N})$, and $A^* = \hat{A} \setminus A = \{p \in \mathbb{N}^* : A \in p\}$ is a basic clopen subset of $\mathbb{N}^*$. If $A, B \subseteq \mathbb{N}$, then $A \subseteq^* B$ means that $A \setminus B$ is finite. In this paper, we shall use the following fact: If $\{A_n : n \in \mathbb{N}\}$ is a family of subsets of $\mathbb{N}$ with the infinite finite intersection property, then there is an infinite subset B of $\mathbb{N}$ such that $B \subseteq^* A_n$, for every $n \in \mathbb{N}$. The set of real numbers will be denoted by $\mathbb{R}$ and the set of positive integers will be denoted by $\mathbb{N}^+$. A pair (X, f) is called a *dynamical system* if X is a Tychonoff space and $f : X \to X$ is a continuous function. If (X, f) is a dynamical system, then the *orbit* of a point $x \in X$ is the set $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{N}\}$. For an infinite set X, we let $[X]^\omega = \{A \subseteq X : |A| = \omega\}$.

Let X be space. Given $p \in \mathbb{N}^*$, a point $x \in X$ is said to be the *p-limit* point of a sequence $(x_n)_{n\in\mathbb{N}}$ in X ($x = p\text{-}\lim_{n\to\infty} x_n$) if for every neighborhood V of x, $\{n \in \mathbb{N} : x_n \in V\} \in p$. The notion of p-limit point was introduced, in the context of non-standard analysis, by R.A. Bernstein [4]. H. Furstenberg [9, p. 179] and E. Atkin [1, p. 5, 61] considered the $\mathcal{F}$-limit points in Dynamical Systems, where $\mathcal{F}$ is a family of nonempty sets with the finite intersection property (for the definition of a $\mathcal{F}$-limit point of a sequence we replace p by $\mathcal{F}$). The p-limit points play a very important role in the study of countably compact spaces. In this paper, we will give some of their applications to Dynamical Systems.

Observe that a point $x \in X$ is an adherent point of a countable set $\{x_n : n \in \mathbb{N}\}$ iff there is $p \in \beta(\mathbb{N})$ such that $x = p\text{-}\lim_{n\to\infty} x_n$. In other words, x is an adherent point of a countable set $\{x_n : n \in \mathbb{N}\}$ iff the set $\{\{n \in \mathbb{N} : x_n \in V\} : V \in \mathcal{N}(x)\}$ is a filter base on $\mathbb{N}$. Notice that $x_n \to x$ iff $x = \mathcal{F}_r\text{-}\lim_{n\to\infty} x_n$, where $\mathcal{F}_r$ is the Frechét filter $\{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$. Hence, we see that $x_n \to x$ iff $x = p\text{-}\lim_{n\to\infty} x_n$ for all $p \in \mathbb{N}^*$. It is not hard to prove that in a compact space the p-limit point of a sequence always exists and is unique (for Hausdorff spaces), for every $p \in \mathbb{N}^*$.

By using p-limit points in metric spaces, we characterize the P-points of $\mathbb{N}^*$ and the selective ultrafilters on $\mathbb{N}$. In the second section, we study the continuity of the functions f^p (for the definition of this function see the abstract) when (X, f) is a dynamical system in which X is a compact metric space. These functions have been also studied in [5], where the author establishes the connection between the algebra of $\beta(\mathbb{N})$ and an arbitrary dynamical system. We consider the particular case when p is a P-point of $\mathbb{N}^*$ and analyze the continuity of the corresponding function f^p. The functions f^p's are very useful to study the limiting behavior of the iterates of the original function f when X is a metric compact space. The fourth section is concerning with some applications to actions of compact metrizable semigroups.

2. p-limit points in metric spaces

Suppose that X is a metric space and $p \in \mathbb{N}^*$. If $x = p\text{-}\lim_{n\to\infty} x_n$, then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \to x$. In general, $x_n \not\to x$

and $\{n_k : k \in \mathbb{N}\} \notin p$. Our first task is to use this remark to characterize the P-points of $\mathbb{N}^*$ and the selective ultrafilters on $\mathbb{N}$. Let us recall a combinatorial definition of a P-point of $\mathbb{N}^*$:

An ultrafilter $p \in \mathbb{N}^*$ is called *P-point* iff for every partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $A_n \notin p$, for each $n \in \mathbb{N}$, there is $A \in p$ such that $A \cap A_n$ is finite for every $n \in \mathbb{N}$.

W. Rudin [13] proved that CH implies the existence of $2^{\mathfrak{c}}$-many P-points in $\mathbb{N}^*$, and years later S. Shelah [6] found a model of ZFC in which $\mathbb{N}^*$ does not have any P-point.

Lemma 2.1. *Let $p \in \mathbb{N}^*$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a space X. If there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$, then $x = p\text{-}\lim_{n\to\infty} x_n$.*

Proof: Let $V \in \mathcal{N}(x)$. By assumption, we know that $\{n_k : k \in \mathbb{N}\} \subseteq^* \{n \in \mathbb{N} : x_n \in V\}$. Hence, we deduce that $\{n \in \mathbb{N} : x_n \in V\} \in p$. This shows that $x = p\text{-}\lim_{n\to\infty} x_n$. $\square$

The next lemma was suggested by the referee and simplifies the original proofs of our main results of this section.

Lemma 2.2. *Let $p \in \mathbb{N}^*$ and let $\{A_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets such that $A_n \notin p$, for all $n \in \mathbb{N}$. Let $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection such that $\sigma[A_n] = \{n\} \times \mathbb{N}$, for every $n \in \mathbb{N}$. If for every $k \in \mathbb{N}$ we have that $x_k = \frac{1}{n} + \frac{1}{a_n+m}$, where $\sigma(k) = (n, m)$ and $n \le a_n \in \mathbb{N}$, then $0 = p\text{-}\lim_{k\to\infty} x_k$.*

Proof: Let $\epsilon > 0$ and assume that $A = \{k \in \mathbb{N} : x_k > \epsilon\} \in p$. Since $A_n \notin p$, for each $n \in \mathbb{N}$, we must have that $\{n \in \mathbb{N} : A \cap A_n \neq \emptyset\}$ is infinite. Hence, we can find $n > \frac{2}{\epsilon}$ such that $A \cap A_n \neq \emptyset$. Pick $k \in A \cap A_n$. Then, $\sigma(k) = (n, m)$ for some $m \in \mathbb{N}$ and we have that $x_k = \frac{1}{n} + \frac{1}{a_n+m} < \frac{2}{n} < \epsilon$, but this is a contradiction. $\square$

Theorem 2.3. *For a point $p \in \mathbb{N}^*$, the following are equivalent.*

(1) *p is a P-point of $\mathbb{N}^*$.*
(2) *In every metric space X, for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$.*
(3) *For every sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers and for every $x \in \mathbb{R}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\lim_{k\to\infty} x_{n_k} = x$.*

Proof: (1) $\Rightarrow$ (2). Necessity. Let $A_n = \{i \in \mathbb{N} : x_i \in B(x, \frac{1}{n})\}$. By assumption, $A_n \in p$ for every $n \in \mathbb{N}$. Then, we can find $A \in p$ so that $A \subseteq^* A_k$ for every $k \in \mathbb{N}$. If we enumerate A as $\{x_{n_k} : k \in \mathbb{N}\}$, then $(x_{n_k})_{k\in\mathbb{N}}$ is the desired subsequence.

Sufficiency. This follows directly from Lemma 2.1.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Suppose that p is not a P-point of $\mathbb{N}^*$. Then, there is a partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $A_n \notin p$, for every $n \in \mathbb{N}$, and for every $A \in p$ there is $n \in \mathbb{N}$ for which $A \cap A_n$ is infinite. Fix a bijection $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ so that $\sigma[A_n] = \{n\} \times \mathbb{N}$, for all $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, we define $x_k = \frac{1}{n} + \frac{1}{n+m}$ provided that $\sigma(k) = (n, m)$. Then, by Lemma 2.2, we know that $0 = p\text{-}\lim_{k\to\infty} x_k$. By assumption, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $B = \{n_k : k \in \mathbb{N}\} \in p$ and $0 = \lim_{k\to\infty} x_{n_k}$. Pick $l \in \mathbb{N}$ so that $B \cap A_l$ is infinite. Then, the sequence $(x_n)_{n\in B\cap A_l}$ must converge to $\frac{1}{l}$ and as a subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ it must converge to 0, which is impossible. □

An ultrafilter $p \in \mathbb{N}^*$ is called *selective* if for every partition $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ with $A_n \notin p$, for each $n \in \mathbb{N}$, there is $A \in p$ such that $|A \cap A_n| \leq 1$, for every $n \in \mathbb{N}$. Every selective ultrafilter is a P-point and under CH we can find $2^{\mathfrak{c}}$-many selective ultrafilters (see [7]).

Theorem 2.4. *For a point $p \in \mathbb{N}^*$, the following are equivalent.*

1. *p is selective.*
2. *In every metric space X, for every sequence $(x_n)_{n\in\mathbb{N}}$ in X and every $x \in X \setminus \{x_n : n \in \mathbb{N}\}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there are a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \leq d(x_{n_k}, x) < \frac{1}{m_k}$, for every $k \in \mathbb{N}$.*
3. *For every sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ and every $x \in \mathbb{R} \setminus \{x_n : n \in \mathbb{N}\}$, we have that $x = p\text{-}\lim_{n\to\infty} x_n$ iff there are a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $\{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \leq |x_{n_k} - x| < \frac{1}{m_k}$, for every $k \in \mathbb{N}$.*

PROOF: (1) $\Rightarrow$ (2). Necessity. Define $A_0 = \{i \in \mathbb{N} : 1 \leq d(x_i, x)\}$ and for every $1 \leq n \in \mathbb{N}$, we let $A_n = \{i \in \mathbb{N} : \frac{1}{n+1} \leq d(x_i, x) < \frac{1}{n}\}$. It is evident that $A_n \notin p$, for each $n \in \mathbb{N}$. Then, we can find $A \in p$ so that $|A \cap A_n| \leq 1$, for every $n \in \mathbb{N}$. Enumerate A as $\{x_{n_k} : k \in \mathbb{N}\}$. Then, for every $k \in \mathbb{N}$ there is a unique $m_k \in \mathbb{N}$ such that $n_k \in A_{m_k}$. Without loss of generality we may assume that the sequence $(m_k)_{k\in\mathbb{N}}$ is increasing. It clear that$(x_{n_k})_{k\in\mathbb{N}}$ is the desired subsequence.

Sufficiency. It is a consequence of Lemma 2.1.

(2) $\Rightarrow$ (3). It is evident.

(3) $\Rightarrow$ (1). Assume that p is not selective. Then there is a partition $\{A_n : n \in \mathbb{N}^+\}$ of $\mathbb{N}$ such that for all $n \in \mathbb{N}^+$, $A_n \notin p$ and for every $A \in p$ there is $n \in \mathbb{N}^+$ with $|A \cap A_n| \geq 2$. Let $\sigma : \mathbb{N}^+ \to \mathbb{N}^+ \times \mathbb{N}^+$ be a bijection such that $\sigma[A_n] = \{n\} \times \mathbb{N}^+$, for each $n \in \mathbb{N}^+$. Put $a_1 = 1$ and for $n > 1$, we let $a_n = n^2 - n$. Observe that if $n > 1$, then $\frac{1}{n} + \frac{1}{a_n} = \frac{1}{n-1}$. Now, for each $k \in \mathbb{N}^+$, we define $x_k = \frac{1}{n} + \frac{1}{a_n+m}$ provided that $\sigma(k) = (n, m)$. By Lemma 2.2, we know that $0 = p\text{-}\lim_{k\to\infty} x_k$. Then, by hypothesis, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$

and an increasing sequence of integers $(m_k)_{k\in\mathbb{N}}$ such that $B = \{n_k : k \in \mathbb{N}\} \in p$ and $\frac{1}{m_{k+1}} \le x_{n_k} < \frac{1}{m_k}$, for every $k \in \mathbb{N}$. Notice that $B \setminus A_1 \in p$. We know that there is $r \in \mathbb{N}$ with $r > 1$ such that $|B \cap A_r| = |(B \setminus A_1) \cap A_r| \ge 2$. Choose $k, l \in \mathbb{N}$ such that $k < l$ and $n_k, n_l \in B \cap A_r$. Put $\sigma(n_k) = (r, s)$ and $\sigma(n_l) = (r, t)$ for some $s, t \in \mathbb{N}^+$. Then, we have that $\frac{1}{r} < x_{n_l} = \frac{1}{r} + \frac{1}{a_r+t} < \frac{1}{m_l}$ and $\frac{1}{r-1} = \frac{1}{r} + \frac{1}{a_r} > \frac{1}{r} + \frac{1}{a_r+s} = x_{n_k} \ge \frac{1}{m_{k+1}}$. Hence, $r - 1 < m_{k+1} \le m_l < r$, which is impossible since r and m_l are natural numbers. □

3. p-limit points and dynamical systems

This section is devoted to study the continuity and discontinuity of the function $f^p : X \to X$, for $p \in \mathbb{N}^*$.

Definition 3.1. Let (X, f) be a dynamical system, where X is a compact space. For a free ultrafilter p on $\mathbb{N}$, the function $f^p : X \to X$ is defined by $f^p(x) = p$-$\lim_{n\to\infty} f^n(x)$, for every $x \in X$. For a point $x \in X$, the function $f_x := p \mapsto f^p(x) : \beta(\mathbb{N}) \to X$ is the Stone extension of the continuous function $n \mapsto f^n(x) : \mathbb{N} \to X$.

We remark that the function $f_x : \beta(\mathbb{N}) \to X$ is continuous for every $x \in X$. Observe that $f_x[\beta(\mathbb{N})] = \mathrm{cl}_X(\mathcal{O}_f(x))$. But, the functions f^p are not always continuous as we shall see in the next example:

Example 3.2. Let $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^+\}$ and define $f : X \to X$ as follows:

$$f(x) = \begin{cases} x & \text{if} \quad x \in \{0, 1\} \\ \frac{1}{n} & \text{if} \quad x = \frac{1}{n+1} \ \text{ and } \ 1 \le n \in \mathbb{N}. \end{cases}$$

It is easy to see that if $p \in \mathbb{N}^*$, then $f^p(x) = 1$ for every $x > 0$ and $f^p(0) = 0$. Thus, f^p is discontinuous at 0, for all $p \in \mathbb{N}^*$. For a connected example, take $X = [0, 1]$ and define $f : [0, 1] \to [0, 1]$ as follows:

$$f(x) = \begin{cases} x & \text{if} \quad x \in [0, \frac{1}{2}] \\ \frac{(n+1)(n+2)x-(2n+1)}{n(n+1)} & \text{if} \quad x \in [\frac{n}{n+1}, \frac{n+1}{n+2}] \text{ and } 1 \le n \in \mathbb{N} \\ 1 & \text{if} \quad x = 1. \end{cases}$$

Observe that f is a homeomorphism between the closed intervals $[\frac{n}{n+1}, \frac{n+1}{n+2}]$ and $[\frac{n-1}{n}, \frac{n}{n+1}]$, for each $1 \le n \in \mathbb{N}$. Then, we have that $f^p[[0, 1)] = [0, \frac{1}{2}]$ and $f^p(1) = 1$, for every $p \in \mathbb{N}^*$. This implies that f^p is discontinuous at 1, for all $p \in \mathbb{N}^*$.

Let us explain one way to extend the ordinary addition on the set of natural numbers to the whole $\beta(\mathbb{N})$ and how to apply this extension to the Theory of Dynamical Systems:

For $p \in \beta(\mathbb{N})$ and $n \in \mathbb{N}$, we define $p+n = p\text{-}\lim_{m\to\infty}(m+n)$ and if $p, q \in \beta(\mathbb{N})$, then we define $p + q = q\text{-}\lim_{n\to\infty} p + n$.

The following theorem is taken from [5].

Theorem 3.3. *Let (X, f) be a dynamical system where X is a compact space. Then*

$$f^p \circ f^q(x) = f^{q+p}(x),$$

for every $x \in X$ and for every $p, q \in \beta(\mathbb{N})$.

Thus, if f^q is continuous at x and f^p is continuous at $f^q(x)$, then f^{q+p} is continuous at x, for $p, q \in \beta(\mathbb{N})$.

The following two theorems are characterizations of the continuity of the function f^p at some point of the given space.

Theorem 3.4. *Let (X, f) be a dynamical system, where X is a compact metric space, and let $p \in \mathbb{N}^*$. For a point $x \in X$, the following are equivalent.*

(1) *f^p is continuous at x.*
(2) *For all $\epsilon > 0$ there is $\delta > 0$ such that for all $y \in X$ if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

Proof: (1) $\Rightarrow$ (2). Let $\epsilon > 0$. So, there is $\delta > 0$ such that if $y \in X$ and $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$. Suppose that $y \in X$ satisfies that $d(x, y) < \delta$. By definition, we have that $A = \{n \in \mathbb{N} : d(f^n(x), f^p(x)) < \frac{\epsilon}{3}\} \cap \{n \in \mathbb{N} : d(f^n(y), f^p(y)) < \frac{\epsilon}{3}\} \in p$. Hence,

$$\begin{aligned} d(f^n(x), f^n(y)) &\leq d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

for every $n \in A$.

(2) $\Rightarrow$ (1). Let $\epsilon > 0$ and let $\delta > 0$ be satisfy the conditions of our hypothesis. Fix $y \in X$ with $d(x, y) < \delta$. Then, we have that $A = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \frac{\epsilon}{3}\} \in p$. Thus,

$$\begin{aligned} d(f^p(x), f^p(y)) &\leq d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(y), f^p(y)) \\ &\leq d(f^p(x), f^n(x)) + \frac{\epsilon}{3} + d(f^n(y), f^p(y)). \end{aligned}$$

We know that $n \in A$ can be chosen so that $d(f^p(x), f^n(x)) < \frac{\epsilon}{3}$, $d(f^n(x), f^n(y)) < \frac{\epsilon}{3}$ and $d(f^n(y), f^p(y)) < \frac{\epsilon}{3}$. Therefore, $d(f^p(x), f^p(y)) < \epsilon$. This shows the continuity of f^p at x. $\square$

Definition 3.5. Let (X, f) be a dynamical system, where X is a metric space, and let $p \in \mathbb{N}^*$. We say that a sequence $(x_k)_{k \in \mathbb{N}}$ in X is *p-proximal* to a point x if $\lim_{k \to \infty} x_k = x$ and for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \in p$, for every $k \in \mathbb{N}$ with $k \geq N$. Two points $x, y \in X$ are said to be *p-proximal* if for every $\epsilon > 0$, $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.

Theorem 3.6. *Let (X, f) be a dynamical system, where X is a compact metric space, and let $p \in \mathbb{N}^*$. For a point $x \in X$ the following are equivalent.*

(1) *f^p is continuous at x.*
(2) *Every sequence $(x_k)_{k \in \mathbb{N}}$ that converges to x is p-proximal to x.*

PROOF: (1) $\Rightarrow$ (2). Let $(x_k)_{k \in \mathbb{N}}$ be a sequence converging to x. Given $\epsilon > 0$, by Theorem 3.4, we can find $\delta > 0$ such that for all $y \in X$, if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$. Let $N \in \mathbb{N}$ such that $d(x_k, x) < \delta$ for every $N \leq k \in \mathbb{N}$. Then, we have that $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \in p$ for every $k \in \mathbb{N}$ with $k \geq N$.

(2) $\Rightarrow$ (1). Let us assume that f^p is not continuous at x. Then, by Theorem 3.4, there is $\epsilon > 0$ such that for every $k \in \mathbb{N}$ there is $x_k \in X$ such that $d(x, x_k) < \frac{1}{k+1}$ and $\{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \epsilon\} \notin p$. It is evident that the sequence $(x_k)_{k \in \mathbb{N}}$ converges to x and it is not p-proximal to x. □

Next we state a classical notion in Dynamical Systems and establish its relation with the concept introduced in Definition 3.5.

Definition 3.7. Let (X, f) be a dynamical system where X is a metric space. We say that two points $x, y \in X$ are *proximal* if for every $\epsilon > 0$, $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\}$ is infinite.

The following result shows that the standard notion "proximal" is included in Definition 3.5.

Theorem 3.8. *Let (X, f) be a dynamical system, where X is a metric space, and let $x, y \in X$. The following conditions are equivalent.*

(1) *x and y are proximal.*
(2) *There is $p \in \mathbb{N}^*$ such that $f^p(x) = f^p(y)$.*
(3) *x and y are p-proximal for some $p \in \mathbb{N}^*$.*

PROOF: The equivalence (1) $\Leftrightarrow$ (2) is stated, for a general case, in [3] and it is proved in [5]. The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3). For every $\epsilon > 0$, we define $A_\epsilon = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\}$. Since the family $\{A_\epsilon : \epsilon > 0\}$ has the finite intersection property, we can find $p \in \mathbb{N}^*$ such that $\{A_\epsilon : \epsilon > 0\} \subseteq p$. It is then evident that x and y are p-proximal. □

The equivalence (2) $\Leftrightarrow$ (3) of the previous theorem can be rewritten as follows.

Theorem 3.9. *Let (X, f) be a dynamical system, where X is a metric space, let $x, y \in X$ and let $p \in \mathbb{N}^*$. The following conditions are equivalent.*

(1) *x and y are p-proximal.*
(2) *$f^p(x) = f^p(y)$.*

It follows from Theorem 3.9 that if $p \in \mathbb{N}^*$ is an idempotent (that is, $p+p=p$), then every $x \in X$ is p-proximal to $f^p(x)$. Indeed, $f^p(x) = f^{p+p}(x) = f^p(f^p(x))$.

Theorem 3.10. *Let (X, f) be a dynamical system, where X is a metric space, and let $x, y \in X$. Then, $\{p \in \mathbb{N}^* : x$ and y are p-proximal$\}$ is a closed subset of $\mathbb{N}^*$.*

Proof: Put $D = \{p \in \mathbb{N}^* : x$ and y are p-proximal$\}$ and let $q \in \mathrm{cl}_{\mathbb{N}^*} D$. Suppose that x and y are not q-proximal. Then, there is $\epsilon > 0$ such that $A = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) \geq \epsilon\} \in q$. Choose $p \in A^* \cap D$. By assumption, $B = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$. But this is impossible since $A \cap B = \emptyset$. Therefore, $D = \mathrm{cl}_{\mathbb{N}^*} D$. □

We remark that the points x and y are p-proximal, for all $p \in \mathbb{N}^*$, iff

$$\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0.$$

The next example shows that the notion of p-proximally could distinguish, in some sense, two proximal points.

Example 3.11. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$, $a_0 = 1$ and $a_{n+1} < a_n$, for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, choose a strictly decreasing sequence $(a_{n,m})_{m\in\mathbb{N}}$ such that

(1) $\lim_{m\to\infty} a_{n,m} = a_n$, for each $n \in \mathbb{N}$, and
(2) $a_n < a_{n,m} < a_{n-1}$, for all $n, m \in \mathbb{N}$; here, $a_{-1} = 2$.

Consider the subspace $X = \{0\} \cup \{a_n : n \in \mathbb{N}\} \cup \{a_{n,m} : n, m \in \mathbb{N}\}$ of $\mathbb{R}$. Then, X is a compact metric space. Now, we shall define a function $f : X \to X$ as follows.

a. $f(a_0) = 0$ and $f(0) = 0$.
b. $f(a_n) = a_{n-1}$, for each $n \in \mathbb{N}$.
c. $f(a_{n,0}) = a_{n+1,0}$, for each $n \in \mathbb{N}$.
d. $f(a_{0,n}) = a_{n,1}$, for each $1 \leq n \in \mathbb{N}$.
e. $f(a_{n-m,m+1}) = a_{n-m-1,m+2}$, for each $m < n \in \mathbb{N}$.

It is not difficult to prove that f is continuous. Let $x = a_{0,0}$ and $y = a_{0,1}$. We define $i_0 = 1$, $j_0 = 2$, $i_1 = 3$, $j_1 = 5$ and if $2 \leq k \in \mathbb{N}$, then we define $i_k = j_{k-1}+1$ and $j_k = j_{k-1} + k + 2$. We know from the definition that $f^{i_0}(a_{0,1}) = a_{1,1}$, $f^{j_0}(a_{0,1}) = a_{0,2}$, $f^{i_1}(a_{0,1}) = a_{2,1}$ y $f^{j_1}(a_{0,1}) = a_{0,3}$. By induction, we can establish that

$$f^{i_k}(a_{0,1}) = f^{j_{k-1}+1}(a_{0,1}) = f(f^{j_{k-1}}(a_{0,1})) = f(a_{0,k+1}) = a_{k+1,1},$$
$$f^{j_k}(a_{0,1}) = f^{j_{k-1}+k+2}(a_{0,1}) = f^{k+1}(f^{i_k}(a_{0,1})) = f^{k+1}(a_{k+1,1}) = a_{0,k+2},$$

and

$$f^i(a_{k,1}) = a_{k-i,i+1},$$

for every $k \in \mathbb{N}$ and for each $1 \leq i \leq k$. Let us define $A = \{i_k : k \in \mathbb{N}\}$ and $B = \{j_k : k \in \mathbb{N}\}$. Then, we have that

$$\lim_{k\to\infty} |f^{i_k}(a_{0,0}) - f^{i_k}(a_{0,1})| = \lim_{k\to\infty} |a_{i_k+1,0} - a_{k+1,0}| = 0.$$

On the other hand,

$$\lim_{k\to\infty} |f^{j_k}(a_{0,0}) - f^{j_k}(a_{0,1})| = \lim_{k\to\infty} |a_{j_k+1,0} - a_{0,k+1}| = 1.$$

These two conditions imply that x and y are p-proximal for all $p \in A^*$ and they are not q-proximal for any $q \in B^*$.

When the function f^p is continuous on the whole space we have the following uniform property:

Theorem 3.12. *Let (X, f) be a dynamical system where X is a compact metric space and let $p \in \mathbb{N}^*$. Then, f^p is continuous iff for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$, if $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

PROOF: Necessity. If f^p is continuous on X, then f^p is uniformly continuous on X and then we follow the proof of Theorem 3.4.

Sufficiency. This follows directly from Theorem 3.4. □

Now, let us study the behavior of the function f_x around a P-point of $\mathbb{N}^*$.

Theorem 3.13. *Let (X, f) be a dynamical system and let $x \in X$, where X is a compact metric space. If $p \in \mathbb{N}^*$ is a P-point, then there is $A \in p$ such that $f_x(p) = f_x(q)$, for every $q \in A^*$.*

PROOF: By the continuity of f_x, for every $k \in \mathbb{N}$ there is $A_k \in p$ such that

$$d(f_x(p), f_x(q)) < \frac{1}{k+1},$$

for all $q \in A_k^*$. Since p is a P-point there is $A \in p$ such that $A \subseteq^* A_k$, for each $k \in \mathbb{N}$. Thus, if $q \in A^*$ and $k \in \mathbb{N}$, then $q \in A_k^*$ and hence $d(f_x(p), f_x(q)) < \frac{1}{k+1}$. This implies that $f_x(p) = f_x(q)$, for every $q \in A^*$. □

For an arbitrary free ultrafilter p on $\mathbb{N}$ we have the following property.

Theorem 3.14. *Let (X, f) be a dynamical system and let $x \in X$, where X is a compact metric space. Then, for every $p \in \mathbb{N}^*$, there is $A \in [\mathbb{N}]^\omega$ such that $f_x(p) = f_x(q)$ for every $q \in A^*$.*

Proof: We know that $f_x(p) \in \mathrm{cl}_X(\{f^n(x) : n \in \mathbb{N}\})$. First suppose that $f_x(p)$ is not an accumulation point of $\mathcal{O}_f(x)$. Then, $f_x(p) = f^p(x) = f^n(x)$ for some $n \in \mathbb{N}$ and there is $\epsilon > 0$ such that $B(f^n(x), \epsilon) \cap \mathcal{O}_f(x) = \{f^n(x)\}$. Since f_x is continuous, there is $A \in p$ such that $f_x(q) \in B(f^n(x), \epsilon)$ for all $q \in A^*$. That is, $f_x(p) = f_x(q) = f^n(x)$ for every $q \in A^*$. Now, assume that there is a non-trivial sequence $(f^{n_k}(x))_{k\in\mathbb{N}}$ for which $\lim_{k\to\infty} f^{n_k}(x) = f_x(p)$ and we also assume that $f^{n_i}(x) \neq f^{n_j}(x)$ for distinct $i, j \in \mathbb{N}$. Put $A = \{n_k : k \in \mathbb{N}\}$ and fix $q \in A^*$. According to Lemma 2.1, we obtain that $f_x(p) = f_x(q)$. □

The proof of Theorem 3.14 with small changes establishes the next result.

Theorem 3.15. *Let (X, f) be a dynamical system and let $x \in X$, where X is a compact metric space. Then, for every $A \in [\mathbb{N}]^\omega$, there is $B \in [A]^\omega$ such that $f_x(p) = f_x(q)$, for every $p, q \in B^*$.*

Now, let us study the continuity of the function f^p when p is a P-point of $\mathbb{N}^*$ and X is a countable metric space.

Theorem 3.16. *Let (X, f) be a dynamical system, where X is a compact metric countable space. If f^p is continuous at $x \in X$, for some P-point $p \in \mathbb{N}^*$, then for every $\epsilon > 0$ there are $\delta > 0$ and $A \in p$ so that for $y \in X$ if $d(x, y) < \delta$, then $d(f^p(y), f^n(y)) < \epsilon$, for all $n \in A$ except finitely many.*

Proof: By definition, we know that $f^p(x) = p\text{-}\lim_{n\to\infty} f^n(x)$. Since X is a metric space, by Theorem 2.3, there is a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers such that $f^p(x) = \lim_{k\to\infty} f^{n_k}(x)$ and $B = \{n_k : k \in \mathbb{N}\} \in p$. Given $\epsilon > 0$, by Theorem 3.4, we may find $\delta > 0$ such that $C_y = \{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \frac{\epsilon}{3}\} \in p$ and $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$, provided that $d(x, y) < \delta$. As p is a P-point, there is $A \in p$ such that $A \subseteq^* C_y \cap B$ for all $y \in X$ with $d(x, y) < \delta$. Fix $y \in X$ with $d(x, y) < \delta$ and $m \in \mathbb{N}$ such that $A \setminus \{0, 1, \ldots, m\} \subseteq C_y$ and $d(f^n(x), f^p(x)) < \frac{\epsilon}{3}$, for every $n \in A \setminus \{0, 1, \ldots, m\}$. Then, for $n \in A \setminus \{0, 1, \ldots, m\}$ we have that

$$
\begin{aligned}
d(f^p(y), f^n(y)) &< d(f^p(y), f^p(x)) + d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}
$$

as required. □

Lemma 3.17. *Let (X, f) be a dynamical system, where X is a compact metric countable space. Suppose that f^p is continuous at $x \in X$ for a P-point p of $\mathbb{N}^*$. Then, for every $\epsilon > 0$ there are $\delta > 0$ and $A \in p$ such that if $y \in X$ satisfies that $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in A$ except finitely many.*

Proof: According to Theorem 3.16, we can find $\delta > 0$ and $B \in p$ so that if $y \in X$ and $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ and $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$ for all $n \in B$ except finitely many. Put $A = \{n \in B : d(f^p(x), f^n(x)) < \frac{\epsilon}{3}\}$. Assume that $y \in X$ satisfies the inequality $d(x, y) < \delta$. By assumption, we can find $m \in \mathbb{N}$ such that $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$, for each $n \in A \setminus m$. Thus, if $n \in A \setminus m$, then we obtain that

$$\begin{aligned} d(f^n(x), f^n(y)) &\leq d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Theorem 3.18. *Let (X, f) be a dynamical system, where X is a compact metric countable space, and let $x \in X$. Suppose that f^p is continuous at $x \in X$ for a P-point p of $\mathbb{N}^*$. Then, there is $A \in p$ such that f^q is continuous at x, for every $q \in A^*$.*

Proof: By Theorem 3.13, we know that there is $B \in p$ such that $f^p(x) = f^q(x)$ for each $q \in B^*$. From the previous lemma, for every $n \in \mathbb{N}$, we can find $\delta_n > 0$ and $A_n \subseteq B$ such that if $d(x, y) < \delta_n$, then $d(f^k(x), f^k(y)) < \frac{1}{n+1}$ for all $k \in A_n$ except finitely many. For every $n \in \mathbb{N}$, let $C_n = \{k \in \mathbb{N} : d(f^p(x), f^k(x)) < \frac{1}{n+1}\}$. We know that $C_n \in p$ for all $n \in \mathbb{N}$. Since p is a P-point, we can find $A \in p$ so that $A \subseteq^* A_n \cap C_n$, for each $n \in \mathbb{N}$. Now, fix $q \in A^*$ and let $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \frac{\epsilon}{3}$. Suppose that $y \in X$ satisfies that $d(x, y) < \delta_n$. Since $D = \{i \in \mathbb{N} : d(f^i(y), f^q(y)) < \frac{1}{n+1}\} \in q$, we can find $k \in D \cap C_n \cap A_n$ for which $d(f^k(x), f^k(y)) < \frac{1}{n+1}$. Then, we have that

$$\begin{aligned} d(f^q(x), f^q(y)) &= d(f^p(x), f^q(y)) \\ &\leq d(f^p(x), f^k(x)) + d(f^k(x), f^k(y)) + d(f^k(y), f^q(y)) \\ &< \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{n+1} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, f^q is continuous at x. □

The next corollary is a direct application of Theorem 3.18.

Corollary 3.19. *Let (X, f) be a dynamical system, where X is a compact metric countable space. If $p \in \mathbb{N}^*$ is a P-point and f^p is continuous on X, then there is $A \in p$ such that f^q is continuous on X, for every $q \in A^*$.*

Proof: According to Theorem 3.18, for every $x \in X$, there is $A_x \in p$ such that f^q is continuous at x, for every $q \in A_x^*$. Choose $A \in p$ so that $A \subseteq^* A_x$, for all $x \in X$. Then, it is evident that f^q is continuous on X, for each $q \in A^*$. □

In the general case, we have the following statement:

Theorem 3.20. *Let (X,f) be a dynamical system, where X is a compact metric space, and let $p \in \mathbb{N}^*$. Suppose that there exist $A \in p$ and $x \in X$ such that*

(1) *$f_x(s) = f_x(t)$ for each $s,t \in A^*$; and*
(2) *f^p is continuous at x.*

If $x = \lim_{n\to\infty} x_n$, then there is $B \in [A]^\omega$ such that $f^q(x) = \lim_{n\to\infty} f^q(x_n)$ for every $q \in B^*$.

Proof: Since f^p is continuous at x, by Theorem 3.4, for every $i \in \mathbb{N}$ there is $K_i \in \mathbb{N}$ such that $B_{k,i} = \{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) < \frac{1}{i+1}\} \in p$, for all $k \geq K_i$. For each $i \in \mathbb{N}$, let $C_i = \{n \in \mathbb{N} : d(f^n(x), f^p(x)) < \frac{1}{i+1}\}$. By definition, we know that $C_i \in p$ for each $i \in \mathbb{N}$. Choose $B \in [A]^\omega$ so that $B \subseteq^* B_{k,i} \cap C_i$, for every $i \in \mathbb{N}$ and for every $k \geq K_i$. Let $q \in B^*$ and let $\epsilon > 0$. Pick $j \in \mathbb{N}$ such that $\frac{1}{j+1} < \frac{\epsilon}{3}$. Fix $k \geq K_j$. We know that $D = \{n \in \mathbb{N} : d(f^n(x_k), f^q(x_k)) < \frac{1}{j+1}\} \in q$. Let $h \in D \cap B_{k,j} \cap C_j$. Then, we have that

$$\begin{aligned} d(f^q(x_k), f^q(x)) &= d(f^q(x_k), f^p(x)) \\ &\leq d(f^q(x_k), f^h(x_k)) + d(f^h(x_k), f^h(x)) + d(f^h(x), f^p(x)) \\ &< \frac{1}{j+1} + \frac{1}{j+1} + \frac{1}{j+1} < \epsilon. \end{aligned}$$

□

Next, we shall study the continuity properties of various functions f^p's at the same time.

Lemma 3.21. *Let (X,f) be a dynamical system, where X is a compact metric, $x,y \in X$ and $p \in \mathbb{N}^*$. If $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ for some $\epsilon > 0$, then $\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \epsilon\} \in p$.*

Proof: We know that $A = \{n \in \mathbb{N} : d(f^p(x), f^n(x)) < \frac{\epsilon}{3}\} \in p$ and $B = \{n \in \mathbb{N} : d(f^p(y), f^n(y)) < \frac{\epsilon}{3}\} \in p$. Then, we have that $A \cap B \in p$ and if $n \in A \cap B$, then

$$\begin{aligned} d(f^n(x), f^n(y)) &\leq d(f^n(x), f^p(x)) + d(f^p(x), f^p(y)) + d(f^p(y), f^n(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Theorem 3.22. *Let (X,f) be a dynamical system, where X is a compact metric space, and let $x \in X$. Let $\{p_n : n \in \mathbb{N}\} \subseteq \beta(\mathbb{N})$ and assume that the family $\{f^{p_n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at x. Then, f^q is continuous at x, for each $q \in \mathrm{cl}_{\mathbb{N}^*}(\{p_n : n \in \mathbb{N}\})$.*

Proof: Fix $q \in \mathrm{cl}_{\mathbb{N}^*}(\{p_n : n \in \mathbb{N}\})$. We know that $q = p\text{-}\lim_{n\to\infty} p_n$ for some $p \in \mathbb{N}^*$. Suppose that f^q is not continuous at x. According to Theorem 3.4, there is $\epsilon > 0$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in X converging to x such that $A_k = \{m \in \mathbb{N} : d(f^m(x), f^m(x_k)) \geq \epsilon\} \in q$, for each $k \in \mathbb{N}$. We know that $B_k = \{n \in \mathbb{N} : A_k \in p_n\} \in p$, for all $k \in \mathbb{N}$. By assumption, there is $\delta > 0$ such that if $y \in X$ and $d(x, y) < \delta$, then $d(f^{p_n}(x), f^{p_n}(y)) < \frac{\epsilon}{3}$, for all $n \in \mathbb{N}$. Choose $l \in \mathbb{N}$ such that $d(x, x_k) < \delta$ for each $k \in \mathbb{N}$ with $l \leq k$. Fix $k \in \mathbb{N}$ with $l \leq k$. So, $d(f^{p_n}(x), f^{p_n}(x_k)) < \frac{\epsilon}{3}$ for all $n \in \mathbb{N}$. By Lemma 3.21, we have that

$$C_n = \{m \in \mathbb{N} : d(f^m(x), f^m(x_k)) < \epsilon\} \in p_n,$$

for every $n \in \mathbb{N}$. Pick $n \in B_k$. It then follows that $A_k \cap C_n \in p_n$, which is impossible. □

Corollary 3.23. *Let (X, f) be a dynamical system, where X is a compact metric space, and let $p \in \mathbb{N}^*$. If $\{f^{p+n} : n \in \mathbb{N}\}$ is uniformly equicontinuous at $x \in X$, then f^{p+q} is continuous at x, for all $q \in \beta(\mathbb{N})$.*

Proof: Let $p \in \mathbb{N}^*$. We know that the function $\lambda_p : \beta(\mathbb{N}) \to \beta(\mathbb{N})$ given by $\lambda_p(q) = p + q$ is continuous (see [11]). Hence, we obtain that $\lambda_p[\mathrm{cl}_{\beta(\mathbb{N})} \mathbb{N}] = \{p + q : q \in \beta(\mathbb{N})\} = \mathrm{cl}_{\beta(\mathbb{N})}(\lambda_p[\mathbb{N}])$. By Theorem 3.22, we conclude that f^{p+q} is continuous at x, for each $q \in \beta(\mathbb{N})$. □

Theorem 3.24. *Let (X, f) be a dynamical system, where X is a compact metric space, and $x \in X$. If $\{q \in \mathbb{N}^* : f^q$ is continuous at $x\}$ is dense in $\mathbb{N}^*$, then f^p is continuous at x for all $p \in \mathbb{N}^*$.*

Proof: Put $D = \{q \in \mathbb{N}^* : f^q$ is continuous at $x\}$. Suppose that f^p is not continuous at x for some $p \in \mathbb{N}^* \setminus D$. Then, by Theorem 3.4, there is $\epsilon > 0$ and for every $k \in \mathbb{N}$ there is $x_k \in X$ such that $x = \lim_{k\to\infty} x_k$ and $A_k = \{n \in \mathbb{N} : d(f^n(x), f^n(x_k)) \geq \epsilon\} \in p$, for each $k \in \mathbb{N}$. We can find $A \in [\mathbb{N}]^\omega$ such that $A \subseteq^* A_k$ for all $k \in \mathbb{N}$. By assumption, there is $q \in A^* \cap D$ for which f^q is continuous at x. Hence, we may chose $N \in \mathbb{N}$ such that $d(f^q(x), f^q(x_k)) < \frac{\epsilon}{3}$, for all $k \in \mathbb{N}$ with $k \geq N$. It then follows from Lemma 3.21 that

$$B_k = \{m \in A : d(f^m(x), f^m(x_k)) < \epsilon\} \in q,$$

for all $k \geq N$. Fix $N \leq i \in \mathbb{N}$. We know that $B_i \subseteq^* A_i$. So, if $m \in B_i \cap A_i$, then $d(f^m(x), f^m(x_i)) < \epsilon$ and $d(f^m(x), f^m(x_i)) \geq \epsilon$, but this is impossible. Therefore, f^p is continuous at x, for all $p \in \mathbb{N}^*$. □

Theorem 3.25. *Let (X, f) be a dynamical system, where X is a compact metric space, and $x \in X$. Let $1 < k \in \mathbb{N}$. For $i < k$, we define $A_i = \{n \in \mathbb{N} : n \cong i \bmod(k)\}$. If there is $j < k$ such that f^q is continuous at x for all $q \in A_j^*$, then f^p is continuous at x for every $p \in \mathbb{N}^*$.*

Proof: First, observe that $\mathbb{N}^* = \bigcup_{i<k} A_i^*$. Let $j \neq i < k$. We define $\phi_i : \mathbb{N} \to \mathbb{N}$ by $\phi_i(n) = |n + i - j|$ for every $n \in \mathbb{N}$. It is not hard to see that ϕ_i is a bijection between A_j and A_i module a finite set. Hence, if $p \in A_i^*$, then there is $q \in A_j^*$ such that $\overline{\phi_i}(q) = q + i - j = p$. Thus, if $i > j$ and f^q is continuous at x, then $f^{q+i-j} = f^p = f^{i-j} \circ f^q$ is continuous at x. If $i < j$, then we consider the function ϕ_{k+i} which is also a bijection between A_j and A_i module a finite set. Thus, for a given $p \in A_i^*$ there is $q \in A_j^*$ such that $\overline{\phi_{k+i}}(q) = q + k + i - j = p$ and then $f^{q+k+i-j} = f^p = f^{k+i-j} \circ f^q$ is continuous at x whenever f^q is continuous at x. □

Let (X, f) be a dynamical system, where X is a metric compact space, and let $x \in X$. The previous corollary assures that if f^p is continuous at x, for all $p \in \{an : n \in \mathbb{N}\}^*$, where $a \in \mathbb{N}$, then f^p is continuous at x, for all $p \in \mathbb{N}^*$.

Lemma 3.26. *Let (X, f) be a dynamical system, where X is a compact metric space, and let $x \in X$ be a fixed point of f. Suppose that there is $\epsilon > 0$ such that for every $k \in \mathbb{N}$ there are $x_k, y_k \in X$ such that $d(x, x_k) < \frac{1}{k+1}$, $\mathcal{O}_f(y_k) \cap B(x, \epsilon) = \emptyset$ and $\mathcal{O}_f(y_k) \cap \mathcal{O}_f(x_k) \neq \emptyset$. Then, f^p is discontinuous at x for every $p \in \mathbb{N}^*$.*

Proof: Fix $k \in \mathbb{N}$. We know that $f^l(x_k) = f^m(y_k)$, for some $l, m \in \mathbb{N}$. Then, $f^{l+a}(x_k) = f^{m+a}(y_k) \in \mathcal{O}_f(y_k)$, for all $a \in \mathbb{N}$. Hence, $\{n \in \mathbb{N} : d(f^n(x_k), x) \geq \epsilon\}$ is a cofinite subset of $\mathbb{N}$ and so

$$\{n \in \mathbb{N} : d(f^n(x_k), f^n(x)) \geq \epsilon\} = \{n \in \mathbb{N} : d(f^n(x_k), x) \geq \epsilon\} \in p,$$

for each $p \in \mathbb{N}^*$. Therefore, f^p is discontinuous at x for every $p \in \mathbb{N}^*$. □

Theorem 3.27. *Let (X, f) be a dynamical system such that X is a compact metric space with only one non-isolated point. Then, either f^p is continuous for all $p \in \mathbb{N}^*$ or f^p is discontinuous for all $p \in \mathbb{N}^*$.*

Proof: Let x be the unique non-isolated point of X. First, suppose that $f(x) \neq x$. Then, we have that $A = \{y \in X : f(y) = f(x)\}$ is cofinite. If $y \in A$ and $n \in \mathbb{N}$, then $f^n(y) = f^n(x)$; hence, we deduce that $f^p(y) = f^p(x)$ for all $y \in A$ and for all $p \in \mathbb{N}^*$. Thus, f^p is continuous, for all $p \in \mathbb{N}^*$. Now, we assume that $f(x) = x$. Let $\epsilon > 0$ and let $X \setminus B(x, \epsilon) = \{x_0, \ldots, x_m\}$. Put $F = \{i \leq m : \mathcal{O}_f(x_i)$ is finite$\}$ and $I = m \setminus F$. We may also assume that $x \notin \mathcal{O}_f(x_i)$ for every $i \in F$. Suppose that the conditions of the previous lemma fail. Then, we can find $\delta > 0$ such that $B(x, \delta) \cap \mathcal{O}_f(x_i) = \emptyset$, for each $i \leq F$, and if $d(x, y) < \delta$, then $\mathcal{O}_f(y) \cap \mathcal{O}_f(z) = \emptyset$, whenever $\mathcal{O}_f(z) \cap B(x, \epsilon) = \emptyset$. Let $y \in X$ such that $d(x, y) < \delta$. If $\mathcal{O}_f(y) \cap \mathcal{O}_f(x_i) \neq \emptyset$ for some $i \in I$, then $\lim_{n\to\infty} f^n(y) = x$ and hence $\{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$ for all $p \in \mathbb{N}^*$. Suppose that $\mathcal{O}_f(y)$ does not intersect any $\mathcal{O}_f(x_i)$, for all $i \leq m$. Then, $\mathcal{O}_f(y) \subseteq B(x, \epsilon)$. So, $\mathbb{N} = \{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$ for all $p \in \mathbb{N}^*$. This shows that $f^p : X \to X$ is continuous, for each $p \in \mathbb{N}^*$. □

Theorem 3.28. *Let (X, f) be a dynamical system such that X is a compact metric space and let $x \in X$ be a fixed point of f. Suppose that there is $m \in \mathbb{N}$ such that $|\mathcal{O}_f(y)| \leq m$, for all $y \in X$. Then, either f^p is continuous at x for all $p \in \mathbb{N}^*$, or f^p is discontinuous at x for all $p \in \mathbb{N}^*$.*

Proof: Suppose that f^p is continuous at x and f^q is discontinuous at x, for some $p, q \in \mathbb{N}^*$. Then there are $\epsilon > 0$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in X converging to x such that $\{n \in \mathbb{N} : d(x, f^n(x_k)) \geq \epsilon\} \in q$, for all $k \in \mathbb{N}$. By the continuity of f^p and Theorem 3.4, there is $\delta > 0$ such that $\delta < \epsilon$ and if $y \in X$ and $d(x, y) < \delta$, then $\{n \in \mathbb{N} : d(x, f^n(y)) < \epsilon\} \in p$. We know that there is $M \in \mathbb{N}$ such that $d(x, x_k) < \delta$ for all $M \leq k \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$ with $k \geq M$, there is $0 < m_k \leq m$ so that $d(x, f^{m_k+1}(x_k)) \geq \epsilon$ and m_k is the minimum positive integer with this property. Without loss of generality, we may assume that there is $l \leq m$ for which $m_k = l$, for each $k \in \mathbb{N} \setminus M$. Since f is continuous we can find $0 < \delta_l < \delta_{l-1} < \cdots < \delta_0 < \epsilon$ such that if $d(x, y) < \delta_i$, then $d(x, f(y)) < \delta_{i-1}$, for every $0 \leq i < l$, and if $d(x, y) < \delta_0$, then $d(x, f(x)) < \epsilon$. Choose $N \in \mathbb{N}$ such that $M < N$ and $d(x, x_k) < \delta_l$, for every $N \leq k \in \mathbb{N}$. Then, we have that $d(x, f^l(x_k)) < \delta_0$, for each $N \leq k \in \mathbb{N}$. But, this is impossible since $d(x, f^{l+1}(x_k)) \geq \epsilon$, for every $N \leq k \in \mathbb{N}$. $\square$

We finish this section with some conditions that are equivalent to the continuity of all functions f^p's.

Theorem 3.29. *Let (X, f) be a dynamical system, where X is a compact metric space. Let us consider the function $F^* : \mathbb{N}^* \times X \to X$ given by $F^*(p, x) = f^p(x)$, for every $(p, x) \in \mathbb{N}^p \times X$. Then, the following conditions are equivalent.*

(1) *f^p is continuous on X, for every $p \in \mathbb{N}^*$ (that is, F^* is separately continuous).*
(2) *There is a dense G_δ-subset D of $\mathbb{N}^*$ such that $F^*|_{D\times X}$ is continuous.*
(3) *There is a dense subset D of $\mathbb{N}^*$ such that $F^*|_{D\times X}$ is continuous.*

Proof: The implication (1) $\Rightarrow$ (2) follows directly from Namioka's Theorem ([2, Theorem III.5.5], [12]), the implication (2) $\Rightarrow$ (3) is trivial and the implication (3) $\Rightarrow$ (1) follows directly from Theorem 3.24. $\square$

4. Dynamical systems and actions of metrizable semigroups

Throughout this section, (X, f) will stand for a dynamical system where X is a compact metric space. From now on to avoid trivial situations we assume that X is infinite and that for every couple of natural numbers (n, m) there exists $x \in X$ such that $f^n(x) \neq f^m(x)$. Our main goal is to establish that the action $F : \beta(\mathbb{N}) \times X \to X$ induced by (X, f) is (in some sense) equivalent to the action of a metrizable semigroup on X. To do this, let us define an equivalent relation $\sim$ on $\beta(\mathbb{N})$ by letting $p \sim q$ if and only if $f^p(x) = f^q(x)$ for every $x \in X$. If d is a compatible metric on X, the real-valued function on $\beta(\mathbb{N}) \times \beta(\mathbb{N})$ defined by

$$\bar{d}(p,q) = \sup_{x \in X} d(f^p(x), f^q(x)) \qquad p,q \in \beta(\mathbb{N}),$$

is a pseudometric on $\beta(\mathbb{N})$ (notice that being X compact, d is bounded). It is clear that $\bar{d}$ induces a metric (also denoted by $\bar{d}$) on the quotient space $\beta(\mathbb{N})/{\sim}$. The following result follows from Theorem 3.3.

Proposition 4.1. *$\beta(\mathbb{N})/{\sim}$ is a semigroup with the addition $+$ defined as*

$$[\,p\,] + [\,q\,] = [\,p + q\,],$$

for each $p, q \in \beta(\mathbb{N})$.

As we deal with actions on metrizable semigroups, a natural question is when the semigroup $(\beta(\mathbb{N})/{\sim}, +)$ equipped with the topology induced by the metric $\bar{d}$ is a topological semigroup; that is, when the operation defined in Proposition 4.1 is continuous. A useful sufficient condition is given in Theorem 4.3 below. Before the statement of this theorem, we prove a lemma.

Lemma 4.2. *Let (X,f) be a dynamical system, where X is a compact metric space. If the family of functions $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous, then the family $\{f^p : p \in \mathbb{N}^*\}$ is also uniformly equicontinuous.*

Proof: By assumption, given $\epsilon > 0$ we can find $\delta > 0$ such that if $x, y \in X$ satisfy that $d(x,y) < \delta$, then $d(f^n(x), f^n(y)) < \frac{\epsilon}{3}$ for all $n \in \mathbb{N}$. Let $x, y \in X$. Assume that $d(x,y) < \delta$ and fix $p \in \mathbb{N}^*$. Choose $n \in \mathbb{N}$ so that $d(f^p(x), f^n(x)) < \frac{\epsilon}{3}$ and $d(f^p(y), f^n(y)) < \frac{\epsilon}{3}$. Then, we obtain that

$$\begin{aligned} d(f^p(x), f^p(y)) &\leq d(f^p(x), f^n(x)) + d(f^n(x), f^n(y)) + d(f^n(x), f^p(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Therefore, the family $\{f^p : p \in \mathbb{N}^*\}$ is also uniformly equicontinuous. □

Theorem 4.3. *Assume that the family $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous, then $\beta(\mathbb{N})/{\sim}$ is a topological semigroup with the topology induced by the metric $\bar{d}$.*

Proof: Let $[\,p\,], [\,q\,] \in \beta(\mathbb{N})/{\sim}$. We know from Lemma 4.2 that the family of functions $\{f^t : t \in \beta(\mathbb{N})\}$ is also uniformly equicontinuous. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that $\delta < \frac{\epsilon}{2}$ and if $x, y \in X$ and $d(x,y) < \delta$, then $d(f^t(x), f^t(y)) < \frac{\epsilon}{2}$ for all $t \in \beta(\mathbb{N})$. Suppose that $r, s \in \beta(\mathbb{N})$ satisfy that

$$\bar{d}(p,r) = \sup\{d(f^p(x), f^r(x)) : x \in X\} < \delta$$

and

$$\bar{d}(q,s) = \sup\{d(f^q(x), f^s(x)) : x \in X\} < \delta.$$

Then, $d(f^s(f^p(x)), f^s(f^r(x))) < \frac{\epsilon}{2}$ and $d(f^q(f^p(x)), f^s(f^p(x))) < \frac{\epsilon}{2}$, for all $x \in X$. Thus,

$$\begin{aligned} d(f^q(f^p(x)), f^s(f^r(x))) &\leq d(f^q(f^p(x)), f^s(f^p(x))) + d(f^s(f^p(x)), f^s(f^r(x))) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all $x \in X$. Therefore,

$$\bar{d}(p+q, r+s) = \sup\{d(f^q(f^p(x)), f^s(f^r(x))) : x \in X\} \leq \epsilon.$$

This shows the theorem. $\square$

Given a dynamical system (X, f) where X is a compact metric space, and an ultrafilter $p \in \beta(\mathbb{N})$, $f^{[p]}$ stands for the function from X into itself defined by $f^{[p]}(x) = f^p(x)$, for every $x \in X$. Let $F : \beta(\mathbb{N}) \times X \longrightarrow X$ be defined by $F(p, x) = f^p(x)$, for all $(p, x) \in \beta(\mathbb{N}) \times X$. We observe that this action F induces a natural action $\widehat{F} : (\beta(\mathbb{N})/{\sim}) \times X \longrightarrow X$ defined as

$$\widehat{F}([p], x) = f^{[p]}(x) \qquad \text{for each } ([p], x) \in (\beta(\mathbb{N})/{\sim}) \times X.$$

Although the authors could not find a specific reference, the following result is probably well known. We include a proof for reader convenience. Given a function $f\colon X \times Y \to Z$ we shall denote by f_x (respectively, by f^y) the function $f_x\colon Y \to Z$ defined by the rule $f_x(y) = f(x, y)$ for every $y \in Y$ (respectively, by the rule $f^y(x) = f(x, y)$ for every $x \in X$). We recall that, if X, Y and Z are topological spaces, then f is said to be *separately continuous* if every f_x and every f^y are continuous functions.

Theorem 4.4. *Let (X, d^1), (Y, d^2) and (Z, d^3) be three compact metric spaces. If $f : X \times Y \to Z$ is a separately continuous function, then the following conditions are equivalent.*

(1) *f is continuous.*
(2) *The family $\{f_x \mid x \in X\}$ is uniformly equicontinuous.*
(3) *The family $\{f^y \mid y \in Y\}$ is uniformly equicontinuous.*

Proof: Obviously we only need to prove that the clauses (1) and (2) are equivalent.

(1) $\Rightarrow$ (2) Consider the space $(C(Y, Z), \|\cdot\|)$ where $\|\cdot\|$ stands for the supremum norm. It is a well-known fact that f continuous implies that the function $g : X \to (C(Y, Z), \|\cdot\|)$ defined as $g(x) = f_x$ is continuous (for a more general result the reader can consult [14, Theorem 3.3]). Let $\varepsilon > 0$. Since X is compact, the

family $g(X) = \{f_x \mid x \in X\}$ is compact so that there exists a finite subfamily $\{f_{x_1}, f_{x_2}, \ldots, f_{x_n}\}$ such that

$$\{f_x \mid x \in X\} \subseteq \bigcup_{i=1}^{n} B(f_{x_i}, \varepsilon/3).$$

Moreover, since each f_{x_i} is uniformly continuous, we can choose $\delta > 0$ such that $d_3(f_{x_i}(y_1), f_{x_i}(y_2)) < \frac{\varepsilon}{3}$ whenever $d_2(y_1, y_2) < \delta$, $i = 1, 2, \ldots, n$.

Now let $x \in X$ and consider f_x. If f_{x_i} satisfies that $f_x \in B(f_{x_i}, \varepsilon/3)$, then

$$\begin{aligned} d_3(f_x(y_1), f_x(y_2)) &\leq d_3(f_x(y_1), f_{x_i}(y_1)) \\ &\quad + d_3(f_{x_i}(y_1), f_{x_i}(y_2)) + d_3(f_{x_i}(y_2), f_x(y_2)) \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

whenever $d_2(y_1, y_2) < \delta$. Thus, the family $\{f_x \mid x \in X\}$ is uniformly equicontinuous.

(2) $\Rightarrow$ (1). Since the family $\{f_x \mid x \in X\}$ is uniformly equicontinuous it is apparent that the function $g : Y \to (C(X, Z), \|\cdot\|)$ defined as $g(y) = f^y$ is continuous. Now to see that f is continuous, consider a point $(x_0, y_0) \in X \times Y$ and $\varepsilon > 0$. Since both g and f^{y_0} are continuous we can choose $\delta > 0$ such that

$$d_3(f(x, y), f(x, y_0)) < \frac{\varepsilon}{2} \quad \text{and} \quad d_3(f(x, y_0), f(x_0, y_0)) < \frac{\varepsilon}{2}$$

whenever $d_1(x, x_0) < \delta$ and $d_2(y, y_0) < \delta$, that is

$$\begin{aligned} d_3(f(x, y), f(x_0, y_0)) &\leq d_3(f(x, y), f(x, y_0)) + d_3(f(x, y_0), f(x_0, y_0)) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

whenever $d_1(x, x_0) < \delta$ and $d_2(y, y_0) < \delta$. Thus, f is continuous at the point $(x_0, y_0) \in X \times Y$. This completes the proof. □

The proof of the following theorem is straightforward.

Theorem 4.5. *Let (X, f) be a dynamical system, where X is a compact metric space, and let $x \in X$. For every $p \in \beta(\mathbb{N})$, the following conditions are equivalent.*

(1) *f^p is continuous at x.*
(2) *$f^{[p]}$ is continuous at x.*

Theorem 4.6. *For a compact metric dynamical system (X, f), the following are equivalent.*

(1) *The set $\{f^n : n \in \mathbb{N}\}$ is uniformly equicontinuous on X.*
(2) *$\bar{d}$ induces the quotient topology on $\beta(\mathbb{N})/\sim$ and F is continuous.*
(3) *The action $\widehat{F}$ is (jointly) continuous.*

Proof: The implication (3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (2) We shall prove that the quotient map $g : \beta(\mathbb{N}) \longrightarrow (\beta(\mathbb{N})/\sim, \bar{d})$ is continuous. Indeed, by Lemma 4.2, we deduce that the family $\{f^p : p \in \mathbb{N}^*\}$ is uniformly equicontinuous. Hence, given $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^p(x), f^p(y)) < \frac{\epsilon}{3}$ for all $p \in \beta(\mathbb{N})$, and Theorem 4.4 tells us that $F : \beta(\mathbb{N}) \times X \to X$ is continuous. It is clear that g is continuous at any point of $\mathbb{N}$. Let $p \in \mathbb{N}^*$. Then, for every $x \in X$ there are $A_x \in p$ and $\delta_x < \delta$ such that if $(q, y) \in A_x^* \times B(x, \delta_x)$, then $d(f^p(x), f^q(y)) < \frac{\epsilon}{3}$. Since X is compact, there are $x_0, \dots, x_k \in X$ such that $X = \bigcup_{i \le k} B(x_i, \delta_{x_i})$. Put $A = \bigcap_{i \le k} A_{x_i}$. Then, $A \in p$. Fix $q \in A^*$ and let $x \in X$. Then, $x \in B(x_j, \delta_{x_j})$, for some $j \le k$. Thus,

$$\begin{aligned} d(f^p(x), f^q(x)) &\le d(f^p(x), f^p(x_j)) + d(f^p(x_j), f^q(x_j)) + d(f^q(x_j), f^q(x)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

So, $\bar{d}([p], [q]) \le \epsilon$, whenever $q \in A^*$. This shows that g is continuous.

(2) $\Rightarrow$ (3) Since $\beta(\mathbb{N})$ is compact, Whitehead's Theorem ([8, Theorem 3.3.17] and [15]) assures that the function $g \times \mathrm{id}_X : \beta(\mathbb{N}) \times X \to (\beta(\mathbb{N})/\sim) \times X$ is a quotient map. Since F is continuous and $F = \widehat{F} \circ (g \times \mathrm{id}_X)$ is continuous, by Proposition 2.4.2 from [8], we get that the function $\widehat{F}$ is continuous. $\square$

The previous theorem establishes a necessary and sufficient condition in order that the induced action $\widehat{F}$ be continuous. This can be applied to obtain that the action F is equivalent to the action $\widehat{F}$ in the sense of Definition 4.7 below. If F is a continuous action of a (compact) topological semigroup S on a compact metric space, we say that (S, X, F) is a *flow*.

Definition 4.7. Let S, T be two compact topological semigroups. Two flows (S, X, F) and (T, Y, G) are said to be *topologically conjugate* (or *equivalent*) if there exists a continuous epimorphism $e : S \longrightarrow T$ and a homeomorphism $h : X \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} S \times X & \xrightarrow{F} & X \\ e \downarrow \quad \downarrow h & & \downarrow h \\ T \times Y & \xrightarrow{G} & Y \end{array}$$

commutes, that is, $h(F(s,x)) = G((e \times h)(s,x))$ for each $(s,x) \in S \times X$.

From Theorem 4.6 we can see that a continuous action of $\beta(\mathbb{N})$ on a compact metric space X is equivalent to a continuous action of a compact metrizable semigroup.

Theorem 4.8. *If X is a compact metric space, then every flow $(\beta(\mathbb{N}), X, F)$ is equivalent to a flow (S, X, G) where S is compact metrizable semigroup.*

Proof: By density, the action F is determined by its restriction to $\mathbb{N} \times X$. So, F is the action induced by the dynamical system (X, f) where f is the function defines as $f(x) = F(1, x)$ for every $x \in X$. Since F is continuous, Theorem 4.4 and Theorem 4.6 assert that $(\beta(\mathbb{N})/\sim, X, \widehat{F})$ is a flow. Hence, the diagram

$$\begin{tikzcd} \beta(\mathbb{N}) \times X \arrow[r, "F"] \arrow[d, "g"', "\mathrm{id}_X"] & X \arrow[d, "\mathrm{id}_X"] \\ S \times X \arrow[r, "\widehat{F}"] & X \end{tikzcd}$$

commutes, where $S = \beta(\mathbb{N})/\sim$ and g is the quotient map. By Proposition 4.1, g is an epimorphism. The proof is done by taking $(S, X, G) = (\beta(\mathbb{N})/\sim, X, \widehat{F})$. $\square$

5. Open questions

We end with some open questions that the authors were unable to solve.

Question 5.1. Given $p, q \in \mathbb{N}^*$ such that $p + n \neq q$, for all $n \in \mathbb{N}$, is there a dynamical system (X, f) and a point $x \in X$ such that X is a compact metric space, f^p is continuous at x and f^q is discontinuous at x ?

Question 5.2. Given $p, q \in \mathbb{N}^*$ such that $p + n \neq q$, for all $n \in \mathbb{N}$, is there a dynamical system (X, f) and a point $x \in X$ such that X is a connected, compact metric space, f^p is continuous at x and f^q is discontinuous at x ?

Acknowledgments. We are grateful to the referee for correcting an error which appeared in an earlier version and for his/her suggestions to improve the paper.

References

[1] Akin E., *Recurrence in Topological Dynamics. Furstenberg Families and Ellis Actions*, The University Series in Mathematics, Plenum Press, New York, 1997.

[2] Arkhangel'skii A.V., *Topological Function Spaces*, Mathematics and its Applications (Soviet Series), vol. 78, Kluwer Academic Publishers, Dordrecht, 1992.

[3] Auslander J., Furstenberg H., *Product recurrence and distal points*, Trans. Amer. Math. Soc. **343** (1994), 221–232.

[4] Bernstein A.R., *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185–193.

[5] Blass A., *Ultrafilters: where topological dynamics = algebra = combinatorics*, Topology Proc. **18** (1993), 33–56.

[6] Blass A., Shelah S., *There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed*, Ann. Pure Appl. Logic **33** (1987), no. 3, 213–243.

[7] Comfort W., Negrepontis S., *The Theory of Ultrafilters*, Springer, Berlin, 1974.

[8] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag, Berlin, 1989.

[9] Furstenberg H., *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton University Press, Princeton, 1981.

[10] Gillman L., Jerison M., *Rings of Continuous Functions*, Graduate Texts in Mathematics, No. 43, Springer, New York-Heidelberg, 1976.

[11] Hindman N., Strauss D., *Algebra in the Stone-Čech Compactification*, Walter de Gruyter, Berlin, 1998.

[12] Namioka I., *Separate continuity and joint continuity*, Pacific J. Math. **51** (1974), 515–531.

[13] Rudin W., *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.

[14] Sanchis M., *Continuous functions on locally pseudocompact groups*, Topology Appl. **86** (1998), 5–23.

[15] Whitehead J.H.C., *A note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. **54** (1948), 1125–1132.

Instituto de Matemáticas (UNAM), Apartado Postal 61-3, Santa María, 58089, Morelia, Michoacán, México

E-mail: sgarcia@matmor.unam.mx

Department de Matemàtiques, Area Científico-Técnica, Campus Riu Sec, 12071-Castelló, Spain

E-mail: sanchis@mat.uji.es

(*Received* September 12, 2006, *revised* November 23, 2006)