Construction of Šindel sequences

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Dedicated to Jan Šindel, the rector of the Prague University in 1410.

Abstract. We found that there is a remarkable relationship between the triangular numbers T_k and the astronomical clock (horologe) of Prague. We introduce Šindel sequences $\{a_i\} \subset \mathbb{N}$ of natural numbers as those periodic sequences with period p that satisfy the following condition: for any $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $T_k = a_1 + \cdots + a_n$. We shall see that this condition guarantees a functioning of the bellworks, which is controlled by the horologe. We give a necessary and sufficient condition for a periodic sequence to be a Šindel sequence. We also present an algorithm which produces the so-called primitive Šindel sequence, which is uniquely determined for a given $s = a_1 + \cdots + a_p$.

Keywords: Jacobi symbol, quadratic nonresidue, clock sequence, primitive Šindel sequences, Chinese remainder theorem, Dirichlet's theorem

Classification: 11A07, 11A51, 01A40

1. Introduction

The origin of a mathematical model of the astronomical clock of Prague is attributed to Joannes Andreae, called *Šindel* (see [2]). He invented this model approximately 600 years ago. In honour of this great achievement we introduce and investigate a new term, the *Šindel sequence*. The clock was realized by the clockmaker Nicholas from Kadaň around 1410. Over the centuries its construction has been renovated several times.

The bellworks of the Prague horologe contains a large gear with 24 slots at increasing distances along its circumference (see Figure 1). This arrangement allows for a periodic repetition of 1–24 strokes of the bell each day. There is also a small auxiliary gear whose circumference is divided by 6 slots into segments of arc lengths 1, 2, 3, 4, 3, 2 (see Figure 1). These numbers constitute a period which repeats after each revolution and their sum is $s = 15$.

At the beginning of every hour a catch rises, both gears start to revolve and the bell chimes. The gears stop when the catch simultaneously falls back into the slots on both gears. The bell strikes $1 + 2 + \cdots + 24 = 300$ times every day. Since this number is divisible by $s = 15$, the small gear is always at the same position at the beginning of each day.

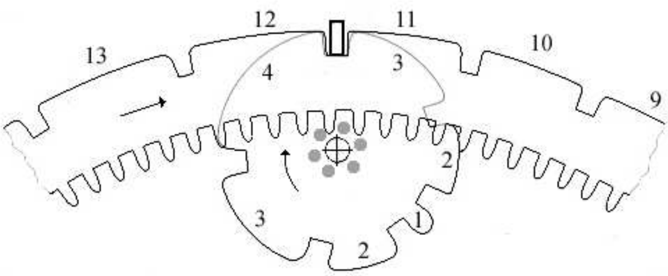

FIGURE 1. The number of bell strokes is denoted by the numbers ..., 9, 10, 11, 12, 13, ... along the large gear. The small gear placed behind it is divided by slots into segments of arc lengths 1, 2, 3, 4, 3, 2. The catch is indicated by a small rectangle on the top.

The large gear has 120 interior teeth which drop into a pin gear with 6 little horizontal bars that surround the centre of the small gear (see Figure 1). The large gear revolves one time per day and therefore, the small gear revolves 20 times per day with approximately 4 times greater circumferential speed, since its circumference is 5 times smaller. Thus, the small gear makes the regulation of strokes sufficiently precise despite the wearing out of the slots on the large gear. Moreover, one stroke of the bell at one a.m. is due only to the movement of the small gear. There is no tooth between the first and second slot of the large gear. Therefore, in this case the catch is in contact only with the tooth of arc length 1 of the small gear, which makes the use of the small gear essential.

When the small gear revolves it generates by means of its slots a periodic sequence whose particular sums correspond to the number of strokes of the bell at each hour:

$$
\begin{aligned}
&1\ 2\ 3\ 4\ \underbrace{3\ 2}_{5}\ \ \underbrace{1\ 2\ 3}_{6}\ \ \underbrace{4\ 3}_{7}\\
&\underbrace{2\ 1\ 2\ 3}_{8}\ \ \underbrace{4\ 3\ 2}_{9}\ \ \underbrace{1\ 2\ 3\ 4}_{10}\ \ \underbrace{3\ 2\ 1\ 2\ 3}_{11}\ \ \underbrace{4\ 3\ 2\ 1\ 2}_{12}\\
&\underbrace{3\ 4\ 3\ 2\ 1}_{13}\ \ \underbrace{2\ 3\ 4\ 3\ 2}_{14}\ \ \underbrace{1\ 2\ 3\ 4\ 3\ 2}_{15}\ \dots
\end{aligned}
\tag{1}
$$

In the next section we show that we could continue in this way until infinity. However, not all periodic sequences have such a nice summation property. For instance, we immediately find that the period 1, 2, 3, 4, 5, 4, 3, 2 could not be used for such a purpose, since $6 < 4 + 3$. Also the period 1, 2, 3, 2 could not be used, since $2 + 1 < 4 < 2 + 1 + 2$.

The astronomical clock of Prague is probably the oldest (see [2, p. 76]) still functioning clock that contains such an apparatus illustrated in Figure 1. Due to the beautiful summation property discussed above, Sloane in [5] and [7, A028355, A028356] calls the sequence $1, 2, 3, 4, 3, 2, 1, 2, 3, 4, \dots$ in (1) the *clock sequence*.

2. Connections with triangular numbers and periodic sequences

In this section we show how the *triangular numbers*

$$T_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}, \quad k = 0, 1, 2, \dots, \tag{2}$$

are related to the astronomical clock. We shall look for all periodic sequences that have a similar property as the clock sequence in (1), i.e., that could be used in the construction of the small gear. Put $\mathbb{N} = \{1, 2, \dots\}$.

A sequence $\{a_i\}_{i=1}^{\infty}$ is said to be *periodic* if there exists $p \in \mathbb{N}$ such that

$$\forall\, i \in \mathbb{N} : \quad a_{i+p} = a_i. \tag{3}$$

The finite sequence $a_1, \dots, a_p$ is called a *period* and p is called the *period length.* The smallest p satisfying (3) is called the *minimal period length* and the associated sequence $a_1, \dots, a_p$ is called the *minimal period.*

Definition 1. Let $\{a_i\} \subset \mathbb{N}$ be a periodic sequence. We say that the triangular number T_k for $k \in \mathbb{N}$ is *achievable* by $\{a_i\}$, if there exists a positive integer n such that

$$T_k = \sum_{i=1}^{n} a_i. \tag{4}$$

The periodic sequence $\{a_i\}$ is said to be a *Šindel sequence* if T_k is achievable by $\{a_i\}$ for every $k \in \mathbb{N}$, i.e.,

$$\forall\, k \in \mathbb{N} \quad \exists\, n \in \mathbb{N} : \quad T_k = \sum_{i=1}^{n} a_i. \tag{5}$$

The triangular number T_k on the left-hand side is equal to the sum $1 + \cdots + k$ of hours on the large gear, whereas the sum on the right-hand side expresses the corresponding rotation of the small gear (see Figure 2). For the kth hour, we have

$$k = T_k - T_{k-1} = \sum_{i=m+1}^{n} a_i, \tag{6}$$

where $T_{k-1} = \sum_{i=1}^{m} a_i$. Since $a_i > 0$, the number n depending on k in (5) is unique. From (2) and (4) we also see that $a_1 = 1$ when $\{a_i\}$ is a Šindel sequence.

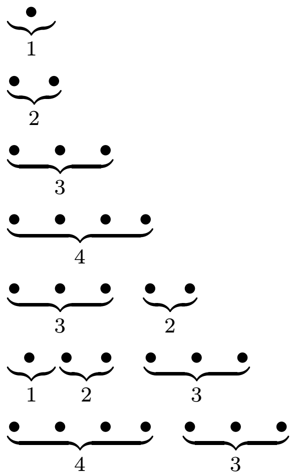

FIGURE 2. Schematic illustration of the triangular number T_7. The bullets in the kth row indicate the number of strokes at the kth hour (see (6)). The numbers denote lengths of segments on the small gear.

The next theorem shows that condition (5) can be replaced by a much weaker condition containing only a finite number of k's. This enables us to perform only a finite number of arithmetic operations to check whether a given period $a_1, \dots, a_p$ yields a Šindel sequence. From now on let

$$s = \sum_{i=1}^{p} a_i \tag{7}$$

denote the sum of the period.

Theorem 1. *A periodic sequence $\{a_i\}$ for s odd is a Šindel sequence if T_k is achievable by $\{a_i\}$ for $k = 1, 2, \dots, (s-1)/2$.*

PROOF: The case $s = 1$ is trivial. So let $s \geq 3$ be odd and suppose that

$$\forall\, k \in \{1, 2, \dots, (s-1)/2\} \quad \exists\, n \in \mathbb{N} : \quad T_k = \sum_{i=1}^{n} a_i. \tag{8}$$

According to (7),

$$1 + 2 + \cdots + (s-1) = \frac{s-1}{2} \sum_{i=1}^{p} a_i, \tag{9}$$

where p is the period length and $(s-1)/2$ is integer. For the corresponding sequence

$$\underbrace{a_1, a_2, \ldots, a_p}_{s},\ \underbrace{a_1, a_2, \ldots, a_p}_{s},\ \ldots,\ \underbrace{a_1, a_2, \ldots, a_p}_{s}, \tag{10}$$

formula (9) expresses that the period $a_1, \ldots, a_p$ in (10) is repeated $(s-1)/2$ times.

Assuming (8), we have to verify equality (4) for all $k \geq (s+1)/2$. For $k = s-1$, which is even, we obtain by (2), (9), and (3)

$$T_k = T_{s-1} = \frac{k}{2}\sum_{i=1}^{p} a_i = \sum_{i=1}^{pk/2} a_i,$$

i.e., $n = pk/2$ in (4) and the number T_{s-1} is achievable.

Suppose now that $k = s - 1 - k'$, where $1 \leq k' \leq (s-3)/2$ and $s > 3$. By assumption (8), there exists $n' \in \mathbb{N}$ such that

$$\frac{k'(k'+1)}{2} = \sum_{i=1}^{n'} a_i. \tag{11}$$

From (2) we observe that

$$T_k = T_{s-1-k'} = \frac{(s-1-k')(s-k')}{2} = \frac{s(s-1-2k')}{2} + \frac{k'(k'+1)}{2}. \tag{12}$$

Since $1 \leq k' \leq (s-3)/2$ and s is odd, it follows that $m = s - 1 - 2k'$ is an even positive integer. Thus, by (12), (7), (11), and (3),

$$T_k = \frac{s-1-2k'}{2}\sum_{i=1}^{p} a_i + \sum_{i=1}^{n'} a_i = \sum_{i=1}^{pm/2+n'} a_i.$$

Next, let $k = qs + k'$ with $q \in \mathbb{N}$ and $0 \leq k' < s$. Then by (2) and (7) we find that

$$T_k = \frac{(qs+k')(qs+k'+1)}{2} = sj + \frac{k'(k'+1)}{2} = \sum_{i=1}^{pj} a_i + T_{k'},$$

where $j = q(qs+1)/2 + qk'$ is integer and $T_{k'} = 0$ for $k' = 0$. By our earlier observation in this proof $T_{k'} = \sum_{i=1}^{n'} a_i$ for some $n' \in \mathbb{N}$ when $0 < k' < s$. $\square$

Remark 1. The number $(s-1)/2$ in (8) cannot be reduced if p is the minimal period length associated with s. To see this consider the periodic sequence $\{a_i\}$ with the minimal period $1, 2, 2, 1, 4, 1, 4$ and $s = 15$. Then, by Definition 1, the triangular numbers $T_1, \ldots, T_6$ are achievable, but T_7 is not.

Example 1. The power of Theorem 1 can be demonstrated on the clock sequence given by (1) with $s = 15$. It is enough to check (5) only for $k \le (s-1)/2 = 7$ (see the first row of (1) and Figure 2) and the achievability of all $k > 7$ follows from Theorem 1.

Similarly, we can easily verify by inspection the assumptions of Theorem 1 for other periods:

1, 2 with $p = 2$ and $s = 3$,
1, 2, 2 with $p = 3$ and $s = 5$,
1, 2, 3, 1 with $p = 4$ and $s = 7$,
1, 2, 3, 3 with $p = 4$ and $s = 9$,
1, 2, 2, 1, 4, 1, 4, 1, 4, 1, 4 with $p = 11$ and $s = 25$.

Example 2. There are also Šindel sequences with s even. We can construct one, e.g., from the period 1, 2, 1, 1, 1:

$$1\ 2\ \underbrace{1\ 1\ 1}_{3}\ \ \underbrace{1\ 2\ 1}_{4}\ \ \underbrace{1\ 1\ 1\ 2}_{5}\ \ \underbrace{1\ 1\ 1\ 1\ 2}_{6}\ldots \tag{13}$$

The factor $(s-1)/2$ appearing on the right-hand side of (9) is not an integer. Therefore, the particular terms expressing the number $s = 6$ in (13) are not in the same order as the given period.

Theorem 2. *A periodic sequence $\{a_i\}$ for an even s in* (7) *is a Šindel sequence if T_k is achievable by $\{a_i\}$ for $k = 1, 2, \ldots, s-1$.*

Proof: Let $s \ge 2$ be even and suppose that

$$\forall\, k \in \{1, 2, \ldots, s-1\} \quad \exists\, n \in \mathbb{N} : \quad T_k = \sum_{i=1}^{n} a_i. \tag{14}$$

From (7) and (3) we get

$$T_{2s-1} = (2s-1)\sum_{i=1}^{p} a_i = \sum_{i=1}^{(2s-1)p} a_i.$$

Now suppose that $k = 2s - 1 - k'$, where $1 \le k' \le s-1$. According to hypothesis (14), there exists $n' \in \mathbb{N}$ such that

$$\frac{k'(k'+1)}{2} = \sum_{i=1}^{n'} a_i.$$

Then, by (2),

$$T_k = T_{2s-1-k'} = \frac{(2s-1-k')(2s-k')}{2} = s(2s-1-2k') + \frac{k'(k'+1)}{2}\,.$$

Hence,

$$T_k = (2s-1-2k')\sum_{i=1}^{p} a_i + \sum_{i=1}^{n'} a_i = \sum_{i=1}^{pm+n'} a_i,$$

where $m = 2s-1-2k'$.

The rest of the proof for $k \geq 2s-1$ is similar to that of Theorem 1. $\square$

Remark 2. The number $s-1$ appearing in (14) is the smallest possible for the minimal period. To see this consider the periodic sequence $\{a_i\}$ with the minimal period $1, 2, 1$ and $s = 4$. Then the triangular numbers T_1 and T_2 are achievable, but T_3 is not.

3. Necessary and sufficient condition for the existence of a Šindel sequence

Let $n \geq 2$ and a be integers. Recall that if the quadratic congruence

$$x^2 \equiv a \pmod{n}$$

has a solution x, then a is called a *quadratic residue modulo n*. Otherwise, a is called a *quadratic nonresidue modulo n*.

Lemma 1. *If f and h are nonnegative integers, then $8f+1$ is a quadratic residue modulo 2^h.*

The proof is a consequence of [4, pp. 105–106].

Theorem 3. *A periodic sequence $\{a_i\}$ is a Šindel sequence if and only if for any $n \in \{1, \dots, p\}$ and any $j \in \{1, 2, \dots, a_n - 1\}$ with $a_n \geq 2$ the number*

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

is a quadratic nonresidue modulo s.

Proof: $\Longleftarrow$: Let a periodic sequence $\{a_i\}$ not be a Šindel sequence. According to (5), there exist positive integers ℓ, m, and j such that $a_m \geq 2$, $j \leq a_m - 1$, and

$$T_\ell = \sum_{i=1}^{m} a_i - j. \tag{15}$$

Let $n \in \{1, \ldots, p\}$ be such that $n \equiv m \pmod p$. Then by (2), (15), (7), and (3),

$$(2\ell+1)^2 = 4\ell^2+4\ell+1 = 8T_\ell+1 = 8\Big(\sum_{i=1}^{m} a_i - j\Big)+1 \equiv 8\Big(\sum_{i=1}^{n} a_i - j\Big)+1 \pmod s,$$

i.e., $8\big(\sum_{i=1}^{n} a_i - j\big) + 1$ is a square modulo s. Thus, the condition given in Theorem 3 is sufficient for $\{a_i\}$ to be a Šindel sequence.

$\Longrightarrow$: Let $\{a_i\}$ be a Šindel sequence with $s = 2^c d$, where $c \ge 0$ and d is odd. Suppose to the contrary that there exist positive integers n, j, and x such that $n \le p$, $a_n \ge 2$, $j \le a_n - 1$, $x \le s$, and

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1 \equiv x^2 \pmod s. \tag{16}$$

From Lemma 1 and (16) there exists y such that

$$\begin{aligned} x^2 &\equiv w \pmod d, \\ y^2 &\equiv w \pmod{2^{c+3}}. \end{aligned} \tag{17}$$

By the Chinese remainder theorem (see [3, p. 15]) there exists an integer $u \ge 3$ such that $u \equiv x \pmod d$ and $u \equiv y \pmod{2^{c+3}}$. Thus, by (17),

$$\begin{aligned} u^2 &\equiv x^2 \equiv w \pmod d, \\ u^2 &\equiv y^2 \equiv w \pmod{2^{c+3}}. \end{aligned}$$

Since $\gcd(d, 2^{c+3}) = 1$, we see that

$$u^2 \equiv w \pmod{2^{c+3}d}. \tag{18}$$

Clearly, u is odd, since w is odd. So let $u = 2\ell + 1$, where $\ell \ge 1$. Then, by (18), $u^2 = 4\ell^2 + 4\ell + 1 = w + 2^{c+3}dg$ for some integer g. Hence, since $u \ge 3$, we find by (2), (18), and (16) that

$$T_\ell = \frac{u^2-1}{8} = \frac{w-1}{8} + 2^c dg \equiv \sum_{i=1}^{n} a_i - j \pmod s.$$

Thus, there exists a positive integer m such that $m \equiv n \pmod p$ and

$$T_\ell = \sum_{i=1}^{m} a_i - j,$$

which contradicts the assumption that $\{a_i\}$ is a Šindel sequence. $\square$

As a byproduct of the proof of Theorem 3, we get the well-known result (see also [1, p. 15] and Figure 3):

Corollary 1. *A positive integer r is a triangular number if and only if $8r+1$ is a square.*

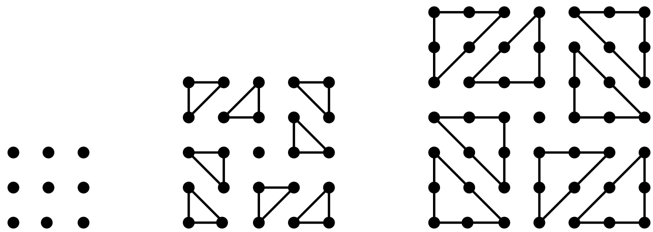

FIGURE 3. The early Pythagoreans knew that if r is a triangular number, then $8r+1$ is a square. This result is mentioned as early as about 100 A.D. in *Platonic Questions* by the Greek historian Plutarch, see [6, p. 4].

Remark 3. In Theorem 3, we require that

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

be a quadratic nonresidue modulo s for various values of n and j when $\{a_i\}$ is a Šindel sequence. A sufficient condition for this to occur is that w be a quadratic nonresidue for some odd prime q dividing s. To see that this condition is not necessary, consider the periodic sequence $\{a_i\}$ given in Example 1 with $p = 11$, $s = 25$, and the period 1, 2, 2, 1, 4, 1, 4, 1, 4, 1, 4. Then

$$8\Big(\sum_{i=1}^{5} a_i - 2\Big) + 1 = 65,$$

which is a quadratic nonresidue modulo 25, but is a quadratic residue modulo 5. Note that 5 is the only odd prime dividing $s = 25$.

Remark 4. Consider the sequence $\{a_i\}$ with period $1, 2, 1, 1, 1, \ldots, 1$. Note that

$$w = 8\Big(\sum_{i=1}^{2} a_i - 1\Big) + 1 = 17.$$

By Theorem 3 and the law of quadratic reciprocity one sees that if s is an odd prime and $s \equiv 1, 2, 4, 8, 9, 13, 15$ or $16 \pmod{17}$ (see [3, pp. 23–24]), then w is a quadratic residue modulo s and thus, $\{a_i\}$ is not a Šindel sequence. Other patterns of the period of periodic sequences $\{a_i\}$ can be similarly investigated.

4. Construction of the primitive Šindel sequence

Definition 2. A Šindel sequence $\{a'_i\}$ with the minimal period length $p+1$ is said to be *composite* if there exists a Šindel sequence $\{a_i\}$ and $\ell \in \mathbb{N}$ such that

$$\begin{aligned} a_i &= a'_i, \quad i = 1, \dots, \ell - 1, \\ a_\ell &= a'_\ell + a'_{\ell+1}, \\ a_i &= a'_{i+1}, \quad i = \ell + 1, \dots, p. \end{aligned}$$

Example 3. The period 1, 2, 3, 2, 2, 3, 2 derived from the period 1, 2, 3, 4, 3, 2 of sequence (1) produces a composite Šindel sequence. In other words, the astronomical clock would also work with the small gear corresponding to this composite Šindel sequence.

Definition 3. A Šindel sequence $\{a_i\}$ is called *primitive* if it is not composite.

Example 4. By inspection, we can verify that all the sequences from Example 1 are primitive.

The proof of the next theorem contains an explicit algorithm for finding a primitive Šindel sequence for a given s.

Theorem 4. *Let s be a positive integer. Then there exists a unique primitive Šindel sequence $\{a_i\}$ such that (7) holds for one of its not necessarily minimal period lengths p.*

Proof: Let $1 \le b_1 < b_2 < \cdots < b_t \le s$ be all the integers such that each $8b_n + 1$ is a square modulo s for $n = 1, \dots, t$. We observe that $b_1 = 1$ and $b_t = s$. Now choose the period as follows: $a_1 = b_1$ and $a_n = b_n - b_{n-1}$ for $n = 2, 3, \dots, t$. Then

$$\forall\, n \in \{1, 2, \dots, t\} : \quad b_n = \sum_{i=1}^{n} a_i.$$

We claim that $\{a_i\}$ is a Šindel sequence. Note that if $n \in \{1, \dots, t\}$, $a_n \ge 2$, and $j \in \{1, 2, \dots, a_n - 1\}$, then

$$b_{n-1} < \sum_{i=1}^{n} a_i - j < b_n.$$

Therefore, $8(\sum_{i=1}^{n} a_i - j) + 1$ is a quadratic nonresidue modulo s, as $8b_1 + 1, \dots,$ $8b_t + 1$ are all the quadratic residues modulo s. It now follows from Theorem 3 that $\{a_i\}$ is a Šindel sequence.

Moreover, one sees that $\{a_i\}$ is a primitive Šindel sequence having a period length $p = t$ and satisfying (7). It is also clear by construction that $\{a_i\}$ is the unique primitive Šindel sequence satisfying (7) for some period length p. $\square$

Definition 4. The sequence 1, 1, 1, ... is called a *trivial* Šindel sequence.

Theorem 5. *The primitive Šindel sequence $\{a_i\}$ is trivial if and only if $s = 2^h$ for $h \geq 0$.*

Proof: $\Longleftarrow$: By the above construction of the period, the primitive Šindel sequence corresponding to s is nontrivial if and only if there exists a positive integer $f \leq s$ such that $8f + 1$ is a quadratic nonresidue modulo s. By Lemma 1, the number $8f+1$ is always a quadratic residue modulo $s = 2^h$ for $h \geq 0$. Hence, the primitive Šindel sequence corresponding to $s = 2^h$ is the trivial Šindel sequence.

$\Longrightarrow$: Conversely, assume that s has an odd prime divisor q. Let d be a quadratic nonresidue modulo q. Since 8 is invertible modulo q, one sees that if z is the inverse of 8 modulo q and $f \equiv z(d-1) \pmod q$, then $8f+1 \equiv d \pmod q$. It now follows that the primitive Šindel sequence corresponding to s is nontrivial. $\square$

We have the following immediate corollaries to Theorems 3, 4, and 5:

Corollary 2. *Let $\{a_i\}$ be a periodic sequence with the minimal length p of the period and $s = 2^m$, where m is a nonnegative integer. Then $\{a_i\}$ is a Šindel sequence if and only if $\{a_i\}$ is the trivial Šindel sequence.*

Corollary 3. *A periodic sequence $\{a_i\}$ is a primitive Šindel sequence if and only if for any $n \in \{1, \dots, p\}$ and any $j \in \{1, 2, \dots, a_n - 1\}$ with $a_n \geq 2$ the number*

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

is a quadratic nonresidue modulo s and

$$v = 8\sum_{i=1}^{n} a_i + 1$$

is a quadratic residue modulo s.

Theorem 6. *For any $k \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and a Šindel sequence $\{a_i\}$ such that $a_\ell = k$.*

Proof: It was stated in Corollary 1 that for $r \in \mathbb{N}$, $8r + 1$ is a square if and only if r is a triangular number. Let $k = T_k - T_{k-1}$ be given (see (6)). Thus it suffices by the proof of Theorem 4 to find a positive integer $s \geq T_k$ such that $8(T_{k-1} + j) + 1$ is a quadratic nonresidue modulo s for $j = 1, 2, \dots, k-1$.

For a fixed $j \in \{1, \dots, k-1\}$ let

$$8(T_{k-1} + j) + 1 = \prod_{i=1}^{v} p_i^{\alpha_i}$$

be the prime power factorization. Since $8(T_{k-1}+j)+1$ is not a square, some α_i is odd. Without loss of generality, we can assume that α_1 is odd. Let c_1 be a quadratic nonresidue modulo p_1. By the Chinese remainder theorem and Dirichlet's theorem on the infinitude of primes in arithmetic progressions, one can find a prime $q_j \geq T_k$ such that $q_j \equiv 1 \pmod 4$, $q_j = c_1 \pmod{p_1}$, and $q_j \equiv 1 \pmod{p_i}$ for $i \in \{2, \dots, v\}$.

Since $q_j \equiv 1 \pmod 4$, it follows from the properties of the Legendre symbol and the law of quadratic reciprocity (see [3, p. 24]) that

$$\Big(\frac{p_1}{q_j}\Big) = \Big(\frac{q_j}{p_1}\Big) = \Big(\frac{c_1}{p_1}\Big) = -1,$$

and

$$\Big(\frac{p_i}{q_j}\Big) = \Big(\frac{q_j}{p_i}\Big) = \Big(\frac{1}{p_i}\Big) = 1 \quad \text{for } \ i = 2, 3, \dots, v,$$

where $\big(\frac{p}{q}\big)$ denotes the Legendre symbol for odd primes p and q. Noting that the Jacobi symbol is multiplicative, we see that

$$\Big(\frac{8(T_{k-1}+j)+1}{q_j}\Big) = \prod_{i=1}^{v} \Big(\frac{p_i}{q_j}\Big)^{\alpha_i} = (-1)^{\alpha_1} \prod_{i=2}^{v} 1^{\alpha_i} = -1,$$

and hence, $8(T_{k-1}+j)+1$ is a quadratic nonresidue modulo q_j. Now simply let s be the product of the distinct q_j's for $j \in \{1, \dots, k-1\}$. □

Example 5. The period 1, 2, 3, 4, 5, 3, 3, 7, 2, 3, 3, 9 with minimal period length $p = 12$ and $s = 45$ yields a primitive Šindel sequence $\{a_i\}$ with a large value of $a_{12} = 9$ relative to s.

Theorem 7. *There exists a primitive Šindel sequence whose period is the minimal period if and only if s given by (7) is odd.*

Proof: $\Longrightarrow$: Let $s = 2^c d$, where $c \geq 1$ and d is odd. Since $8f+1$ is a quadratic residue modulo 2^c for all nonnegative integers f by Lemma 1, we have by the Chinese remainder theorem that $8f+1$ is a square modulo s if and only if $8f+1$ is a square modulo d. It now follows from the construction given in the proof of Theorem 4 that the primitive Šindel sequence corresponding to $s = 2^c d$ has the same period, not necessarily minimal, as the period of the primitive Šindel sequence corresponding to $s = d$. Hence, we see that for s even, the primitive Šindel sequence corresponding to s does not have the associated period as its minimal period.

$\Longleftarrow$: Now let s be odd. If $s = 1$, the result is trivial. So assume that $s \geq 3$ and let $\{a_i\}$ be the unique primitive Šindel sequence corresponding to s and having period length p. Let p' be the minimal period length of the sequence $\{a_i\}$ and let

$$s' = \sum_{i=1}^{p'} a_i.$$

Suppose to the contrary that $p' < p$, i.e., $s' < s$. For $k \in \mathbb{N}$, we let $w_k = \sum_{i=1}^{k} a_i$. To obtain a contradiction, it suffices by the proof of Theorem 4 to find a positive integer $n \le p$ such that $8w_n+1$ is a quadratic nonresidue modulo s. To accomplish this, we need only find a divisor f of s such that $8w_n+1$ is a quadratic nonresidue modulo f.

Since $\gcd(8, s) = 1$, there exists a unique integer b such that $0 \le b \le s-1$ and $8b+1 \equiv 0 \pmod{s}$. Simply let $b \equiv -z \pmod{s}$, where z is the inverse of 8 modulo s. Since 0 is a square modulo s, we see by the construction in the proof of Theorem 4 that $8w_k + 1 \equiv 0 \pmod{s}$ for some $k \in \{1, 2, \ldots, p\}$. Let m be an integer such that $1 \le m \le p'$ and $m \equiv k \pmod{p'}$. Then $8w_m + 1 \equiv 0 \pmod{s'}$. Since $s' < s$, there exists an odd prime q such that $q \mid \frac{s}{s'} = \frac{p}{p'}$. First, suppose that $q \nmid s'$. Consider the q integers

$$8w_m + 1, 8w_{m+p'} + 1, 8w_{m+2p'+1}, \ldots, 8w_{m+(q-1)p'} + 1. \tag{19}$$

Noting that

$$(8w_{m+jp'} + 1) - (8w_{m+ip'} + 1) = 8(j-i)s' \tag{20}$$

for $0 \le i < j \le q-1$ and that $\gcd(8s', q) = 1$, we find that the q numbers in (19) are incongruent modulo q. Let e be a quadratic nonresidue modulo q. Then $8w_{m+jp'}+1 \equiv e \pmod{q}$ for some $j \in \{0, 1, 2, \ldots, q-1\}$, which is a contradiction, since $q \mid s$ and $m + jp' \le m + (q-1)p' \le p$.

Finally, we treat the remaining case in which $q^\alpha \| s'$ and $q^{\alpha+1} \mid s$ for some odd prime q and integer $\alpha \ge 1$, where $q^\alpha \| s'$ means that $q^\alpha \mid s'$ but $q^{\alpha+1} \nmid s'$. Then $8w_m+1 \equiv 0 \pmod{q^\alpha}$. By (20) and the fact that $q^\alpha \| s'$, we see that the q integers in (19) are congruent to

$$0, q^\alpha, 2q^\alpha, \ldots, (q-1)q^\alpha \pmod{q^{\alpha+1}} \tag{21}$$

in some order. To complete the proof, it suffices to demonstrate that at least one of the q numbers in (21) is a quadratic nonresidue modulo $q^{\alpha+1}$. If α is odd, then clearly q^α is a quadratic nonresidue modulo $q^{\alpha+1}$.

Now suppose that α is a positive even integer, $q^\alpha \| r$, and r is a quadratic residue modulo $q^{\alpha+1}$. Then

$$r \equiv \left(aq^{\alpha/2}\right)^2 = a^2 q^\alpha \pmod{q^{\alpha+1}}$$

for some integer a such that $q \nmid a$. If

$$a^2 q^\alpha \equiv h q^\alpha \pmod{q^{\alpha+1}},$$

then

$$a^2 \equiv h \pmod{q}.$$

Let u be a quadratic nonresidue modulo q and suppose that $0 \le u \le q-1$. Then uq^α is a quadratic nonresidue modulo $q^{\alpha+1}$. □

5. Computer results

We developed a program that generates the primitive Šindel sequence for a given s. It is based on the numerical algorithm presented in the proof of Theorem 4. By this theorem we know that the primitive Šindel sequence is uniquely determined for each positive integer s.

s	Primitive Šindel sequences															
1	1															
2	1	1														
3	1	2														
4	1	1	1	1												
5	1	2	2													
6	1	2	1	2												
7	1	2	3	1												
8	1	1	1	1	1	1	1	1								
9	1	2	3	3												
10	1	2	2	1	2	2										
11	1	2	1	2	4	1										
12	1	2	1	2	1	2	1	2								
13	1	1	1	3	2	2	3									
14	1	2	3	1	1	2	3	1								
15	**1**	**2**	**3**	**4**	**3**	**2**										
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
17	1	1	1	1	2	4	1	4	2							
18	1	2	3	3	1	2	3	3								
19	1	1	1	3	1	2	1	5	2	2						
20	1	2	2	1	2	2	1	2	2	1	2	2				
21	1	2	3	1	3	3	2	6								
22	1	2	1	2	4	1	1	2	1	2	4	1				
23	1	2	2	1	3	1	3	2	5	1	1	1				
24	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
25	1	2	2	1	4	1	4	1	4	1	4					

The above table shows values of all primitive Šindel sequences for $s = 1, \ldots, 25$. From this table we observe the property guaranteed by Theorem 5, namely that trivial primitive Šindel sequences appear only when $s = 2^h$ for some $h \geq 0$. We also see that there does not exist a primitive Šindel sequence with the minimal period for s even, which is stated by Theorem 7. The structure of sequences corresponding to $s = 13$ and $s = 19$ is discussed in Remark 4. We verified that no primitive Šindel sequence for $s \leq 1000$ and $s \neq 15$ has such a nice symmetry property as the clock sequence in (1), which was used to construct the bellworks of the Prague horologe (see Figure 4).

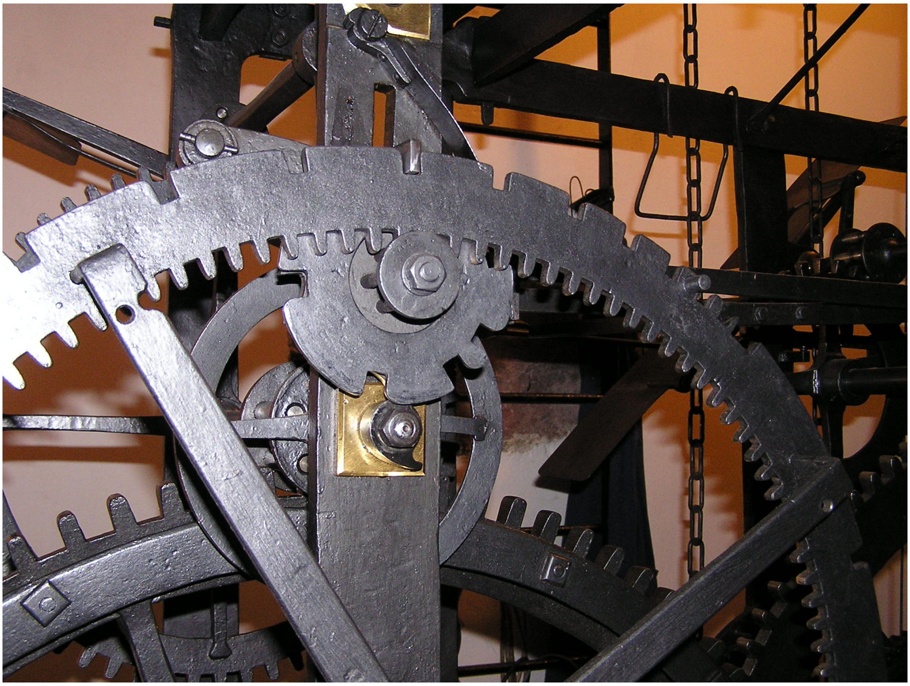

FIGURE 4. A detail of the bellworks of the astronomical clock. The catch is in the slot between the segments corresponding to 8 and 9 hours on the large gear.

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