On non-normality points and metrizable crowded spaces

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Abstract. $\beta X - \{p\}$ is non-normal for any metrizable crowded space X and an arbitrary point $p \in X^*$.

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1. Introduction

We investigate non-normality points in Čech-Stone remainders $X^* = \beta X - X$ of metrizable spaces.

There are several simple proofs that, under CH, $\omega^* - \{p\}$ is not normal for any $p \in \omega^*$ [7], [8]. "Naively" it is known only for special points of ω^*. If p is an accumulation point of some countable discrete subset of ω^*, or if p is a *strong R-point*, or if p is a *Kunen's point*, then $\omega^* - \{p\}$ is not normal (Blaszczyk and Szymanski [1], Gryzlov [2], van Douwen respectively).

What about realcompact crowded spaces? Is $\beta X - \{p\}$ non-normal whenever X is realcompact and crowded and $p \in X^*$? Probably, but we are unaware of any counterexample. On the other hand, the answer is "yes" if X is a locally compact Lindelöf separable crowded space with $\pi w(X) \leq \omega_1$ and p is remote [5]. It is also "yes" if X is a second countable crowded space and either X is locally compact, or X is zero-dimensional, or p is remote [3], [4], [6]. Using the regular base of Arhangel'skiĭ J. Terasawa has omitted the separability condition in the last two cases. He has obtained the affirmative answer in case if X is a metrizable crowded space and either X is strongly zero-dimensional or p is remote [10]. Here, introducing p-filters into this construction, we answer affirmatively for all metrizable crowded spaces.

B. Shapirovskij [9] has defined a *butterfly-point* (or *b-point*) in a space X. We call $p \in X^*$ a *butterfly-point in* βX, if $\{p\} = \mathrm{Cl}\ F \cap \mathrm{Cl}\ G$ for some $F, G \subset X^* - \{p\}$ with $\mathrm{Cl}\,(F \cup G) \subset X^*$.

Theorem. *Let X be a non-compact metrizable crowded space. Then any point $p \in X^*$ is a butterfly-point in βX. Hence $\beta X - \{p\}$ is not normal.*

2. Proofs

From now on a space X is non-compact, metrizable and crowded, i.e. X has no isolated points, and $p \in X^*$ is an arbitrary point. We denote by cl- and Cl- the closure operations in X and βX respectively, $3 = \{0, 1, 2\}$.

Let π and σ be an arbitrary families. A set $U \in \pi$ is called a *maximal member* of the family π if $U \subsetneq V$ for no $V \in \pi$. If members of π are mutually disjoint (with closure), then π is called (*strongly*) *cellular*. We write $\pi \prec \sigma$ if $U \cap V \neq \emptyset$ implies $U \supsetneq V$ for any $U \in \pi$ and $V \in \sigma$. We denote by $\operatorname{Exp} \pi$ the set of subfamilies $\{F : F \subset \pi\}$. We define a *projection* f^π_σ from $\operatorname{Exp} \pi$ to $\operatorname{Exp} \sigma$ by $f^\pi_\sigma F = \{V \in \sigma : \bigcup F \cap V \neq \emptyset\}$ for every $F \in \operatorname{Exp} \pi$.

A maximal locally finite cellular family of open sets is called *nice*. The introduced in [6] *cellular refinement* $\operatorname{Cel}(\pi) = \{\bigcap \phi - \operatorname{cl} \bigcup(\pi - \phi) : \phi \subset \pi\}$ of π is nice, if π is an open locally finite cover of X.

Let π and σ be nice families. A collection $\mathcal{F} = \{F\}$ of subfamilies $F \subseteq \pi$ is called a *p-filter on* π, if $p \in \operatorname{Cl} \cup \bigcap_{k=0}^n F_k$ for any finite subcollection $\{F_0, \dots, F_n\} \subset \mathcal{F}$. Obviously, the union of any increasing family of *p-filters* is also a *p-filter*. So by Zorn's lemma there are maximal p-filters or *p-ultrafilters* $\mathcal{F}'$ *on* π, that is $\mathcal{F}' = \mathcal{G}$ for any p-filter $\mathcal{G}$ with $\mathcal{F}' \subseteq \mathcal{G}$. Adding step-by-step new subfamilies from $\operatorname{Exp} \pi - \mathcal{F}$ to $\mathcal{F}$, while possible, we can embed any *p-filter* $\mathcal{F}$ into some *p-ultrafilter* $\mathcal{F}'$. If p is not a remote point, distinct p-ultrafilters $\mathcal{F}'$ may exist. But each of them contains $\pi(O) = \{V \in \pi : V \cap O \neq \emptyset\}$ for any neighborhood O of p and its image $f^\pi_\sigma \mathcal{F} = \{f^\pi_\sigma F : F \in \mathcal{F}\}$ is a p-filter on σ. We write $\pi \prec_{\mathcal{F}} \sigma$, if there is $F \in \mathcal{F}$ with $F \prec \sigma$. We denote $\bigcap \mathcal{F}^* = \bigcap\{\operatorname{Cl} \bigcup F : F \in \mathcal{F}\}$.

For every $i \in \mathbb{N}$ we fix an open locally finite cover $\mathcal{P}_i$ of X so that $\operatorname{diam} U \leq \frac{1}{i}$ for any $U \in \mathcal{P}_i$ and $\{V \in \mathcal{P}_j : V \cap U \neq \emptyset\}$ is finite for each $j < i$. Then it is easy to see that

$$\mathcal{P} = \bigcup_{i \in \mathbb{N}} \mathcal{P}_i$$

is a regular base of Arhangel'skiĭ, i.e. for any point $x \in X$ and for any its neighborhood $O \subset X$ there is another neighborhood $O' \subset X$ of x with the following properties: $O' \subset O$ and at most finitely many members of $\mathcal{P}$ meet booth O' and $X - O$ simultaneously. Moreover, for any cover $\pi \subset \mathcal{P}$ the family of its maximal members is a locally finite subcover of X.

By induction (see, also, [6]) we define the families of non-empty open sets $\mathcal{D}_k$ and $\mathcal{W}_k \subset \mathcal{P}$ for all $k \in \mathbb{N}$ as follows:

$$\mathcal{D}_1 = \operatorname{Cel}(\mathcal{P}_1).$$

If a nice family $\mathcal{D}_k = \{U\}$ has been constructed, then

$$\mathcal{W}_k = \{U(\nu) : U \in \mathcal{D}_k \text{ and } \nu \in 3\}$$

is strongly cellular with $\mathrm{cl}\, U(\nu) \subset U$ for any its member and

$$\mathcal{D}_{k+1} = \mathrm{Cel}\,(\mathcal{D}_k \cup \mathcal{W}_k \cup \mathcal{P}_{k+1}).$$

By our construction, if $U, V \in \bigcup_{k\in\mathbb{N}} \mathcal{D}_k$ are not disjoint, then either $U \subseteq V$ or $U \supseteq V$. For any $U \in \mathcal{P}_k$ the family $\hat{U} = \{V \in \mathcal{D}_k : V \cap U \neq \emptyset\}$ is locally finite and nice in U. For any locally finite cover $\pi \subset \mathcal{P}$ we denote $\sigma(\pi)$ all maximal members of the family $\bigcup\{\hat{U} : U \in \pi\}$. Then $\sigma(\pi)$ is nice. Define

$$\Sigma = \{\sigma(\pi) : \pi \subset \mathcal{P} \text{ is a locally finite cover of } X\}$$

and put $\sigma(\nu) = \{U(\nu) : U \in \sigma\}$ for any $\sigma \in \Sigma$ and $\nu \in 3$.

Lemma 1. *If π is an open locally finite cover of X, then $\mathrm{Cel}\,(\pi)$ is nice.*

Proof: Let $\phi \subset \pi$. If $\bigcap \phi \neq \emptyset$, then ϕ is finite. So $\bigcap \phi$ and, hence, $\bigcap \phi - \mathrm{cl}\,(\pi - \phi)$ is open.

Let $\phi, \phi' \subset \pi$ be different and $U \in \phi - \phi'$. Then $\bigcap \phi \subset U$ and $\bigcap \phi' \cap U = \emptyset$, because $U \in \pi - \phi'$.

Let a neighborhood O of $x \in X$ meet finitely many members of π, say $U_1, \dots, U_k$. If $\phi \subset \pi$ contains some $U \in \pi - \{U_1, \dots, U_k\}$, then $\bigcap \phi \subseteq U \subseteq X - O$. So O meets at most 2^k members of $\mathrm{Cel}\,(\pi)$.

As π is a locally finite family of open sets, $K = \bigcup\{\mathrm{cl}\, U - U : U \in \pi\}$ is nowhere dense. Let $x \notin K$ and $\phi = \{U \in \pi : x \in U\}$. Then $U \notin \phi$ implies $x \notin \mathrm{cl}\, U$. So $x \in \bigcap \phi - \mathrm{cl} \bigcup(\pi - \phi)$, because π is conservative, and $\mathrm{Cel}\,(\pi)$ is maximal. Our proof is complete. □

Lemma 2. *There is a well-ordered chain $\{\sigma_\alpha : \alpha < \lambda\} \subset \Sigma$ and p-ultrafilters $\mathcal{F}_\alpha$ on σ_α with the following properties for all $\alpha < \beta < \lambda$ and $f^\alpha_\beta = f^{\sigma_\alpha}_{\sigma_\beta}$:*

(1) *$p \notin \mathrm{Cl}\, U$ for each $U \in \sigma_0$;*
(2) *$f^\alpha_\beta \mathcal{F}_\alpha \subset \mathcal{F}_\beta$;*
(3) *$\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma_\beta$;*
(4) *for any $\sigma \in \Sigma - \{\sigma_\alpha : \alpha < \lambda\}$ there is $\alpha < \lambda$ with $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$.*

Proof: Let π be all maximal members of the cover $\{U \in \mathcal{P} : p \notin \mathrm{Cl}\, U\}$ and let $\mathcal{F}_0$ be any p-ultrafilter on $\sigma_0 = \sigma(\pi)$.

For any ordinal β assume p-ultrafilters $\mathcal{F}_\alpha$ on $\sigma_\alpha \in \Sigma$ have been constructed for all $\alpha < \beta$. If some $\sigma \in \Sigma - \{\sigma_\alpha : \alpha < \beta\}$ satisfies the condition $\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma$ for all $\alpha < \beta$, then we put $\sigma_\beta = \sigma$ and embed the p-filter $\bigcup_{\alpha<\beta} f^\alpha_\beta \mathcal{F}_\alpha$ into some p-ultrafilter $\mathcal{F}_\beta$ on σ_β. Otherwise our construction is complete. □

Lemma 3. $\bigcap \mathcal{F}^*_0 \subset X^*$.

Proof: Let $x \in X$ be an arbitrary point. Then $F = \{U \in \sigma_0 : x \notin \mathrm{cl}\, U\}$ satisfies, obviously, $x \notin \mathrm{Cl} \bigcup F$ and $F \in \mathcal{F}_0$. □

Lemma 4. *If $\alpha < \beta < \lambda$, then $\bigcap \mathcal{F}^*_\beta \subset \bigcap \mathcal{F}^*_\alpha$.*

Proof: There is $F \in \mathcal{F}_\alpha$ with $F \prec \sigma_\beta$ by (3). For any $G \in \mathcal{F}_\alpha$ we have $G \cap F \in \mathcal{F}_\alpha$ and $G \cap F \prec \sigma_\beta$. But then

$$\bigcap \mathcal{F}^*_\beta \subset \mathrm{Cl}\ f^\alpha_\beta(G \cap F) \subset \mathrm{Cl}\,(G \cap F) \subset \mathrm{Cl}\ G.$$

□

Lemma 5. *For any neighbourhood O of p in βX there is $\alpha < \lambda$ with $\bigcap \mathcal{F}^*_\alpha \subset O$.*

Proof: Let $\mathrm{Cl}\ O' \subset O$ for a neigbourhood O' of p and let π be all maximal members of the cover $\{U \in \mathcal{P} : U \cap O' \neq \emptyset \Rightarrow U \subset O\}$. For $\sigma = \sigma(\pi)$ there is $\alpha < \lambda$ with $\neg(\sigma_\alpha \prec_{\mathcal{F}_\alpha} \sigma)$ by (3) or (4). As $\sigma_\alpha(O') \in \mathcal{F}_\alpha$ then $F = \{V \in \sigma_\alpha(O') : V \subseteq U$ for some $U \in \sigma\}$ also belongs $\mathcal{F}_\alpha$. So $\bigcap \mathcal{F}^*_\alpha \subset \mathrm{Cl} \bigcup F \subset \mathrm{Cl} \bigcup \sigma(O') \subset \mathrm{Cl}\ O$.

□

Proposition 6. *For any $\alpha < \lambda$ and $\nu \in 3$ there is a point $p_\alpha(\nu) \in \bigcap \mathcal{F}^*_\alpha$ such that $p_\alpha(\nu) \in \mathrm{Cl} \bigcup \sigma_\beta(\nu)$ for all $\beta \in \lambda - \alpha$.*

Proof: Let $\alpha < \beta_0 < \ldots < \beta_n < \lambda$ be any finite sequence and $F \in \mathcal{F}_\alpha$. Our idea is to find non-empty $W \in \bigcup_{i \leq n} \sigma_{\beta_i}$ so that

$$W(\nu) \subseteq \bigcap_{i \leq n} \bigcup \sigma_{\beta_i}(\nu) \cap \bigcup F.$$

At the first step of induction we put $\Delta_0 = \{\sigma_{\beta_i} : i \leq n\}$, $\Theta_0 = \emptyset$ and choose $W_0 \in \bigcup \Delta_0$ as follows: We may assume $F \prec \sigma_{\beta_0}$. For any $i < n$ there is $G_i \in \mathcal{F}_{\beta_i}$ with $G_i \prec \sigma_{\beta_{i+1}}$. We denote $F_0 = f^\alpha_{\beta_0} F \cap G_0$ and $F_{i+1} = f^{\beta_i}_{\beta_{i+1}} F_i \cap G_{i+1}$. Then $F_{i+1} \succ F_i$ and $\bigcup F_{i+1} \subseteq \bigcup F_i$. Any pairwise intersecting $U_i \in F_i$ make up an embedded sequence $U_n \subseteq \ldots \subseteq U_0 \subseteq \bigcup F$. We define $W_0 = U_0$.

For any $m < n$ let $\Delta_m, \Theta_m \subset \Delta_0$ and $W_m \in \bigcup \Delta_m$ has been constructed so that

(1) $\Delta_m \cap \Theta_m = \emptyset$;
(2) $\Delta_m \cup \Theta_m = \Delta_0$;
(3) $W_m \subseteq \bigcup F$;
(4) $W_m \subseteq \bigcup \sigma(\nu)$ for any $\sigma \in \Theta_m$;
(5) for any $\sigma \in \Delta_m$ there is $U_\sigma \in \sigma$ with $U_\sigma \subseteq W_m$.

Let $\Omega_m = \{\sigma \in \Delta_m : U_\sigma = W_m\}$.

If $\Delta_m \neq \Omega_m$, then we put $\Delta_{m+1} = \Delta_m - \Omega_m$ and $\Theta_{m+1} = \Theta_m \cup \Omega_m$. As $\sigma \in \Delta_{m+1}$ are nice, we can choose $U'_\sigma \in \sigma$ so that $\bigcap \{U'_\sigma : \sigma \in \Delta_{m+1}\} \cap W_m(\nu) \neq \emptyset$. Then $U_\sigma \subsetneq W_m$ implies $U'_\sigma \subseteq W_m(\nu)$ by our construction. We define W_{m+1} to be the maximal member of embedded sequence $\{U'_\sigma : \sigma \in \Delta_{m+1}\}$.

If, finally, $\Delta_m = \Omega_m$, then W_m is as required. □

Proof of Theorem: Define $F_\nu = \{p_\alpha(\nu) : \alpha < \lambda\}$ for all $\nu \in 3$. By our construction, $F_\nu \subset \bigcap \mathcal{F}_0^* \subset X^*$ and for any neighbourhood O of p there is $\alpha < \lambda$ with $\{p_\beta(\nu) : \beta \in \lambda - \alpha\} \subset \bigcap \mathcal{F}_\alpha^* \subset O$. Then the condition $\{p_\beta(\nu) : \beta < \alpha\} \subset \mathrm{Cl} \bigcup \sigma_\alpha(\nu)$ implies that the sets $\mathrm{Cl}\, F_\nu - \{p\}$ are pairwise disjoint and $p \in F_\nu$ for no more then one unique F_ν. The other two ensure that p is a b-point in βX.

Our proof is complete. □

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