Characterization of the strict convexity of the Besicovitch-Musielak-Orlicz space of almost periodic functions

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Abstract. We introduce the new class of Besicovitch-Musielak-Orlicz almost periodic functions and consider its strict convexity with respect to the Luxemburg norm.

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1. Introduction

We denote by $C^0a.p.$ the linear set of all continuous almost periodic functions ($u.a.p.$). Let A be the subspace of $C^0a.p.$ whose elements are the generalized trigonometric polynomials i.e.,

$$A = \left\{ P_n(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

The class $C^0a.p.$ is in fact the closure of A in the uniform norm of $C_b(\mathbb{R})$ (the space of continuous and bounded functions on $\mathbb{R}$).

This topological characterization is used to define widest classes of almost periodic functions as the closure of the linear set A with respect to some specific norms.

The first extension was obtained by A.S. Besicovitch (cf. [2]) in the context of L^p spaces. Namely he defined the $S^q_{a.p.}$, $W^q_{a.p.}$ and $B^q_{a.p.}$ spaces (resp. Stepanoff, Weyl and Besicovitch spaces of almost periodic functions). Later on, T.R. Hillmann (cf. [5]) used a similar approach to obtain an extension in the context of Orlicz spaces.

Most of the Hillmann's work concerns topological and structural properties of the new spaces.

In [9], [10], [11], there are considered the fundamental geometric properties of the Besicovitch-Orlicz spaces of almost periodic functions.

In this paper, we consider the natural extension of almost periodicity to the context of Besicovitch-Musielak-Orlicz spaces, in particular the case when the function φ generating the space depends on a parameter.

The theory of spaces of generalized almost periodic functions was since its beginning a subject of great interest. This was essentially motivated by the development of the theory of differential and partial differential equations with almost periodic terms (cf. [1], [8], [13]).

Actually this interest is still in growth and is enlarged to cover new domains of applications.

2. Preliminaries

In the sequel $\varphi : \mathbb{R}\times [0,+\infty[\;\rightarrow\; [0,+\infty[$ will be a continuous function on $\mathbb{R}\times [0,+\infty[$ satisfying:

(i) For every $t \in \mathbb{R}, \varphi(t,0) = 0$.
(ii) For each $t \in \mathbb{R}, \varphi(t,u)$ is convex with respect to $u \in [0,+\infty[$.
(iii) For every $u \in [0,+\infty[$, $\varphi(t,u)$ is periodic with respect to $t \in \mathbb{R}$, the period τ being fixed and independent of $u \in [0,+\infty[$. Without loss of generality we may suppose that $\tau = 1$.
(iv) For each $\alpha > 0$, we have $\inf_{t\in\mathbb{R}} \ \varphi(t,\alpha) = \phi(\alpha) > 0$.

We denote by $M(\mathbb{R})$ the space of all real valued Lebesgue measurable functions. The functional

$$\begin{aligned} \rho_\varphi : M(\mathbb{R}) &\rightarrow [0,+\infty] \\ f &\mapsto \rho_\varphi(f) = \overline{\lim_{T\rightarrow+\infty}} \ \frac{1}{2T}\int_{-T}^{+T} \varphi(t,|f(t)|)\,dt \end{aligned}$$

is a convex pseudomodular (cf. [10], [12]).

We define the Besicovitch-Musielak-Orlicz space associated to this pseudomodular by

$$\begin{aligned} B^\varphi(\mathbb{R}) &= \left\{ f \in M(\mathbb{R}) : \lim_{\alpha\rightarrow 0} \rho_\varphi(\alpha f) = 0 \right\} \\ &= \left\{ f \in M(\mathbb{R}) : \rho_\varphi(\alpha f) < +\infty, \quad \text{for some} \ \ \alpha > 0 \right\}. \end{aligned}$$

The space $B^\varphi(\mathbb{R})$ is naturally endowed with the pseudonorm

$$\|f\|_\varphi = \inf\left\{ k > 0 : \rho_\varphi\left(\frac{f}{k}\right) \leq 1 \right\}, \qquad f \in B^\varphi(\mathbb{R}).$$

Let A be the set of all generalized trigonometric polynomials, i.e.,

$$A = \left\{ P_n(t) = \sum_{j=1}^{n} a_j e^{i\lambda_j t}, a_j \in \mathbb{C}, \lambda_j \in \mathbb{R}, n \in \mathbb{N} \right\}.$$

We denote by $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ (resp. $B^{\varphi}_{a.p.}(\mathbb{R})$) the closure of A with respect to the pseudomodular ρ_φ (resp. with respect to the pseudonorm $\|.\|_\varphi$), more precisely:

$$\tilde{B}^{\varphi}_{a.p.}(\mathbb{R}) = \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists\, f_n \in A, \exists\, k_0 > 0, \lim_{n\to+\infty} \rho_\varphi\left(k_0\left(f_n - f\right)\right) = 0 \right\},$$

$$\begin{aligned} B^{\varphi}_{a.p.}(\mathbb{R}) &= \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists\, f_n \in A, \forall\, k > 0, \lim_{n\to+\infty} \rho_\varphi\left(k\left(f_n - f\right)\right) = 0 \right\} \\ &= \left\{ f \in B^{\varphi}(\mathbb{R}) : \exists\, f_n \in A, \lim_{n\to+\infty} \|f_n - f\|_\varphi = 0 \right\}. \end{aligned}$$

$\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ and $B^{\varphi}_{a.p.}(\mathbb{R})$ will be called Besicovitch-Musielak-Orlicz spaces of almost periodic functions.

It is clear that

$$B^{\varphi}_{a.p.}(\mathbb{R}) \subseteq \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}) \subseteq B^{\varphi}(\mathbb{R}).$$

When $\varphi(t, |x|) = |x|$, we denote by $B^1(\mathbb{R})$ and $B^1 a.p.(\mathbb{R})$ the respective spaces. The notation ρ_1 is used for the associated pseudomodular.

Recall that the function φ is said to be strictly convex if $\varphi(t, \lambda u + (1-\lambda)v) < \lambda\varphi(t,u) + (1-\lambda)\varphi(t,v)$ for almost all $t \in \mathbb{R}$ and for every $0 \le u < v < +\infty$, $0 < \lambda < 1$.

A normed linear space $(X, \|.\|)$ is strictly convex if $\left\|\frac{x+y}{2}\right\| < 1$ whenever $\|x\| = \|y\| = 1$ and $\|x - y\| > 0$.

We say that φ satisfies the Δ_2-condition ($\varphi \in \Delta_2$) if there exist $k > 1$ and a measurable nonnegative function h such that $\rho_\varphi(h) < +\infty$ and $\varphi(t, 2u) \le k\varphi(t,u)$ for almost all $t \in \mathbb{R}$ and all $u \ge h(t)$.

3. Auxiliary results

The space $B^{\varphi}_{a.p.}(\mathbb{R})$ can be regarded as a subspace of measurable functions on $\mathbb{R}$ with respect to Lebesgue measure. However, the theory of $B^{\varphi}_{a.p.}(\mathbb{R})$ spaces is different from that of $L^{\varphi}(\mathbb{R})$ spaces: the usual convergence results of the Lebesgue measure theory are not valid in the $B^{\varphi}_{a.p.}(\mathbb{R})$ spaces (see [11]).

To handle $B^{\varphi}_{a.p.}(\mathbb{R})$ spaces as $L^{\varphi}(\mathbb{R})$ ones, we introduce the set function $\bar{\mu}$.

Let $\Sigma = \Sigma(\mathbb{R})$ be the σ-algebra of all Lebesgue measurable subsets of $\mathbb{R}$. We denote by $\bar{\mu}$ the set function defined on Σ by

$$\bar{\mu}(A) = \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{+T} \chi_A(t)\, dt = \overline{\lim_{T\to+\infty}} \frac{1}{2T} \mu(A \cap [-T, +T]),$$

where μ denotes the Lebesgue measure on $\mathbb{R}$.

It is easily seen that the set function $\bar{\mu}$ is not σ-additive.

A sequence $\{f_n\} \subset B^{\varphi}(\mathbb{R})$ is said to be $\bar{\mu}$-convergent to some $f \in B^{\varphi}(\mathbb{R})$ (in symbol $f_n \xrightarrow{\bar{\mu}} f$) when, for every $\alpha > 0$, we have

$$\lim_{n\to+\infty} \bar{\mu}\,\{x \in \mathbb{R} : |f_n(x) - f(x)| > \alpha\} = 0.$$

We give here some technical results that are the key arguments in the proof of the main theorem.

Lemma 1. *Let $\nu(A) = \overline{\lim}_{T\to+\infty} \frac{1}{2T}\int_{-T}^{+T} \varphi(t, \chi_A(t))\,dt$. Then the set function $\bar{\mu}$ is absolutely continuous with respect to ν, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(A \in \Sigma, \nu(A) < \delta) \Rightarrow (\bar{\mu}(A) < \varepsilon). \tag{3.1}$$

Proof: Suppose that (3.1) is false. Then for some $\varepsilon_0 > 0$ we will have the following:

for each $n \in \mathbb{N}$, there exists $E_n \in \Sigma$ s.t. $\nu(E_n) < \frac{1}{2^n}$ and $\bar{\mu}(E_n) > \varepsilon_0$. Thus

$$\begin{aligned}
\nu\,(E_n) &= \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{-T}^{+T} \varphi\,(t, \chi_{E_n}(t))\ dt \\
&= \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{-T}^{+T} \varphi(t,1)\chi_{E_n}(t)\,dt \\
&\geq \phi(1)\bar{\mu}\,(E_n) \geq \phi(1)\varepsilon_0,
\end{aligned}$$

a contradiction. □

Lemma 2. *Let $\{f_n\}_{n\geq 1} \subset B^{\varphi}_{a.p.}(\mathbb{R})$ be a sequence modular convergent to $f \in B^{\varphi}_{a.p.}(\mathbb{R})$, i.e., $\lim_{n\to+\infty} \rho_{\varphi}(f_n - f) = 0$. Then $f_n \xrightarrow{\bar{\mu}} f$.*

Proof: Notice first that we have also $\lim_{n\to+\infty} \rho_{\phi}(f_n - f) = 0$. Then from a similar result for functions without parameter (cf. [10]) it follows that $f_n \xrightarrow{\bar{\mu}} f$. □

Lemma 3. *Let $h \in B^{\varphi}(\mathbb{R})$ be such that $\rho_{\varphi}(h) = a > 0$. Then for every $\bar{\theta} \in (0,1)$ there exist constants $\beta > 0$, $T_0 > 0$ and a set $\bar{G} = \{t \in \mathbb{R}, |h(t)| \leq \beta\}$ such that*

$$\mu\left\{\bar{G} \cap [-T,+T]\right\} \geq \bar{\theta}2T, \quad \textit{for} \ \ T \geq T_0. \tag{3.2}$$

Proof: It is clear that $h \in B^{\phi}(\mathbb{R})$. Then if $\rho_{\phi}(h) > 0$ the conclusion follows from a similar result for the function ϕ without parameter (cf. [10]). The conclusion is immediate if $\rho_{\phi}(h) = 0$. □

Lemma 4. *Let $g \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then for all $\varepsilon > 0$ there exist $\delta > 0$ and $T_0 > 0$ such that $\rho_\varphi(g\chi_Q) \leq \varepsilon$, for all $Q \in \Sigma$ satisfying $\mu\{Q \cap [-T, +T]\} \leq 2\delta T$, $T \geq T_0$.*

Proof: We may suppose $\rho_\varphi(g) > 0$.

Let $\varepsilon > 0$ and $P_\varepsilon \in A$ be such that $\rho_\varphi(2(g - P_\varepsilon)) < \frac{\varepsilon}{2}$. Using the properties of φ we have $\varphi(t, 2|P_\varepsilon(t)|) \in C^0 a.p.$ (cf. [4]). We then put $M_\varepsilon = \sup_{t\in\mathbb{R}} \varphi(t, 2|P_\varepsilon(t)|)$.

We choose $\bar{\theta} \in (0,1)$ satisfying $M_\varepsilon(1 - \bar{\theta}) < \frac{\varepsilon}{2}$. Then by Lemma 3 there exist $\beta > 0$ and a set $\bar{G} = \{t \in \mathbb{R}, |g(t)| \leq \beta\}$ for which $\mu\{\bar{G} \cap [-T, +T]\} \geq 2\bar{\theta}T$, $\forall\, T \geq T_0$, for some $T_0 > 0$. Hence, denoting by $\bar{G}'$ the complement of $\bar{G}$, we will have for all $T \geq T_0$,

$$\begin{aligned}
&\frac{1}{2T}\int_{\bar{G}'\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \\
&\leq \frac{1}{2}\left(\frac{1}{2T}\int_{\bar{G}'\cap[-T,+T]} \left[\varphi\left(t, 2\left|g(t) - P_\varepsilon\left(t\right)\right|\right) + \varphi\left(t, 2\left|P_\varepsilon(t)\right|\right)\right] dt\right) \\
&\leq \frac{\varepsilon}{4} + \frac{1}{4T} M_\varepsilon\left(1 - \bar{\theta}\right) 2T \leq \frac{\varepsilon}{2}\,.
\end{aligned} \tag{3.3}$$

We put $\delta = \frac{\varepsilon}{2\sup_{t\in\mathbb{R}} \varphi(t,\beta)}$ and let $Q \subset \mathbb{R}$ be such that $\mu\{Q \cap [-T, +T]\} \leq 2\delta T$ for $T \geq T_0$.

Then if $Q_1 = Q \cap \bar{G}$ and $Q_2 = Q \cap \bar{G}'$, we will have

$$\begin{aligned}
\frac{1}{2T}\int_{Q_1\cap[-T,T]} \varphi(t, |g(t)|)\, dt &\leq \frac{1}{2T}\int_{Q_1\cap[-T,T]} \varphi(t, \beta)\, dt \\
&\leq \frac{1}{2T}\mu\left(Q_1\right) \sup_{t\in\mathbb{R}} \varphi(t, \beta) \\
&\leq \delta \sup_{t\in\mathbb{R}} \varphi(t, \beta) \leq \frac{\varepsilon}{2}\,.
\end{aligned}$$

Similarly using (3.3) we get

$$\frac{1}{2T}\int_{Q_2} \varphi(t, |g(t)|)\, dt \leq \frac{1}{2T}\int_{\bar{G}'\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \leq \frac{\varepsilon}{2}\,.$$

Finally for all $T \geq T_0$, we have

$$\frac{1}{2T}\int_{Q\cap[-T,+T]} \varphi(t, |g(t)|)\, dt \leq \varepsilon,$$

which means that $\rho_\varphi(g\chi_Q) \leq \varepsilon$. □

Proposition 1. *Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then $\varphi(t, |f(t)|) \in B^{1}_{a.p.}(\mathbb{R})$ and consequently the limit $\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{+T} \varphi(t, |f(t)|)\, dt$ exists and is finite.*

Proof: Let $\{f_n\}$ be a sequence of trigonometric polynomials such that $\|f_n - f\|_{\varphi} \to 0$. Then using Lemma 2 we have also $f_n \xrightarrow{\bar{\mu}} f$.

Let $\bar{\theta} \in (0,1)$. In view of Lemma 3, there exist $\beta > 0$ and $T_0 > 0$ for which $\bar{\mu}(\bar{G}) \geq \bar{\theta}$ with $\bar{G} = \{t \in \mathbb{R} : |f(t)| \leq \beta\}$.

Let $\alpha > 0$ and $A^{\alpha}_n = \{t \in \mathbb{R} : |f_n(t) - f(t)| > \alpha\}$. It is easily seen that $|f_n(t)| \leq \beta + \alpha, \forall\, t \in \bar{G} \cap (A^{\alpha}_n)'$.

Now, the function φ being continuous on $\mathbb{R} \times [0, +\infty[$, is also uniformly continuous on $[0,1] \times [0, \alpha+\beta]$. Moreover, using the periodicity of $\varphi(t,u)$ with respect to $t \in \mathbb{R}$, it follows that φ is uniformly continuous on $\mathbb{R} \times [0, \alpha+\beta]$.

Then for every $\eta > 0$ there exists $\alpha_\eta > 0$ such that

$$\forall\, t \in \bar{G} \cap (A^{\alpha}_n)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta \Longrightarrow |f_n(t) - f(t)| > \alpha_\eta.$$

Hence, since $f_n \xrightarrow{\bar{\mu}} f$ we get also

$$\lim_{n\to+\infty} \bar{\mu}\left\{t \in \bar{G} \cap (A^{\alpha}_n)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\right\} = 0.$$

Consequently,

$$\begin{aligned}
&\bar{\mu}\{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \\
&\leq \bar{\mu}\left\{t \in \bar{G} \cap (A^{\alpha}_n)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\right\} \\
&\quad + \bar{\mu}\left\{t \in (\bar{G})' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\right\} \\
&\quad + \bar{\mu}\{t \in A^{\alpha}_n : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \\
&\leq \bar{\mu}\left\{t \in \bar{G} \cap (A^{\alpha}_n)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\right\} \\
&\quad + \bar{\mu}\left((\bar{G})'\right) + \bar{\mu}(A^{\alpha}_n) \\
&\leq \bar{\mu}\left\{t \in \bar{G} \cap (A^{\alpha}_n)' : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\right\} \\
&\quad + \left(1 - \bar{\theta}\right) + \bar{\mu}(A^{\alpha}_n).
\end{aligned}$$

Letting n tend to infinity, we will have

$$\overline{\lim_{n\to+\infty}}\, \bar{\mu}\{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} \leq \left(1 - \bar{\theta}\right).$$

Finally, since $\bar{\theta} \in (0,1)$ is arbitrary, we deduce that for all $\eta > 0$

$$\lim_{n\to+\infty} \bar{\mu}\{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \eta\} = 0. \tag{3.4}$$

On the other hand, using Lemma 4, it is easy to see that given $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following implication holds

$$(Q \in \Sigma, \bar{\mu}(Q) \leq \delta) \Longrightarrow \max\left(\rho_\varphi\left(f\chi_Q\right), \rho_\varphi\left(f_n\chi_Q\right)\right) \leq \varepsilon.$$

Let $E_n^\varepsilon = \{t \in \mathbb{R} : |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)| \geq \varepsilon\}$. Then since by (3.3), $\bar{\mu}(E_n^\varepsilon) \leq \delta$ for $n \geq n_0$, we get

$$\begin{aligned}
&\overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{+T} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)|\, dt \\
&\leq \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{E_n^\varepsilon\cap[-T,T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)|\, dt \\
&\quad + \overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{(E_n^\varepsilon)'\cap[-T,T]} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)|\, dt \\
&\leq 2\varepsilon + \varepsilon = 3\varepsilon.
\end{aligned}$$

Finally by $\varepsilon > 0$ being arbitrary we deduce that

$$\lim_{n\to+\infty}\overline{\lim_{T\to+\infty}} \frac{1}{2T}\int_{-T}^{+T} |\varphi(t, |f_n(t)|) - \varphi(t, |f(t)|)|\, dt = 0.$$

It remains to see that $\varphi(t, |f_n(t)|) \in C^0 a.p.$ This follows from the properties of the function φ and the fact that $f_n \in A$ (see [4]). □

Lemma 5. *Let $\{f_n\}_n \subset B^1_{a.p.}(\mathbb{R})$ be such that $f_n \xrightarrow{\bar{\mu}} f \in B^1_{a.p.}(\mathbb{R})$. Suppose there exists $g \in B^1_{a.p.}(\mathbb{R})$ for which $\max(|f_n(t)|, |f(t)|) \leq g(t)$, $t \in \mathbb{R}$. Then $\rho_1(f_n) \to \rho_1(f)$.*

Proof: Take $\varepsilon > 0$ and let $\delta > 0$ be associated to g as in Lemma 4. We put $A_n^\varepsilon = \{t \in \mathbb{R} : |f_n(t) - f(t)| \geq \frac{\varepsilon}{2}\}$. Then since $f_n \xrightarrow{\bar{\mu}} f$ it follows that $\bar{\mu}(A_n^\varepsilon) \leq \delta$ for all $n \geq n_0$ and then by Lemma 4

$$\rho_1\left(|f_n - f|\,\chi_{A_n^\varepsilon}\right) \leq \rho_1\left(2g\chi_{A_n^\varepsilon}\right) \leq \frac{\varepsilon}{2}.$$

Consequently, for all $n \geq n_0$ we have

$$\begin{aligned}
\rho_1(|f_n - f|) &\leq \rho_1\left(|f_n - f|\,\chi_{A_n^\varepsilon}\right) + \rho_1\left(|f_n - f|\,\chi_{(A_n^\varepsilon)'}\right) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

i.e., $\lim_{n\to+\infty} \rho_1(f_n) = \rho_1(f)$. □

Lemma 6. *Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then the functional $\lambda \mapsto \rho_\varphi\left(\frac{f}{\lambda}\right)$ is continuous on* $]0,+\infty[$.

Proof: First, notice that since $f \in B^{\varphi}_{a.p.}(\mathbb{R})$ we have $\rho_\varphi(\alpha f) < +\infty$ for each $\alpha > 0$. Indeed, f being in $B^{\varphi}_{a.p.}(\mathbb{R})$ there exists a sequence $\{f_n\}_n \subset A$ such that $\lim_{n\to\infty} \|f-f_n\|_\varphi = 0$ or equivalently $\lim_{n\to\infty} \rho_\varphi(\alpha(f-f_n)) = 0$ for every $\alpha > 0$.

Let $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that $\rho_\varphi(2\alpha(f - f_{n_0})) \leq 1$. Then

$$\rho_\varphi(\alpha f) \leq \frac{1}{2}\rho_\varphi\left(2\alpha\left(f - f_{n_0}\right)\right) + \frac{1}{2}\rho_\varphi\left(2\alpha f_{n_0}\right),$$

consequently, using the fact that the trigonometric polynomial f_{n_0} is uniformly bounded, it follows that $\rho_\varphi(\alpha f) < +\infty$.

Let now $\lambda_0 \in]0,+\infty[$ and $\{\lambda_n\}$ be a sequence of real numbers which converges to λ_0. We have

$$\rho_\varphi\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) \leq \left|\frac{1}{\lambda_n} - \frac{1}{\lambda_0}\right| \rho_\varphi(f) \text{ for every } n \geq n_0.$$

Then $\lim_{n\to+\infty} \rho_\varphi\left(\frac{f}{\lambda_n} - \frac{f}{\lambda_0}\right) = 0$.

Now, using Lemma 2 we get $\frac{f}{\lambda_n} \xrightarrow{\bar{\mu}} \frac{f}{\lambda_0}$ and then $\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right) \xrightarrow{\bar{\mu}} \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)$ (see the proof of Proposition 1). Furthermore

$$\max\left(\varphi\left(t, \frac{|f(t)|}{\lambda_n}\right), \varphi\left(t, \frac{|f(t)|}{\lambda_0}\right)\right) \leq \varphi\left(t, \frac{2}{\lambda_0}|f(t)|\right)$$

and by Proposition 1 we have $\varphi\left(t, \frac{2}{\lambda_0}|f(t)|\right) \in B^1_{a.p.}(\mathbb{R})$. Consequently, using Lemma 5 we deduce

$$\rho_\varphi\left(\frac{f}{\lambda_n}\right) \to \rho_\varphi\left(\frac{f}{\lambda_0}\right).$$

This means that $\lambda \mapsto \rho_\varphi\left(\frac{f}{\lambda}\right)$ is continuous on $]0,+\infty[$. □

Corollary 1. *Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$. Then*

(1) $\|f\|_\varphi \leq 1$ *if and only if* $\rho_\varphi(f) \leq 1$;
(2) $\|f\|_\varphi = 1$ *if and only if* $\rho_\varphi(f) = 1$.

Proof: We prove briefly (2), the assertion (1) follows then easily.

Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$ with $\|f\|_\varphi = 1$. Then for $\varepsilon > 0$ we will have $\rho_\varphi\left(\frac{f}{1+\varepsilon}\right) \leq 1$ and using Lemma 6 it follows that $\rho_\varphi(f) \leq 1$.

We have also $\rho_\varphi\left(\frac{f}{1-\varepsilon}\right) \geq 1$ and again by Lemma 6 we get $\rho_\varphi(f) \geq 1$. Finally, $\rho_\varphi(f) = 1$.

The converse implication is known for a general modular space. □

Remark 1. We recall that a similar result holds in classical Musielak-Orlicz spaces under the additional Δ_2-condition. This condition is not necessary in our case since Lemma 6 holds with the restriction $f \in B^{\varphi}_{a.p.}(\mathbb{R})$.

Lemma 7. *Let $f \in B^{\varphi}_{a.p.}(\mathbb{R})$ with $\|f\|_{\varphi} = 1$. Then there exist real numbers $0 < \alpha < \beta$ and $\theta \in (0,1)$ such that if $G_1 = \{t \in \mathbb{R} : \alpha \leq |f(t)| \leq \beta\}$ we have $\bar{\mu}(G_1) \geq \theta$.*

Proof: Let $\bar{\theta} \in (0,1)$. Then from Lemma 3 there exist $\beta > 0$ and $T_0 > 0$ such that $\mu\{\bar{G} \cap [-T,+T]\} \geq \bar{\theta}2T, \forall\, T \geq T_0$, where $\bar{G} = \{t \in \mathbb{R} : |f(t)| \leq \beta\}$.

We claim that the following is also true:

- for each $\delta \in (0,1)$ there exist $\tilde{\theta} \in (0,1)$, $T_0 > 0$ and a set $\tilde{G} = \{t \in \mathbb{R},\ \varphi(t, |f(t)|) \leq 1 - \delta\}$ such that for $T \geq T_0$

$$\mu\left\{\tilde{G} \cap [-T,+T]\right\} < \tilde{\theta}2T. \tag{3.5}$$

For, let $\delta \in (0,1)$ and P_n be a sequence of trigonometric polynomials approximating f, i.e., $\|f - P_n\|_{\varphi} \to 0$. We take P_δ such that $\rho_{\varphi}(2|f - P_\delta|) < \frac{\delta}{4}$ and put $M = \sup_{t\in\mathbb{R}} \varphi(t, 2P_\delta(t))$.

Let $\varepsilon > 0$ be such that $\left(\frac{\delta}{4} + M\varepsilon\right) < \delta$ and suppose that (3.5) is not satisfied. Then taking $\tilde{\theta} = 1 - \varepsilon$, there will exists a sequence $\{T_n\}$ increasing to infinity for which $\mu\{\tilde{G} \cap [-T_n, +T_n]\} \geq \tilde{\theta}2T_n$. We then get

$$\begin{aligned}
\frac{1}{2T_n}\int_{-T_n}^{+T_n} \varphi\left(t, |f\left(t\right)|\right)\, dt &= \frac{1}{2T_n}\int_{\tilde{G}\cap[-T_n,+T_n]} \varphi\left(t, |f\left(t\right)|\right)\, dt \\
&\quad + \frac{1}{2T_n}\int_{(\tilde{G})'\cap[-T_n,+T_n]} \varphi\left(t, |f\left(t\right)|\right)\, dt \\
&\leq (1-\delta) + \frac{1}{2T_n}\int_{(\tilde{G})'\cap[-T_n,+T_n]} \varphi\left(t, |f\left(t\right)|\right)\, dt.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\frac{1}{2T_n}\int_{(\tilde{G})'\cap[-T_n,+T_n]} \varphi\left(t, |f\left(t\right)|\right)\, dt \\
&\leq \frac{1}{2}\left[\frac{1}{2T_n}\int_{(\tilde{G})'\cap[-T_n,+T_n]} \varphi\left(t, 2\left|f\left(t\right) - P_\delta(t)\right|\right)\, dt\right. \\
&\quad \left. + \frac{1}{2T_n}\int_{(\tilde{G})'\cap[-T_n,+T_n]} \varphi\left(t, 2\left|P_\delta(t)\right|\right)\, dt\right] \\
&\leq \frac{1}{2}\left[\frac{\delta}{4} + M\varepsilon\right] \leq \frac{\delta}{2}\,.
\end{aligned}$$

Then

$$\frac{1}{2T_n}\int_{-T_n}^{+T_n}\varphi\left(t,|f(t)|\right)\,dt \le 1-\delta+\frac{\delta}{2}\le 1-\frac{\delta}{2}\,.$$

Hence, letting n tend to infinity we will have $\rho_\varphi(f)\le 1-\frac{\delta}{2}$. Finally, using Corollary 1 it follows $\|f\|_\varphi<1$. This contradicts the fact that $\|f\|_\varphi=1$.

We now show the statement of the lemma. Let $\delta\in(0,1)$ and $\alpha>0$ be such that $\sup_{t\in\mathbb{R}}\varphi(t,\alpha)\le 1-\delta$. We choose $\tilde{\theta}$ as in (3.5) and then take $\bar{\theta}>\tilde{\theta}$ as in Lemma 3. If $\beta>\alpha$ is a fixed number we define the set $G_1=\{t\in\mathbb{R}:\alpha\le|f(t)|\le\beta\}$. Then since

$$(G_1)'\cap[-T,T]=\{t\in[-T,T]:|f(t)|\le\alpha\}\cup\{t\in[-T,T]:f(t)\ge\beta\}\subset\tilde{G}\cup\left(\bar{G}\right)',$$

it follows that for $T\ge T_0$ we have

$$\begin{aligned}\mu\left((G_1)'\cap[-T,T]\right)&\le\mu\left(\tilde{G}\cap[-T,T]\right)+\mu\left(\left(\bar{G}\right)'\cap[-T,T]\right)\\&\le\tilde{\theta}2T+\left(1-\bar{\theta}\right)2T=\left(1-\left(\bar{\theta}-\tilde{\theta}\right)\right)2T,\end{aligned}$$

or equivalently

$$\mu\left(G_1\cap[-T,T]\right)\ge\left(\bar{\theta}-\tilde{\theta}\right)2T,\quad\text{for}\quad T\ge T_0.$$

□

Lemma 8. *Let $\{a_n\}_n$, $a_n>0$ be a sequence of real numbers and $\alpha\in(0,1)$. To each n we associate a measurable set A_n such that*

(i) *$A_i\cap A_j=\phi$, for $i\ne j$ and $\bigcup_{n\ge1}A_n\subset[0,\alpha[$, $\alpha<1$;*

(ii) *$\sum_{n\ge0}\int_0^1\varphi(t,a_n\chi_{A_n}(t))\,dt<+\infty$.*

Consider the function $f=\sum_{n\ge1}a_n\chi_{A_n}$ on $[0,1]$ and let $\tilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$ (with period $\tau=1$). Then $\tilde{f}\in\tilde{B}^{\varphi}_{a.p.}$.

Proof: Let us first remark that since $\int_0^1\varphi(t,a_n)\,dt<+\infty$, for $n\ge1$ there exists a set $A_n\subset[0,\alpha[$ for which $\int_0^1\varphi(t,a_n\chi_{A_n}(t))\,dt<\frac{1}{n^2}$. It is also clear that we may choose the A_n's so that the conditions of the lemma are satisfied. Now, for an arbitrary $\varepsilon>0$ we fix n_0 such that $\sum_{n\ge n_0}\int_0^1\varphi(t,a_n\chi_{A_n}(t))\,dt\le\frac{\varepsilon}{3}$ and put $f_1=\sum_{i=1}^{n_0}a_i\chi_{A_i}$ on $[0,1[$. Let then $M=\max_{i\le n_0}\sup_{t\in[0,1]}\varphi(t,2a_i)$ and $\delta\le\frac{\varepsilon}{3M}$ (remark that we may suppose $1-\alpha>\delta$).

Let f_1^r denote the restriction of f_1 to $[0,1-\delta]$. Then by Luzin's theorem there exists a continuous function g_ε^r on $[0,1-\delta]$ such that

$$\mu\left\{t\in[0,1-\delta]:\varphi\left(t,|f_1^r(t)-g_\varepsilon^r(t)|\right)>0\right\}\le\frac{\varepsilon}{3M}\,.$$

Moreover since f_1 is bounded so is g_ε^r (with the same bound).

Let now g_ε be a linear extension of g_ε^r to $[0,1]$, more precisely g_ε is such that $g_\varepsilon = g_\varepsilon^r$ on $[0, 1-\delta]$, g_ε is linear between $1-\delta$ and 1 and satisfies $g_\varepsilon(1) = g_\varepsilon^r(0)$.

We then get

$$\begin{aligned}
&\int_0^1 \varphi\left(t, \frac{|f(t) - g_\varepsilon(t)|}{2}\right) dt \\
&\leq \int_0^1 \varphi\left(t, \frac{|f(t) - f_1(t)| + |f_1(t) - g_\varepsilon(t)|}{2}\right) dt \\
&\leq \frac{1}{2}\int_0^1 \varphi(t, |f(t) - f_1(t)|)\, dt + \frac{1}{2}\int_0^1 \varphi(t, |f_1(t) - g_\varepsilon(t)|)\, dt \\
&\leq \frac{1}{2}\int_0^1 \varphi\left(t, \sum_{n\geq n_0} a_n \chi_{A_n}(t)\right) dt \\
&\quad + \frac{1}{2}\int_0^{1-\delta} \varphi(t, |f_1^r(t) - g_\varepsilon^r(t)|)\, dt + \frac{1}{2}\int_{1-\delta}^1 \varphi(t, |f_1(t) - g_\varepsilon(t)|)\, dt \\
&\leq \frac{1}{2}\sum_{n\geq n_0}\int_0^1 \varphi(t, a_n\chi_{A_n}(t))\, dt + \frac{1}{2}M\frac{\varepsilon}{3M} + \frac{1}{2}M\frac{\varepsilon}{3M} \\
&\leq \frac{\varepsilon}{2}\,.
\end{aligned}$$

Finally, the continuous function $g_\varepsilon : [0,1] \to \mathbb{R}$ satisfies

$$g_\varepsilon(0) = g_\varepsilon(1) \quad\text{and}\quad \int_0^1 \varphi\left(t, \frac{|f(t) - g_\varepsilon(t)|}{2}\right) dt \leq \frac{\varepsilon}{2}\,.$$

Let now $\tilde{f}$ and $\tilde{g}_\varepsilon$ be the respective periodic extensions of f and g_ε to the whole $\mathbb{R}$ (with the period $\tau = 1$). Clearly $\tilde{g}_\varepsilon$ is *u.a.p.* and then it is also in $B^\varphi_{a.p.}(\mathbb{R})$.

Consequently, there exists $P_\varepsilon \in A$ for which $\rho_\varphi\left(\frac{\tilde{g}_\varepsilon - P_\varepsilon}{2}\right) \leq \frac{\varepsilon}{2}$.

On the other hand $\tilde{f}$ and $\tilde{g}$ being periodic with period $\tau = 1$, using the periodicity of φ (with $\tau = 1$), we get

$$\begin{aligned}
\rho_\varphi\left(\frac{\tilde{f} - \tilde{g}_\varepsilon}{2}\right) &= \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{-T}^{+T} \varphi\left(t, \frac{\left|\tilde{f}(t) - \tilde{g}_\varepsilon(t)\right|}{2}\right) dt \\
&= \int_0^1 \varphi\left(t, \frac{|f(t) - g_\varepsilon(t)|}{2}\right) dt \leq \frac{\varepsilon}{2}\,.
\end{aligned}$$

Finally,

$$\rho_\varphi\left(\frac{\tilde{f} - P_\varepsilon}{4}\right) \leq \frac{1}{2}\left[\rho_\varphi\left(\frac{\tilde{f} - \tilde{g}_\varepsilon}{2}\right) + \rho_\varphi\left(\frac{\tilde{g}_\varepsilon - P_\varepsilon}{2}\right)\right] \leq \varepsilon,$$

i.e., $\tilde{f} \in \tilde{B}^\varphi_{a.p.}$. □

4. Results

Lemma 9. *Let $\varphi(t,u)$ be strictly convex with respect to $u \geq 0$ and $f_n, g_n \in B^{\varphi}_{a.p.}(\mathbb{R})$ be sequences such that, for some $r > 0$, we have*

$$\rho_{\varphi}\left(f_n\right) \leq r, \rho_{\varphi}\left(g_n\right) \leq r \quad \text{and} \quad \lim_{n\to\infty} \rho_{\varphi}\left(\frac{f_n + g_n}{2}\right) = r.$$

Then $(f_n - g_n) \xrightarrow{\bar{\mu}} 0$.

Proof: Suppose that $\lim_{n\to\infty}(f_n - g_n) \neq 0$ in the $\bar{\mu}$-convergence sense. Then there exist $\varepsilon > 0$, $\sigma > 0$ and $n_k \nearrow \infty$ such that if $E_k = \{t \in \mathbb{R} : |f_{n_k}(t) - g_{n_k}(t)| \geq \sigma\}$ we have $\bar{\mu}(E_k) > \varepsilon$.

Take a number $k_\varepsilon > 1$ such that (see Lemma 1) there holds

$$\bar{\mu}(E) \geq \frac{\varepsilon}{4} \Rightarrow \rho_{\varphi}\left(\chi_E\right) > \frac{r}{k_\varepsilon},$$

where $r > 0$ is the constant from the lemma.

Then putting

$$\begin{aligned} A_k &= \left\{t \in \mathbb{R} : |f_{n_k}(t)| > k_\varepsilon\right\}, \\ B_k &= \left\{t \in \mathbb{R} : |g_{n_k}(t)| > k_\varepsilon\right\} \end{aligned}$$

we obtain

$$\begin{aligned} r &\geq \rho_{\varphi}\left(f_{n_k}\right) \\ &= \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{-T}^{+T} \varphi\left(t, |f_{n_k}(t)|\right) dt \\ &\geq \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{A_k\cap[-T,T]} \varphi\left(t, k_\varepsilon\right) dt \\ &\geq k_\varepsilon \overline{\lim_{T\to+\infty}} \frac{1}{2T} \int_{A_k\cap[-T,T]} \varphi(t,1)\, dt = k_\varepsilon \rho_{\varphi}\left(\chi_{A_k}\right). \end{aligned}$$

It follows that $\rho_{\varphi}(\chi_{A_k}) \leq \frac{r}{k_\varepsilon}$ and then $\bar{\mu}(A_k) \leq \frac{\varepsilon}{4}$.

In the same way we show that $\bar{\mu}(B_k) \leq \frac{\varepsilon}{4}$.

Now, define the set

$$Q = \{(u,v) \in \mathbb{R}^2 / \, |u| \leq k_\varepsilon, |v| \leq k_\varepsilon, |u - v| \geq \sigma\},$$

and consider the function

$$F(t,u,v) = \frac{2\varphi\left(t, \frac{u+v}{2}\right)}{\varphi\left(t,u\right) + \varphi\left(t,v\right)}.$$

Since φ is strictly convex we have $F(t,u,v) < 1$, for all $(t,u,v) \in \mathbb{R} \times Q$. Then using the continuity of φ on $\mathbb{R} \times Q$ (where Q is a compact set of $\mathbb{R}^2$) and its periodicity with respect to t, it follows that

$$\sup_{\mathbb{R}\times Q} F(t,u,v) = 1-\delta \quad \text{for some} \quad \delta > 0.$$

More precisely, for $(t,u,v) \in \mathbb{R}\times Q$ we have

$$\varphi\left(t, \frac{u+v}{2}\right) \leq (1-\delta)\, \frac{\varphi(t,u)+\varphi(t,v)}{2}\,.$$

Let now $t \in E_k\backslash(A_k \cup B_k)$. Then $f_{n_k}(t), g_{n_k}(t) \in Q$ and consequently

$$\varphi\left(t, \frac{|f_{n_k}(t)+g_{n_k}(t)|}{2}\right) \leq (1-\delta)\, \frac{\varphi\left(t,|f_{n_k}(t)|\right)+\varphi\left(t,|g_{n_k}(t)|\right)}{2}.$$

Hence

$$\begin{aligned}
& r-\rho_\varphi\left(\frac{f_{n_k}+g_{n_k}}{2}\right)\\
& \geq \frac{\rho_\varphi\left(f_{n_k}\right)+\rho_\varphi\left(g_{n_k}\right)}{2} - \rho_\varphi\left(\frac{f_{n_k}+g_{n_k}}{2}\right)\\
& \geq \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]}\\
& \quad \left[\frac{\varphi\left(t,|f_{n_k}(t)|\right)+\varphi\left(t,|g_{n_k}(t)|\right)}{2} - \varphi\left(t,\frac{|f_{n_k}(t)+g_{n_k}(t)|}{2}\right)\right] dt\\
& \geq \frac{\delta}{2}\overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]} \left[\varphi\left(t,|f_{n_k}(t)|\right)+\varphi\left(t,|g_{n_k}(t)|\right)\right] dt\\
& \geq \delta\ \overline{\lim_{T\to+\infty}}\ \frac{1}{2T}\int_{[E_k\backslash(A_k\cup B_k)]\cap[-T,+T]} \varphi\left(t,\frac{|f_{n_k}(t)-g_{n_k}(t)|}{2}\right) dt\\
& \geq \delta\varphi\left(\frac{\sigma}{2}\right)\left(\varepsilon-\frac{\varepsilon}{4}-\frac{\varepsilon}{4}\right) = \delta\frac{\varepsilon}{2}\varphi\left(\frac{\sigma}{2}\right).
\end{aligned}$$

Finally,

$$r-\rho_\varphi\left(\frac{f_n+g_n}{2}\right) \geq \delta\frac{\varepsilon}{2}\phi\left(\frac{\sigma}{2}\right) > 0,$$

a contradiction with the hypothesis $\rho_\varphi\left(\frac{f_n+g_n}{2}\right) \to r$. $\square$

Theorem 1. *$\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ is strictly convex if and only if φ is strictly convex and φ satisfies the Δ_2-condition.*

Proof: Sufficiency. Suppose that φ is strictly convex and satisfies the Δ_2-condition but $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ is not strictly convex. Then for some f and $g \in \tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ we will have $\|f\|_\varphi = \|g\|_\varphi = 1$ and $\|f-g\|_\varphi > 0$ but $\left\|\frac{f+g}{2}\right\|_\varphi = 1$. From Corollary 1 we will have also $\rho_\varphi(f) = \rho_\varphi(g) = \rho_\varphi\left(\frac{f+g}{2}\right) = 1$. Then from Lemma 9 it follows that for each $\alpha > 0$, $\bar{\mu}\{t \in \mathbb{R} : |f-g| > \alpha\} = 0$. Finally, using Lemma 7 we get $\rho_\varphi(f-g) = 0$. Contradiction.

Necessity. Let $L^\varphi = L^\varphi([0,1]) = \{f \in M(\mathbb{R}) : \int_0^1 \varphi(t, \lambda|f(t)|)\,dt < +\infty$ for some $\lambda > 0\}$ be the usual Musielak-Orlicz space and $\|.\|_{L^\varphi}$ its associated Luxemburg norm.

We consider the injection map

$$i : L^\varphi \hookrightarrow \tilde{B}^{\varphi}_{a.p.}(\mathbb{R}), \quad i(f) = \widetilde{f},$$

where $\widetilde{f}$ is the periodic extension (with period $\tau = 1$) of f to $\mathbb{R}$. We show first that $i(L^\varphi) \subset \tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$.

Let $f \in L^\varphi([0,1])$. Then there exists $\lambda > 0$ such that $\varphi(t, \lambda|f(t)|) \in L^1([0,1])$. From usual arguments of Lebesgue theory we have $\lim_{N\to+\infty} \mu(V_N) = 0$, where

$$V_N = \{t \in [0,1] : \varphi(t, \lambda|f(t)|) \geq N\}.$$

Let $E_N = \{t \in [0,1] : |f(t)| \geq N\}$. Then for $t \in E_N$ we have

$$\varphi(t, \lambda|f(t)|) \geq \varphi(t, \lambda N) \geq \lambda N \varphi(t,1) \geq \lambda N \phi(1),$$

where $\phi(1) = \inf_{t\in[0,1]} \varphi(t,1)$, $\phi(1) > 0$ (we may suppose $\phi(1) = 1$). It follows that $E_N \subset V_{\lambda N}$ and then we get $\lim_{N\to+\infty} \mu(E_N) = 0$.

Consider the following functions for $N \in \mathbb{N}$,

$$f_N(t) = \begin{cases} f(t) & \text{if } f(t) \leq N \\ N & \text{if } f(t) \geq N. \end{cases}$$

It is clear that the sequence $\{f_N\}$ is increasing and $f_N \leq f$. Moreover, since $\lim_{N\to+\infty} \mu(E_N) = 0$ we have $\lim_{N\to+\infty} \int_{E_N} \varphi(t, \lambda|f(t)|)\,dt = 0$.

Then for a given $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that

$$\int_0^1 \varphi(t, \lambda|f(t) - f_{N_\varepsilon}(t)|)\,dt \leq \int_{E_{N\varepsilon}} \varphi(t, \lambda|f(t)|)\,dt \leq \varepsilon.$$

Now for f_{N_ε} being bounded there exists a sequence of simple functions $(S_{N_\varepsilon})_n$ uniformly convergent to f_{N_ε}. In particular, there exists a simple function S_{N_ε} such that $\sup_{t\in[0,1]}|\lambda\,(f_{N_\varepsilon}(t)-S_{N_\varepsilon}(t))|\le\varepsilon$ and then

$$\begin{aligned}&\int_0^1\varphi\left(t,\frac{\lambda}{2}\,|f(t)-S_{N_\varepsilon}(t)|\right)dt\\&\le\frac{1}{2}\int_0^1\varphi\left(t,\lambda\,|f(t)-f_{N_\varepsilon}(t)|\right)dt+\frac{1}{2}\int_0^1\varphi\left(t,\lambda\,|f_{N_\varepsilon}(t)-S_{N_\varepsilon}(t)|\right)dt\le\varepsilon.\end{aligned}$$

We denote by $\widetilde{f}$, $\widetilde{f}_{N_\varepsilon}$ and $\widetilde{S}_{N_\varepsilon}$ the respective periodic extensions (with period $\tau=1$) of the functions f, f_{N_ε} and S_{N_ε}. We have from the periodicity properties of φ, $\widetilde{f}$, $\widetilde{f}_{N_\varepsilon}$ and $\widetilde{S}_{N_\varepsilon}$:

$$\begin{aligned}\rho_\varphi\left(\frac{\lambda}{2}\left(\widetilde{f}-\widetilde{S}_{N_\varepsilon}\right)\right)&=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{+T}\varphi\left(t,\frac{\lambda}{2}\left|\widetilde{f}\,(t)-\widetilde{S}_{N_\varepsilon}\,(t)\right|\right)dt\\&=\int_0^1\varphi\left(t,\frac{\lambda}{2}\,|f(t)-S_{N_\varepsilon}\,(t)|\right)dt\le\varepsilon.\end{aligned}$$

Moreover, from Lemma 8 we have $\widetilde{S}_{N_\varepsilon}\in\widetilde{B}^{\varphi}_{a.p.}(\mathbb{R})$. Then there exists $P_\varepsilon\in A$ for which $\rho_\varphi\left(\frac{1}{4}(\widetilde{S}_{N_\varepsilon}-P_\varepsilon)\right)\le\varepsilon$ (see the proof of Lemma 8).

Finally, putting $\alpha=\min\left(\lambda,\frac{1}{4}\right)$ we get

$$\rho_\varphi\left(\frac{\alpha}{2}\left(\widetilde{f}-P_\varepsilon\right)\right)\le\frac{1}{2}\left\{\rho_\varphi\left(\frac{\lambda}{2}\left(\widetilde{f}-\widetilde{S}_{N_\varepsilon}\right)\right)+\rho_\varphi\left(\frac{1}{4}\left(\widetilde{S}_{N_\varepsilon}-P_\varepsilon\right)\right)\right\}\le\varepsilon.$$

This means that $\widetilde{f}\in\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$.

Now, since $i:L^\varphi([0,1])\hookrightarrow\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ is an isometry, the strict convexity of $\tilde{B}^{\varphi}_{a.p.}(\mathbb{R})$ implies the strict convexity of $L^\varphi([0,1])$.

Consequently $\varphi(t,u)$, $t\in[0,1]$, $u\ge 0$ is strictly convex and satisfies the Δ_2-condition for Musielak-Orlicz spaces (see [6], [7]) i.e., there exist $k\ge 1$ and $h\ge 0$ with $\int_0^1 h(t)\,dt<\infty$ such that $\varphi(t,2u)\le k\varphi(t,u)+h(t)$ for all $u\ge 0$ and almost all $t\in[0,1]$. The periodically (with $\tau=1$) extended functions $\varphi(t,u)$, $t\in\mathbb{R}$, $u\ge 0$ and $\widetilde{h}(t)$, $t\in\mathbb{R}$ satisfy the conditions $\widetilde{h}\in B^1(\mathbb{R})$ and $\varphi(t,2u)\le k\varphi(t,u)+\widetilde{h}(t)$ for $u\ge 0$ and almost all $t\in\mathbb{R}$.

Now, putting $f(t)=\sup\{u\ge 0:\varphi(t,u)\le\widetilde{h}(t)\}$ it follows that f is measurable and $\varphi(t,f(t))=\widetilde{h}(t)$ for $t\in\mathbb{R}$. Finally, we get

$$\varphi(t,2u)\le k\varphi(t,u)+\widetilde{h}(t)\le(k+1)\varphi(t,u)$$

for $u\ge f(t)$ and almost all $t\in\mathbb{R}$, i.e., φ satisfies the Δ_2-condition for Besicovitch-Musielak-Orlicz spaces.

$\square$

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