Sign-changing solutions and multiplicity results for some quasi-linear elliptic Dirichlet problems

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Abstract. In this paper we show some results of multiplicity and existence of sign-changing solutions using a mountain pass theorem in ordered intervals, for a class of quasi-linear elliptic Dirichlet problems. As a by product we construct a special pseudo-gradient vector field and a negative pseudo-gradient flow for the nondifferentiable functional associated to our class of problems.

Keywords: sign-changing, mountain-pass theorem, ordered intervals

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N > 1)$ be an open and bounded domain with sufficiently smooth boundary $\partial\Omega$. We consider a quasi-linear elliptic boundary value problems of the form:

$$(P_\lambda) \qquad \begin{aligned} \operatorname{div}(A(x,u)\nabla u) &= f(\lambda, x, u), \quad && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., is measurable with respect to $x \in \Omega$ for all $(\lambda, s) \in \mathbb{R}^2$ and continuous in (λ, s) for almost every $x \in \Omega$), such that $f(\lambda, x, 0) = 0$ for almost every (in short a.e.) $x \in \Omega$, $\lambda \in \mathbb{R}$, and $f(\lambda, x, s)s > 0$ for $s \neq 0$. We suppose that f satisfies

- (f$_1$) for every bounded set $\Lambda \subset (0, \infty)$ and for $2 < r < 2^*$, $|f(\lambda, x, u)| \le C(1 + |u|^r)$ for all $\lambda \in \Lambda$, and a.e. $x \in \Omega$,
- (f$_2$) there exist $\theta > 2$, $M > 0$ such that $0 < \theta F(\lambda, x, u) \le u f(\lambda, x, u)$ for all $|u| \ge M$, $\lambda \in \Lambda$ and a.e. $x \in \Omega$, where $F(\lambda, x, u) = \int_0^u f(\lambda, x, s)\, ds$,

and that there exist constants $0 < \alpha < \beta$ such that

- (f$_3$) $\lim_{s \to 0} \frac{f(\lambda,x,s)}{s} > \beta\mu_1$ uniformly in $(\lambda, x) \in \mathbb{R}^+ \times \Omega$,
- (f$_4$) $\limsup_{s \to \infty} \frac{f(\lambda,x,s)}{s} < \alpha\mu_1$ uniformly in $(\lambda, x) \in \mathbb{R}^+ \times \Omega$,

where $\mu_1 > 0$ is the first eigenvalue of the Laplacian operator (with Dirichlet condition).

Let $A : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

(A$_1$) $|A(x,u)| \le \beta$, for every $u \in \mathbb{R}$ and a.e. $x \in \Omega$,

(A$_2$) $\forall\,(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, $A(x,s)\xi \cdot \xi \ge \alpha|\xi|^2$,

(A$_3$) there exists a continuous function $\omega : \mathbb{R}^+ \to \mathbb{R}$ such that $\omega(0) = 0$, $\int_0^\infty \frac{ds}{\omega(s)} = +\infty$ and $|A(x,s) - A(x,t)| \le \omega(|s-t|)$, for s and $t \in \mathbb{R}$,

(A$_4$) the function $u \to A(x,u)$ has continuous and bounded derivative for a.e. $x \in \Omega$, and there exists $u_0 > 0$ such that $A(x,u)$ is nondecreasing in $u \in [0,u_0]$.

We can consider the matrix $A(x,u) = (a_{ij}(x,u))$, $i,j = 1,2,\dots,N$, with Carathéodory coefficients $a_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $a_{ij} = a_{ji}$ and $s \mapsto a_{ij}(x,s)$ is C^1 for a.e. $x \in \Omega$, $a_{ij}(x,u)$ and $\frac{\partial a_{ij}}{\partial s}(x,s) \in L^\infty(\Omega \times \mathbb{R}, \mathbb{R})$.

The problem (P_λ) has been extensively studied in semilinear case, including the case $A = 1$, see [1]–[6], by means of bifurcation, variational methods, subsolution and supersolution method according to the behavior of the function f (see Ambrosetti at al. [4]–[6] for related topics). In this case, the existence of multiple and sign-changing solutions has been considered by many authors (cf. Li Shujie and Wang [15], Dancer and Du Yihon [12], Li Shujie and Zang Zhitao [17], Alama and Del Pino [1], and references therein). However, it seems that very few results have been reported on the quasi-linear case (see for instance [7] and [8]), and at least to the best of our knowledge, sign-changing solutions have not been considered yet.

Our purpose is to contribute to some nontrivial and sign-changing solutions for the problem (P_λ), when f satisfies (f$_1$)–(f$_4$). The main difficulty in this problem lies in deriving a min-max critical value for the Euler functional associated to (P_λ), since this functional is continuous, but may not be Lipschitz continuous and therefore nondifferentiable. In order to overcome this difficulty, we use some technical tools used by Struwe [17] for nondifferentiable functionals in Banach spaces. We also use a mountain-pass theorem in ordered intervals, in the spirit of Li Shujie and Wang [15], and differential equations theory in Banach spaces to define a critical point value and to show the existence of sign-changing solutions. Our results generalize or improve many results obtained for C^1 functionals and for nondifferentiable functionals in [5], [15] and [7], as shown in Theorems 5, 7 and Remark 6.

2. Main result

Let us consider the functional $J_\lambda : W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega A(x,u)|\nabla u|^2\,dx - \int_\Omega F(\lambda,x,u)\,dx.$$

The functional J_λ has been considered in [7] by Arcoya and Boccardo, and in [9] by Artola and Boccardo, where it has been shown that J_λ has a directional derivative

$J'_\lambda(u)(v)$ at each $u \in W_0^{1,2}(\Omega)$ along any direction $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, with

$$J'_\lambda(u)(v) = \int_\Omega A(x,u)\nabla u \nabla v\, dx + \int_\Omega \frac{1}{2} A'_u(x,u)|\nabla u|^2 v\, dx - \int_\Omega f(\lambda,x,u)v\, dx,$$

where $A'_u(x,u) = \frac{\partial}{\partial u}A(x,u)$. Clearly for fixed $u \in W_0^{1,2}(\Omega)$, the function $J'_\lambda(u)(v)$ is linear in $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, and for every fixed direction $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $J'_\lambda(u)(v)$ is continuous in $u \in W_0^{1,2}(\Omega)$. Hence, a critical point of $J_\lambda(u)$ is a function $u \in W_0^{1,2}(\Omega)$ such that $J'_\lambda(u)(v) = 0$ for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Therefore, for every $\lambda \in \Lambda$, a nontrivial critical point of J_λ is a nontrivial solution of the boundary problem

$$(P'_\lambda) \qquad -\operatorname{div}(A(x,u)\nabla u) + \frac{1}{2}A'_u(x,u)|\nabla u|^2 = \frac{\partial}{\partial u}F(\lambda,x,u) = f(\lambda,x,u).$$

For $u \in W_0^{1,2}(\Omega)$, and in this case for a solution of (P'_λ), we consider $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega A(x,u)\nabla u \nabla v\, dx + \frac{1}{2}\int_\Omega A'_u(x,u)|\nabla u|^2 v\, dx = \int_\Omega f(\lambda,x,u)v\, dx,$$

for all $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Let us consider the space $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ endowed with the norm $\|\cdot\|_\infty$, and the set

$$M = \Big\{u \in Y \setminus \{0\} : \int_\Omega A'_u(x,u)|\nabla u|^2 v\, dx = 0, \forall\, v \in Y\Big\}.$$

On M, $J'_\lambda(u)(v)$ has the form

$$J'_\lambda(u)(v) = \int_\Omega A(x,u)\nabla u \nabla v\, dx - \int_\Omega f(\lambda,x,u)v\, dx,$$

for all $u \in Y$. From (A_1)–(A_2) and (f_1), a solution of $J'_\lambda(u)(v) = 0$ is a weak solution of the boundary value problem (see also [9])

$$\begin{aligned} \operatorname{div}(A(x,u)\nabla u) &= f(\lambda,x,u), \quad && x \in \Omega,\\ u &= 0, && x \in \partial\Omega. \end{aligned}$$

Indeed, for all $h \in W^{-1,2}(\Omega)$ (the dual space of $W_0^{1,2}(\Omega)$), a weak solution of

$$\begin{aligned} \operatorname{div}(A(x,u)\nabla u) &= h(x), \quad && x \in \Omega,\\ u &= 0, && x \in \partial\Omega, \end{aligned}$$

is a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega A(x,u)\nabla u \nabla v \, dx = \int_\Omega h(x) v \, dx, \quad \forall\, v \in W_0^{1,2}(\Omega).$$

Moreover, if (A_3) is satisfied, then this solution is unique (see [9]). Therefore, the divergent operator

$$Q(u) = -\operatorname{div}(A(x,u)\nabla u), \quad u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega),$$

is invertible with continuous inverse.

Let us denote $K(u) = Q^{-1}(f(\lambda, x, v))$. If u is a solution of (P_λ) then u satisfies

$$u = K(u). \tag{1}$$

Thus, the solutions of (1) are zeros of $J'_\lambda(u)(v) = 0$, and therefore, critical points of J_λ on M.

Remark 1. (a) If $u \in W_0^{1,2}(\Omega)$, then by (f_1), $f(\lambda, x, u) \in L^r(\Omega)$ and by regularity theorems, $Q^{-1}(f(\lambda, x, u)) \in W_0^{1,2}(\Omega)$. Thus, $K : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega)$.

(b) Under (A_1)–(A_3) and (f_1), it can be shown (see [8, Lemma 3.1]) that if $u \in W_0^{1,2}(\Omega)$ is a solution of $\int_\Omega A(x,u)|\nabla u|^2 \, dx - \int_\Omega f(\lambda, x, u) u \, dx = 0$, then $u \in L^\infty(\Omega)$. Therefore $K : Y \to Y$ is well defined and $I - K : Y \to Y$ is well defined as well.

Lemma 2. *Assume that A satisfies (A_1)–(A_3). Let $v \in Y$ be a nonnegative function. Then u is a positive solution of $Q(u) = v$ if and only if*

$$u - Q^{-1}(v) = 0. \tag{2}$$

Proof: Note that $u - Q^{-1}(u) = 0$ if and only if

$$\int_\Omega A(x,u)\nabla u \nabla \varphi \, dx = \int_\Omega u\varphi \, dx, \quad \forall\, \varphi \in Y.$$

We claim that $u \geq 0$. In fact, since $v \geq 0$, using (A_2) and taking $\varphi \equiv u^-$ as a test function, we obtain

$$\alpha \|u^-\|_x^2 \leq \int_\Omega A(x,u)\nabla u^- \nabla u^- \, dx = \int_\Omega u u^- \, dx \leq 0.$$

This implies that $u^- \equiv 0$. $\square$

Remark 3. (a) K is continuous and compact, by (f_1).

(b) Let $\overset{\circ}{P}$ the interior of the positive cone P. If in addition $f(\lambda,\cdot,\cdot)$ is $C^1(\Omega\times\mathbb{R})$ and $A(x,\cdot)$ is C^1, then $u\in\overset{\circ}{P}$ (see [9, Remarks 2]). We observe that the regularity condition for f can be relaxed by taking $f(\lambda,x,s)+ms$ increasing in $[0,s_0]$ for some $m>0$ and for every $s_0>0$, and we shall use this relaxation throughout all the paper if necessary, so we will not require explicitly this regularity condition. Using the strong maximum principle, we can prove that K is strongly order preserving for some $\lambda\in\Lambda$.

Remark 4. (A_3) implies that, for a fixed direction $\lambda\in Y$, the function $J'_\lambda(u)(v)$ is continuous in $u\in W_0^{1,2}(\Omega)$.

Our main result is the following:

Theorem 5. *Suppose that* (A_1)–(A_4) *and* (f_1)–(f_5) *hold. Then there exists* $\lambda_0>0$ *such that, for every* $\lambda\in(0,\lambda_0]$ *and* $\lambda\notin\sigma(Q)$*, the problem* (P_λ) *has at least six nontrivial solutions. More precisely:*

(i) (P_λ) *has at least two positive solutions* u_1^+ *and* u_2^+ *with* $u_1^+>u_2^+$, $J_\lambda(u_2^+)<0$ *and* u_2^+ *is a local minimizer of* $J_\lambda(u)$;

(ii) (P_λ) *has at least two negative solutions* u_3^- *and* u_4^- *with* $u_3^-<u_4^-$, $J_\lambda(u_4^-)<0$ *and* u_4^- *is a local minimizer of* $J_\lambda(u)$;

(iii) (P_λ) *has at least two sign-changing solutions* u_5 *and* u_6 *with* $J_\lambda(u_5)<0$, u_5 *is a mountain-pass point of* J_λ, *and* u_6 *is outside of* $[u_4^-,u_2^+]$.

Remark 6. We do not require f to be C^1 as in many applications, however, we assume that f is sublinear and we obtain the same results as in [15], where the C^1 condition is required and

$$f(\lambda,x,u)=\lambda|u|^{q-1}u+g(u)$$

with $g(s)=o(|s|)$ at 0 and $g'(s)>-a$ for some $a>0$. Our Theorem 5 also generalizes a result of [5], where the following strong assumptions were made:

(G1) $G\in C^2(\mathbb{R},\mathbb{R})$, $sG'(s)\ge\alpha G(s)\ge 0$ for all $s\in\mathbb{R}$ with $2<\alpha<2^*$, where $2^*=2N/(N-2)$, if $N>2$;

(G2) $s^2G''(s)\ge\alpha G'(s)s$, for all $s\in\mathbb{R}$;

(G3) $s^2G''(s)\le C_1|s|^\alpha$, for all $s\in\mathbb{R}$ (for some $C_1>0$),

where $G(s)=\int_0^s g(t)\,dt$. In comparison with [5] and [15], we get a much stronger and general result with more information than in [5], using weaker assumptions.

We observe that our result is also valid for $A\equiv 1$, in which case $\alpha=\beta=1$. Our assumptions (f_3), (f_4) are the same as in [12], where the author obtains at least two positive solutions.

In [7], the authors consider the same assumptions as us for the function A, with $g(s)$ convex and obtain at least two solutions, using a variant of the usual mountain-pass theorem; their results, however, do not give any information on sign-changing solutions.

To deal with the superlinear case, we consider the following hypothesis.

(f_2') There exists $2 < \theta < 2^*$ such that $0 < \theta F(\lambda, x) \le uf(\lambda, x, u)$ for all $u \in \mathbb{R}$, $\lambda \in \Lambda$ and for almost every $x \in \Omega$.

(f_4') There exist a number $k > 0$ and $\lambda_0 > 0$ such that for all $0 < \lambda < \lambda_0$, and for a.e. $x \in \Omega$, $|f(\lambda, x, u) < k|$ for $u \in [-c, c]$, where $c = \max_\Omega e(x)$ and $e(x)$ satisfies the boundary value problem $-\triangle e(x) = k$ in Ω and $e(x) = 0$ on $\partial\Omega$.

Under the above assumptions we have the following result.

Theorem 7. *The results of Theorem* 5 *still hold under the hypotheses* (A_1)–(A_4), (f_1), (f_2') *and* (f_4').

3. Abstract results

Let us start by presenting some abstract results. The abstract framework is derived from the following hypothesis.

Let $(X, \|\cdot\|_X)$ be a Hilbert space and $Y \subset X$ a normed subspace of X endowed with the norm $\|\cdot\|_Y$, and densely embedded in X.

Let $P_X \subset X$ be a convex cone and $P = Y \cap P_X$. Assume that P has a nonempty interior $\overset{\circ}{P}$, and that any ordered interval in X is finitely bounded. Let $\Phi : X \to \mathbb{R}$ be a functional in X, which is continuous in $(Y, \|\cdot\| + \|\cdot\|_Y)$ and satisfies the following assumptions.

(Φ_1) The functional Φ has directional derivative $\Phi'(u)(v)$ at each $u \in X$, through any bounded direction $v \in Y$. For fixed $u \in X$, the function $\Phi'(u)(v)$ is linear in v, and for fixed $v \in Y$, the function $\Phi'(u)(v)$ is continuous in $u \in X$.

(Φ_2) The functional Φ is bounded from below on any ordered interval in X.

(Φ_3) For any fixed direction $v \in Y$, the function $\Phi'(u)(v)$ is of the form $u - K_X(u)$ for each $u \in K$, where $K_X : X \to X$ is compact, $K_X(Y) \subset Y$ and $K = K_{X|Y} : Y \to Y$ is continuous and strongly order preserving, i.e. $u > v \Leftrightarrow K(v) \gg K(v)$ for all $u, v \in Y$, where $u \gg v \Leftrightarrow u - v \in \overset{\circ}{P}$.

(Φ_4) The functional Φ satisfies the Palais-Smale ((PS) for short) condition in X, the deformation property in Y and has only finitely many isolated critical points.

We shall consider the following compactness conditions.

(H) There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in Y such that for some sequences $(K_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $(\varepsilon_n) \to 0$, the following are satisfied

$$(3) \qquad (\Phi(u_n))_{n\in\mathbb{N}} \ \text{ is bounded,}$$
$$(4) \qquad \|u_n\|_Y \leq 2K_n, \quad \forall\, n \in \mathbb{N}$$

and

$$(5) \qquad |\Phi'(u_n)(v)| \leq \varepsilon\bigg(\frac{\|v\|_Y}{K_n} + \|v\|_X\bigg), \quad \text{for all} \ \ v \in Y.$$

(C) Any sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{dom}(\Phi) \subset Y$ satisfying for some $(K_n) \subset \mathbb{R}^+$ and $(\varepsilon_n) \to 0$ the conditions (3), (4) and (5), possesses a convergent subsequence in X.

We state some abstract results, which will be used to solve the problem (P_λ).

Theorem 8. *Assume that Φ satisfies (Φ_1), (Φ_2), (Φ_3), (Φ_4) and $\underline{u} < \overline{u}$ is a pair of subsolution and supersolution for $\Phi'(u)(v) = 0$, for any bounded direction $v \in Y$. Then, $[\underline{u}, \overline{u}]$ is positively invariant under the negative gradient flow of Φ, and $u - \Phi'(u)(v)$ belongs to the interior of $[\underline{u}, \overline{u}]$ through any fixed and bounded direction $v \in Y$. Moreover, if $\underline{u} < \overline{u}$ is a pair of strict subsolution and supersolution, then* $\deg(\mathrm{id} - K, [\underline{u}, \overline{u}], 0) = 1$.

Corollary 9. *If $\underline{u} < \overline{u}$ is a pair of strict subsolution and supersolution for $\Phi'(\cdot)(v) = 0$ in X, then $K_\Phi \cap \partial[\underline{u}, \overline{u}] = \emptyset$, where $K_\Phi = \{u \in X : \Phi'(u)(v) = 0\}$ is the critical point set of Φ.*

Now, we give a suitable version of the mountain-pass theorem in ordered interval in this framework.

Theorem 10. *Assume that Φ satisfies (Φ_1), (Φ_2), (Φ_3) and (Φ_4). Suppose that there exist four points in Y : $v_1 < v_2$, $\omega_1 < \omega_2$, $v_1 < \omega_2$, satisfying $[v_1, v_1] \cap [\omega_1, \omega_2] = \emptyset$ with $v_1 < Kv_1$, $v_2 > Kv_2$, $\omega_1 < K\omega_1$ and $\omega_2 > K\omega_2$. Then, Φ has a mountain-pass point $u_0 \in [v_1, \omega_2] \setminus ([v_1, \omega_1])$.*

Proofs of abstract results

Proof of Theorem 8: Since $[\underline{u}, \overline{u}]$ is finitely bounded and K compact, $\deg(\mathrm{id} - K, [\underline{u}, \overline{u}], 0)$ is well defined. Consider the negative gradient flow of Φ in X satisfying

$$\begin{cases} \frac{\partial}{\partial t}\eta(t,u) = -\Phi'(\eta(t,u)(v)), \\ \eta(0,u) = u, \end{cases}$$

for any fixed direction $v \in Y$ and for all $u \in Y$. Then, from (Φ_4), $u(t,u) \in Y$. It suffices to show that for $y \in P - \{0\}$, and for any bounded direction $v \in Y$,

$\eta(t, \underline{u}+y) \in \underline{u}+\overset{\circ}{P}$ and $\eta(t, \overline{u}-y) \in \overline{u}-\overset{\circ}{P}$ for all $t > 0$. By (Φ_1), we know that for any given direction in Y, $\Phi(u+\cdot)$ is differentiable in Y, for all $u \in \mathrm{dom}(\Phi) \subset X$.

Hence, for any given $y \in P - \{0\}$,

$$\underline{u} + y - \Phi'_Y(\underline{u}+y)(v) = K(\underline{u}+y) \gg K(u) > u,$$

where $\Phi'_Y(\underline{u}+y) = \frac{\partial}{\partial y}\Phi(\underline{u}+y)$. Similar arguments imply that $\overline{u} - y - \Phi'_Y(\overline{u} - y)(v) = K(\overline{u}-y) \ll K(\overline{u}) < \overline{u}$. Therefore, $\eta(t, \underline{u}+y) \in \underline{u}+\overset{\circ}{P}$ and $\eta(t, \overline{u}-y) \in \overline{u}-\overset{\circ}{P}$ for any bounded direction v in Y. Hence, $[\underline{u}, \overline{u}]$ is positively invariant and for all $u \in [\underline{u}, \overline{u}]$, $u - \Phi'(u)(v)$ belongs to the interior of $[\underline{u}, \overline{u}]$ for any bounded direction v in Y. Now, using this invariance property, the fact that K is strongly order preserving and compact, $\overset{\circ}{P} \neq \emptyset$, and arguments of H. Amann [2], it is easy to see that K has a fixed point in $[\underline{u}, \overline{u}]$. If this fixed point is isolated, then its index is 1, and by the excision property of the Schauder degree, we are done. □

Proof of Theorem 10: Since Φ is bounded from below and satisfies the deformation property, it has at least one local minimizer in each ordered interval. Let v_0 be the minimizer of Φ in $[v1, v2]$, and ω_0 the minimizer of Φ in $[\omega_1, \omega_2]$. Let $\eta(t, u)$ denote the negative gradient flow of Φ, and let

$$\begin{aligned} \Gamma = \Big\{ & x(t) : x(t) \in C([0,1], [v_1, \omega_2]) \text{ is such that} \\ & x(t) \in [v_1, \omega_2] \setminus ([v_1, v_2] \cap [\omega_1, \omega_2]), \quad \text{if } \ t \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ & x(t) = \eta\left(\frac{1}{3} - t, x\left(\frac{1}{3}\right)\right), \quad \text{if } \ 0 \le t \le \frac{1}{3}, x\left(\frac{1}{3}\right) \in \partial[v_1, v_2], \\ & x(t) = \eta\left(t - \frac{2}{3}, x\left(\frac{2}{3}\right)\right), \quad \text{if } \ \frac{2}{3} \le t \le 1, x\left(\frac{2}{3}\right) \in \partial[\omega_1, \omega_2] \Big\}. \end{aligned} \tag{6}$$

Then $\Gamma \neq \emptyset$ is a complete metric space for the metric $\rho(x, y) = \max_{t \in [0,1]} \|x(t) - y(t)\|$. Let

$$c = \inf_{x \in \Gamma} \sup_{t \in [0,1]} \Phi(x(t)). \tag{7}$$

Then c is a critical value. In fact, let

$$K_c^v = \{u \in [v_1, \omega_2] : \Phi'(u)(v) = 0, \, \Phi(u) = c\}$$

for all $v \in Y$; it suffices to prove that $K_c^v \cap \{[v_1, \omega_2] \setminus ([v_1, v_2] \cup [\omega_1, \omega_2])\}$. Let

$$F(x(t)) = \max_{t \in [0,1]} \Phi(x(t)), \quad x(t) \in \Gamma.$$

Then, F is a continuous function, bounded from below on Γ. Define

$$F'(x)(v) = \limsup_{\theta \to 0} \frac{F(x+\theta h) - F(x)}{\theta},$$

and let $\mathbb{B}(x) = \{s \in [0,1] : \Phi(x(s)) = F(x(s)) \equiv \max_{t\in[0,1]} \Phi(x(t))\}$. From Ekeland's variational principle, we have for any given sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, that there exists a sequence $\{x_n\} \subset \Gamma$ such that

$$c \le F(x_n) \le c + \varepsilon_n \quad \text{and} \quad F(x) \ge F(x_n) - \varepsilon_n \rho(x, x_n),$$
$$\forall\, x \ne x_n,\ x \in \Gamma,\ n = 1, 2, 3 \ldots .$$

Hence, for any $h \in \Gamma$ and $\theta \in \,]0,1[\,$, we have

$$\frac{F(x_n+\theta h) - F(x_n)}{\theta} \ge -\varepsilon_n \max_{t\in[0,1]} \|h(t)\|.$$

Taking limit as θ decreases to zero, we get

$$F'(x_n)(h) = \limsup_{\theta\to 0} \frac{F(x_n+\theta h) - F(x_n)}{\theta} \ge -\varepsilon_n \max_{t\in[0,1]} \|h(t)\|.$$

Therefore, there exists $(s_n) \subset \mathbb{B}(x)$ such that $|\Phi'(x_n(s_n))(v)| \le \varepsilon_n \|v\|$ for all $v \in Y$, which implies that $\Phi'(x_n(s_n))(v) \to 0$ as $n \to \infty$. From the definition of Γ and Theorem 1, $x_n(s_n) \in [v_1, \omega_2] - ([v_1, v_2] \cup [\omega_1, \omega_2])$, and from (3), (4), (5) and the deformation property, $(x_n(s_n))$ has a subsequence converging in Y to some $u_0 \in [v_1, \omega_2] - ([v_1, v_2] \cap [v_1, \omega_2])$. Since Φ is continuous in X, we have $\Phi(u_0) = c$ and $\Phi'(u_0)(v) = 0$. Thus c is a critical value. Substituting v_0, ω_0 for v_1, ω_2, and using the proof of Theorem 8, it is easy to see that $v_0 \ll u_0 \ll \omega_0$. From the deformation property, Theorem 8, and Corollary 9 we have

$$\inf_{u\in\partial[v_1,v_2]} \{\Phi(u)\} > \Phi(v_0), \quad \inf_{u\in\partial[\omega_1,\omega_2]} \{\Phi(u)\} > \Phi(\omega_0).$$

Therefore, $c > \max\{\Phi(v_0), \Phi(\omega_0)\}$ and u_0 is an isolated critical point of Φ. Let V be a neighborhood of u_0 such that for any open neighborhood $W \subset V$ of u_0, $\mathring{\Phi}^c \cap W \ne \emptyset$ and is not path-connected. Let us define critical groups of u_0 by

$$C_n(\Phi, v_0) = H_n(\Phi^c \cap W,\ \Phi^c \cap W \setminus \{u_0\}), \quad n = 1, 2, \ldots,$$

where $\mathring{\Phi}^c \cap W \setminus \{0\} \ne \emptyset$, and is not path-connected either. Thus, from the definition of critical group

$$C_0(\Phi, u_0) = H_n(\Phi^c \cap W,\ \Phi^c \cap W \setminus \{u_0\}) = 0.$$

Using arguments of [11] and [14], we can prove that $C_1(\Phi, u_0) \ne 0$, and from the argument given in [10], we know that u_0 is a mountain-pass point. $\square$

4. Some lemmas and definitions

We first prove the existence of two pairs of subsolutions and supersolutions.

Lemma 11. *Assume* (A_1)–(A_4) *and* (f_1)–(f_4) *hold. Then, there exist two pairs of strict subsolutions and supersolutions of* (P_λ).

Proof: Let φ_1 be the eigenfunction associated to μ; then we may take $\|\varphi_1\|_\infty = 1$. For $r > 0$ sufficiently small, (f_3) implies that $f(\lambda, x, r\varphi_1) > (\beta\mu_1 - \varepsilon(r))r\varphi_1$, for some $\varepsilon(r) > 0$. Using (A_1), we obtain

$$-\operatorname{div}(A(x, r\varphi)\nabla(r\varphi_1)) < f(\lambda, x, r\varphi_1).$$

This means that $r\varphi_1$ is a strict subsolution of (P_λ) for $r > 0$ sufficiently small. Hence, by continuity, there exists $r_0 > 0$ such that this is true for $0 < r < 2r_0$. Now, suppose that u is a nonnegative and nontrivial solution of (P_λ). Then by the maximum principle, $u > 0$ in Ω and there exists $\varepsilon \in (0, 2r_0)$ small enough such that $u \geq \varepsilon\varphi_1$. Using a sweeping argument, we can prove that $u \geq r\varphi_1$, for all $r \in (0, 2r_0]$. In particular $u \geq 2r_0\varphi_1 > r_0\varphi_1$, and we can take $\underline{u} = r_0\varphi_1$.

By (f_4), there exist constants $C > 0$ and $\tilde{\alpha}$, with $0 < \tilde{\alpha} < \alpha$ such that

$$f(\lambda, x, u) < \tilde{\alpha}\mu_1 u + C. \tag{8}$$

Therefore, any nontrivial solution of (P_λ) satisfies

$$-\operatorname{div}(A(x, u)\nabla u) < \tilde{\alpha}\mu_1 u + C. \tag{9}$$

Using (A_2), we find that $Q(u) = -\operatorname{div}(A(x, u)\nabla u) - \tilde{\alpha}\mu_1 u$ has a bounded inverse $Q_\alpha^{-1}C$ in Y, which is strongly order preserving, thus from (9) it follows that every solution of (P_λ) satisfies $u < \phi = Q_\alpha^{-1}C$. Hence, by (8)

$$-\operatorname{div}(A(x, \phi)\triangle\phi) < f(\lambda, x, \phi) \ \text{ in } \ \Omega, \ \phi_{|\partial\Omega}.$$

This means that ϕ is a strict supersolution of (P_λ), and we can take $\overline{\omega} = \phi$ if $\phi > r_0\varphi_1$; but this can be easily achieved by enlarging C if necessary. Using similar arguments, we obtain another pair of strict subsolution and supersolution $\underline{v} < \overline{v} < 0$, where $\overline{v} = -r_0\varphi_1$ and $\underline{v} = -\phi$. □

Now, we prove that J_λ satisfies the compactness condition (C).

Lemma 12. *Assume that* (A_1)–(A_3), (f_1), (f_2) *hold. Then* J_λ *satisfies the compactness condition* (C).

Proof: Let (u_n) a sequence such that $(J_\lambda(u_n))$ is bounded and $J'_\lambda(u)(v) \to 0$ as $n \to \infty$, for all $v \in Y$. Then, there exist $(\varepsilon_n) \subset \mathbb{R}^+$, $\varepsilon_n \to 0$, $(K_n) \subset \mathbb{R}^+$, such that $|J'_\lambda(u)(v)| \leq \varepsilon_n \left[\left\|\frac{v}{K_n}\right\|_Y + \|v\|\right]$ for all $v \in Y$.

We now prove that (u_n) is bounded in Y. By (f_2), for any $2 < \theta$, we have

$$
\begin{aligned}
C_0 + \varepsilon_n \left[\|u_n\|_Y + \|u_n\|\right] &> \theta J_\lambda(u_n) - J'_\lambda(u_n)(u_n) \\
&= (\theta - 1) \int_\Omega A(x, u_n)|\nabla u_n|^2 \, dx \\
&\quad - \int_\Omega \{\theta F(\lambda, x, u_n) - f(\lambda, x, u_n)u_n\} \, dx \\
&\geq (\theta - 1)\alpha\|u_n\|^2 - o\big(\|u_n\|_Y^2\big) \\
&> (\theta - 1)r\|u_n\|^2 - o\big(\|u_n\|_Y^2\big),
\end{aligned}
$$

and for $r > 0$ sufficiently small, (u_n) is bounded in Y and, therefore, there exists a subsequence (u_{nk}) of (u_n) converging in $W_0^{1,2}(\Omega)$ to some $u \in Y$. Since the function $J'_\lambda(u)(v)$ is continuous in $u \in Y$ for any fixed $v \in Y$, we have $J'_\lambda(u_n)(v) \to J'_\lambda(u)(v)$.

Consider $(u_n) \subset M$ such that $J'_\lambda(u_n)(v) = 0$ for every $v \in Y$. Since M is closed, if $u \in Y$ is such that $u_{nk} \to u$, then $u \in M$ and $J'_\lambda(u)(v) = 0$ for any $u \in Y$. Hence, u is a critical point of J_λ. $\square$

We recall the following definitions in the regular case (see for instance [14]).

Definition 13. Let Φ be a C^1 functional on a Banach space X. Denote by $\mathrm{Reg}(\Phi) = \{u \in x : \Phi'(u) \neq 0\}$. A *pseudo-gradient vector field* for Φ on $\mathrm{Reg}(\Phi)$ is a locally Lipschitz continuous mapping $v : \mathrm{Reg}(\Phi) \to X$ such that

$$
\|v(u)\| \leq \|\Phi'(u)\| \quad \text{and} \quad \langle \Phi'(u), v(u) \rangle \geq \|\Phi'(u)\|^2. \tag{10}
$$

Let $K = \{u \in X : \Phi(u) = 0\}$ be the set of critical points of Φ. Consider the initial value problem

$$
\frac{du}{dt} = -v(u), \quad u(0) = u_0 \in X \setminus K.
$$

Since $v(u)$ is locally Lipschitz continuous in $X \setminus K$, the initial value problem has a unique solution $u : [0, t(u_0)[\to X \setminus K$ with $t(u_0)$ maximal.

Definition 14. Let $N \subset X$. We say that N is an *invariant set of descent flow* of Φ if the set $\{u(t_0, u_0, t), t \in [0, t(u)), \, u_0 \in N \setminus K\} \subset N$.

In this paper, the functional J_λ is not C^1, it has only directional derivatives at any direction $v \in Y$ and, for any fixed direction $v \in Y$, the function (directional derivative) $J'_\lambda(u)(v)$ is continuous in $u \in W_0^{1,2}(\Omega)$, and linear in $v \in Y$ for fixed $u \in W_0^{1,2}(\Omega)$. Using these properties, we shall construct a pseudo-gradient vector field for J_λ in M.

Definition 15. We define the set

$$\mathrm{Reg}(J_\lambda) = \left\{u \in W_0^{1,2}(\Omega) : J'_\lambda(u)(v) \neq 0, \quad \text{for any direction} \quad v \in Y\right\} \quad \text{and}$$
$$K_v = \left\{u \in W_0^{1,2}(\Omega) : J'_0(u)(v) = 0, \quad \text{for any direction} \quad v \in Y\right\}.$$

Lemma 16. *Assume that* (A$_1$)–(A$_4$) (f$_1$)–(f$_4$) *hold. Then, there exists a pseudo-gradient flow for* J_λ *such that* P, $-P$, *and* $[\underline{u}, \overline{u}]$ *are invariant sets of descent flow of* J_λ, *where* $\{\underline{u}, \overline{u}\}$ *is a pair of strict subsolution and supersolution of* J_λ *in* M.

Proof: We construct a pseudo-gradient vector field for J_λ in M, and show that the flow under that vector field satisfies required invariance property.

For all $u \in [\underline{u}, \overline{u}]$ we have

$$u - J'_\lambda(u)(v) = K(u) \gg K(u) > \underline{u}\,, \quad K(u) \ll K(\overline{u}) < \overline{u}. \tag{11}$$

We observe that through any fixed direction $v \in Y$, the directional derivative function $J'_\lambda(u)(v)$ is continuous and $u - J'_\lambda(u)(v) \in \mathrm{int}([\underline{u}, \overline{u}])$, by (11). For all $u_0 \in M \setminus K^c$, there exists $y_0 \in Y$ with $\|y_0\|_Y = 1$ (we can normalize y_0 if necessary), such that $\langle J'_\lambda(u_0), y_0\rangle > \frac{2}{3}\|J'_\lambda(u_0)\|^2_Y$. If $u_0 \in [\underline{u}, \overline{u}]$ then by (11), we may require $u_0 + y_0 \in \mathrm{int}([\underline{u}, \overline{u}])$. Let $v_0 = \frac{2}{3}(\|J'_\lambda(u_0)\|_Y, y_0)$; then $\|v_0\|_y < 2\|J'_\lambda(u)\|_Y$ and $\langle J'_\lambda(u_0), v_0\rangle > \|J'_\lambda(u_0)\|^2_Y$.

From the continuity of the directional derivative for a fixed direction in Y, there exists a neighborhood $\tilde{U}(u_0)$ of u_0 in $\overline{M}$ such that

$$\|v_0\|_y < 2\|J'_\lambda(u)\|_Y \quad \text{and} \quad \langle J'_\lambda(u_0), v_0\rangle > \|J'_\lambda(u_0)\|^2_Y. \tag{12}$$

For all $\tilde{U}(u_0)$, take

$$U(u_0) = \begin{cases} \tilde{U}(u_0) & \text{if} \;\; u_0 \in [\underline{u}, \overline{u}], \\ \tilde{U}(u_0) \cap (\overline{M} \setminus [\underline{u}, \overline{u}]) & \text{if} \;\; u_0 \in \overline{M} \setminus [\underline{u}, \overline{u}], \end{cases}$$

where $\overline{M} = M \setminus K^v_c$. Let $u \in P$, $u \neq 0$. Then $K(u) \gg 0$, since K is strongly order preserving. Hence, for any $u_1 \in P$, we can assume that $u_1 + y \in \overset{\circ}{P}$ for any $y \in Y$ such that $\|y\|_Y = 1$. If we take such u_1 in $P \setminus K^v_c$, and let $v_1 = \frac{3}{2}\|J'_\lambda(u_1)\|_Y y$, then $\|v_1\|_Y < 2\|J'_\lambda(u_1)\|_Y$, and $\langle J'_\lambda(u), v_1\rangle > \|J'_\lambda(u)\|^2_Y$.

From the continuity of the directional derivative for a fixed direction in Y, there exits a neighborhood $U(u_1)$ of u_0 in $\overline{M}$ such that

$$\|v_1\|_y < 2\|J'_\lambda(u_1)\|_Y \quad \text{and} \quad \langle J'_\lambda(u), v_1\rangle > \|J'_\lambda(u)\|^2_Y, \tag{13}$$

for all $u \in U(u_1)$. Using similar arguments, we can show the existence of a pseudo-gradient vector field v_2 for J_λ and, get an open covering $U(u_2)$ for any $u_2 \in P \setminus K^v_c$.

Since $Y \backslash K_c^v$ is paracompact, there exists a family $U = \{U(u) : u \in Y \backslash K_c^v\}$, which is an open covering of $P \backslash K_c^v$. Hence, U has a locally finite refinement $(U(u_1))_{i \in I}$, and each $u \in \overline{M}$ has a neighborhood $\beta(u)$ such that $\beta(u) \subset (U(u_1))_{i \in I}$, for a finite number of $i \in I$. Let us define

$$\rho_i(u) = \mathrm{dist}(u, \overline{M} \setminus U(u_i)), \qquad i \in I, \tag{14}$$

for any $u \in M$, and define

$$v(u) = \sum_{i \in I} \frac{\rho_i(u)}{\sum_{j \in I} \rho_j(u)} v_i. \tag{15}$$

Since $(U(u_1))_{i \in I}$ is locally finite, all sums in (15) are finite. Therefore, $v(u)$ is locally Lipschitz continuous. Since ρ_i vanishes outside $U(u_i)$, $v(u)$ is a convex combination of finite elements satisfying (10). Clearly, it is a pseudo-gradient vector field for J_λ. For the sets $[\underline{u}, \overline{u}]$, P and $-P$, we may define

$$v(u) = \sum_{i=0}^{2} \frac{\rho_i(u)}{\sum_{j=0}^{2} \rho_j(u)} v_i. \tag{16}$$

Clearly, $v(u)$ satisfies (10) and is locally Lipschitz continuous.

Let $N_\varepsilon^c = \{u \in M : |J_\lambda(u) - c| < \varepsilon;\ \|J'_\lambda(u)(z)\|_Y < \varepsilon$, for any fixed $z \in Y\}$, and consider cut-off functions φ and ψ such that $0 \le \varphi$, $\psi \le 1$, $\varphi(u) = 0$ on N_ε^c, $\varphi(u) = 1$ on $M - N_\varepsilon^c$, $|J_\lambda(\varphi(u)) - c| < \varepsilon$ $\varphi(s) = 0$ for $|s - c| \ge 2\varepsilon_0$ and $\varphi(s) = 1$ for $|s - c| < \varepsilon_0$, for a given ε_0 such that $0 < \varepsilon_0 < \varepsilon$. For every $u \in \mathrm{Reg}(J_\lambda)$, i.e. u such that $J'_\lambda(u)(z) \neq 0$ for any fixed direction $z \in Y$, we define

$$v_*(u) = \begin{cases} -\psi(J_\lambda(u))\varphi(u) \frac{v(u)}{\|v(u)\|_Y} & \text{if } u \in M \cap \mathrm{Reg}(J_\lambda), \\ 0, & \text{otherwise.} \end{cases}$$

Then $J_\lambda : \mathrm{Reg}(J_\lambda) \to Y$ is a pseudo-gradient vector field and an odd continuous function of u. Consider the initial value problem

$$\frac{\partial \sigma(t,u)}{\partial t} = v_*(\sigma(t,u)),\ \sigma(0,u) = u, \tag{17}$$

for $u \in M$ and $t \ge 0$. Then, $\sigma(t,u)$ is a global, unique and maximal solution of (17) for $0 \le t < t^*(u)$, where the interval $]-t^*(u), t^*(u)[$ is maximal. Moreover, the solution $\sigma(t,u)$ is continuous on $\mathbb{R} \times M$. Let $\eta \in C([0,1] \times M, M)$ be defined by

$$\eta(t,u) = \sigma(2\varepsilon t, u). \tag{18}$$

Then by (17), $\eta(t,u) = u + \int_0^{2\varepsilon t} v_*(\sigma(\tau,u))\,d\tau$. For $u_0 \in P \setminus K_c^v$ and for any fixed $y_0 \in \beta(u_0) \cap P$, we have

$$\begin{aligned}\eta(t,y_0) &= y_0 + \int_0^{2\varepsilon t} v_*(\sigma(\tau,y_0))\,d\tau \\ &= y_0 + \int_0^{2\varepsilon t} v_*(y_0)\,d\tau \\ &= y_0 + 2\varepsilon t v_*(y_0) \in \overset{\circ}{P}\end{aligned} \tag{19}$$

for $t \in [0,1]$ by (9). Therefore, P is an invariant set.

Note that for $u_0 \in \overset{\circ}{P} \setminus K_c^v$ such that $\operatorname{dist}(U(u), \partial P) > 0$, we also get (19) for every initial value in $\beta_\varepsilon(u_0)$, for all $\varepsilon > 0$, where $\beta_\varepsilon(u_0)$, a small neighborhood of u_0 in M. Thus $\overset{\circ}{P}$ is an invariant set of descent flow of J_λ. By similar arguments, we prove that $-P$ and $-\overset{\circ}{P}$ are invariant sets of J_λ. As for $[\underline{u}, \overline{u}]$, by (11) and the definition of $v_*(u)$, if $u_0 \in [\underline{u}, \overline{u}]$, then

$$\eta(t,u) = u_0 + 2\varepsilon t v_*(u_0) \in [\underline{u}, \overline{u}]$$

for $\varepsilon > 0$ small and $t \in [0,1]$. Thus, $[\underline{u}, \overline{u}]$ is invariant set of descent flow for J_λ. □

Remark 17. The same proof as above shows that under the above pseudo-gradient flow for any subsolution $\overline{u}$ and supersolution $\underline{u}$, $\underline{u} + P$ and $\overline{u} + P$ are invariant.

Lemma 18 (Deformation Lemma). *Assume that* (A_1)–(A_4), (f_1)–(f_4) *are satisfied. Then for every* $\overline{\varepsilon} > 0$, *for every* $c > 0$ *and every neighborhood* N *of* K_c^v, *there exist* $\varepsilon \in [0, \overline{\varepsilon}[$ *and a continuous family of odd continuous maps such that*

(i) $\eta(t,u) = u$ *if* $J'_\lambda(u)(z) = 0$ *for any direction* $v \in Y$ *or* $t = 0$ *or if* $|J_\lambda(u) - c| \geq \varepsilon$;
(ii) $J_\lambda(\eta(t,u))$ *is nonincreasing in* t *for any* $u \in M$;
(iii) $\|\eta(t,u) - u\|_Y \leq \delta$, *for* $t \in [0,1]$, $u \in M$ *and some* $\delta > 0$;
(iv) $\eta(1, J_\lambda^{c-\varepsilon} \setminus N) \subset J_\lambda^{c-\varepsilon}$;
(v) η *has the invariance properties of Lemma* 4.

Proof: Let $v : \operatorname{Reg}(J_\lambda) \to Y$ be an odd, continuous pseudo-gradient vector field for J_λ such that

$$\|v(u)\|_y < 2\|J'_\lambda(u)\|_Y, \quad \langle J'_\lambda(u), v(u)\rangle > \|J'_\lambda(u)\|_Y^2$$

for all $u \in M \setminus K_c^v$ and any fixed $z \in Y$. Let $N_\varepsilon^c := \{u \in M : |J_\lambda(u) - c| < \varepsilon,\ \|J'_\lambda(u)\| < \sqrt{\varepsilon}\}$. Hence, $N_\varepsilon^c \subset N$. Let φ be a continuous cut-off function such that $\varphi(u) = \varphi(-u)$, $\varphi(u) = 0$ on N_ε^c and $\varphi(u) = 1$ outside N_ε^c, in particular $\varphi(u) = 1$ for $u \notin N$ such that $|J_\lambda(u) - c| < \varepsilon$. Then, we may define a locally Lipschitz, odd continuous vector field $v_*(u) = -\varphi(u)\frac{v(u)}{\|v(u)\|_Y}$ for $u \in M \setminus N$ and $v_*(u) = 0$ elsewhere. Consider the initial value problem

$$\begin{aligned}\frac{\partial}{\partial t}\sigma(t,u) &= v_*(\sigma(t,u)),\\ \sigma(0,u) &= u.\end{aligned}$$

It has a unique global solution $\sigma(t,u)$ which is continuous on $\mathbb{R} \times M$. Define $\eta(t,u) = \sigma(2\sqrt{\varepsilon}t,u)$; since $\|v_*(u)\|_Y = 1$, we obtain $\|\sigma(t,u) - u\|_Y = \|\int_0^t v_*(\sigma(t,u))\,d\tau\|_Y \le t$. Hence, for $\delta \ge 2\varepsilon$ we have $\frac{\delta}{2\varepsilon} \ge 1$ and $\|v_*(u)\|_y \le \frac{\delta}{2\varepsilon}$. Therefore, $\|\eta(t,u) - u\|_Y \le \delta$ for $0 \le t \le 1$, which is (iii).

For any fixed $u \in M$, we can write $\eta(t,u) = u + 2\sqrt{\varepsilon}tv_*(u)$, thus the directional derivative of $J_\lambda(\eta(t,u))$ in the direction of v_* is equivalent to the derivative of the function $J_\lambda \circ g(t)$ at 0, where $g : \mathbb{R} \to \mathbb{R}$ is defined by $g(t) = u + 2\sqrt{\varepsilon}tv_*(u)$. Hence,

$$\frac{d}{dt}(J_\lambda \circ g(t))\big|_{t=0} = \Big\langle J'_\lambda(\eta(t,u)), \frac{\partial}{\partial t}\eta(t,u)\Big\rangle\big|_{t=0} = 2\sqrt{\varepsilon}\langle J'_\lambda(u), v_*(u)\rangle. \tag{20}$$

If $u \in J_\lambda^{c+\varepsilon} \setminus N$, then $\|J'_\lambda(u)\|_Y \ge \sqrt{\varepsilon}$. In integrating (20) for $0 \le t \le 1$, we get

$$\begin{aligned}J_\lambda(\eta(1,u)) - J_\lambda(u) &= 2\sqrt{\varepsilon}\int_0^1 \langle J'_\lambda(u), v_*(u)\rangle\,dt\\ &= 2\sqrt{\varepsilon}\int_0^1 \langle J'_\lambda(u), -\varphi(u)v(u)\rangle\,dt\\ &= -2\sqrt{\varepsilon}\int_0^1 \varphi(u)\Big\langle J'_\lambda(u), \frac{v(u)}{\|v(u)\|_Y}\Big\rangle\,dt\\ &= -2\sqrt{\varepsilon}\int_0^1 \|J'_\lambda(u)\|_y\,dt \le -2\varepsilon.\end{aligned}$$

Here we used our choice of φ. Hence,

$$J_\lambda(\eta(1,u)) \le c + \varepsilon - 2\varepsilon = c - \varepsilon.$$

Therefore, $\eta(1, J_\lambda^{c+\varepsilon} \setminus N) \subset J_\lambda^{c-\varepsilon}$ which is (iv). From (20), we have for any fixed u in M,

$$\begin{aligned}\frac{d}{dt}J_\lambda(\eta(t,u))\big|_{t=0} &= \frac{d}{dt}J_\lambda(u + 2\sqrt{\varepsilon}tv_*(u))\big|_{t=0}\\ &= 2\sqrt{\varepsilon}\big\langle J'_\lambda(u), v_*(u)\big\rangle\\ &\le -2\sqrt{\varepsilon}\varphi(u)\|J'_\lambda(u)\|_Y^2 < 0.\end{aligned}$$

Hence, $\frac{d}{dt}J_\lambda(\eta(t,u)) \le -2\varepsilon\varphi(u)$ for $t \ge 0$, so we get (ii) by our choice of φ. $\square$

Lemma 19. *Assume that A satisfies* (A_1)–(A_4), *and f satisfies* (f_1), (f_3), (f_4). *Then, there is a path $L_0 \subset M$ connecting $\overset{\circ}{P}$ and $-\overset{\circ}{P}$ such that $J_\lambda(\varphi) < 0$, for all $\varphi \in L_0$.*

Proof: Let $\varphi \in M$ be a solution of (P_λ) such that φ changes sign. Then, $\varphi = \varphi^+ - \varphi^-$. Multiplying $-\operatorname{div}(A(x,\varphi)\nabla\varphi) = f(\lambda,x,\varphi)$ by φ^+ (φ^-, respectively) and integrating, we get

$$\int_\Omega A(x,\varphi)|\nabla\varphi^+| = \int_\Omega f(\lambda,x,\varphi)\varphi^+\,dx \tag{21}$$

and

$$\int_\Omega A(x,\varphi)|\nabla\varphi^-| = \int_\Omega f(\lambda,x,\varphi)\varphi^-\,dx. \tag{22}$$

By (A_1), (21), (22) and (f_4) we have

$$\int_\Omega f(\lambda,x,\varphi)\varphi^+\,dx < \alpha\mu_1\int_\Omega \varphi\cdot\varphi^+\,dx = \alpha\mu_1\int_\Omega(\varphi^+)^2\,dx$$

and

$$\int_\Omega f(\lambda,x,\varphi)\varphi^-\,dx < \alpha\mu_1\int_\Omega \varphi\cdot\varphi^-\,dx = \alpha\mu_1\int_\Omega(\varphi^-)^2\,dx.$$

Thus, $\forall\, t \in [0,1]$, if $\varphi_t = t\varphi^+ + (1-t)\varphi^-$, then there exists $R > 0$ such $|\varphi_t| \geq R$ and

$$\begin{aligned}
J_\lambda(\varphi_t) &= \frac{1}{2}\int_\Omega A(x,\varphi_t)|\nabla\varphi_t|^2\,dx - \int_\Omega F(\lambda,x,\varphi_t)\,dx\\
&= \frac{t^2}{2}\int_\Omega A(x,\varphi_t)|\nabla\varphi^+|^2\,dx + \frac{(1-t)^2}{2}\int_\Omega A(x,\varphi_t)|\nabla\varphi^-|^2\,dx\\
&\quad - \int_\Omega F(\lambda,x,\varphi_t)\,dx\\
&\leq \frac{t^2}{t}\int_\Omega A(x,\varphi_t)|\nabla\varphi^+|^2\,dx - \frac{t^2\alpha\mu_1}{2}\int_\Omega(\varphi^+)^2\,dx - \frac{(1-t)^2\alpha\mu_1}{2}\int_\Omega(\varphi^-)^2\,dx\\
&\quad + \beta\mu_1\int_\Omega \varphi_t^2\,dx + \frac{(1-t)^2}{2}\int_\Omega A(x,\varphi)|\nabla\varphi^-|^2\,dx - \int_\Omega F(\lambda,x,\varphi_t)\,dx\\
&< 0.
\end{aligned}$$

Since $\{t\varphi^+ + (1-t)\varphi^-,\ t \in [0,1]\}$ is compact in Y, we can choose in M a path $L_0 = \{l_0(t) : t \in [0,1]\}$ such that $l_0(t)$ is very close to φ_t for all $t \in [0,1]$, $\varphi \in M$,

$l_0(0) \in \overset{\circ}{P}$, $l_0(1) \in -\overset{\circ}{P}$ and $J_\lambda(l_0(t)) < 0$. Hence, for every $\varphi \in L_0$ and for $t > 0$ sufficiently small we have

$$\begin{aligned} J_\lambda(t\varphi) &= \frac{1}{2}\int_\Omega A(x,t\varphi)|\nabla t\varphi|^2\,dx - \int_\Omega F(\lambda,x,t\varphi)\,dx \\ &= \frac{t^2}{2}\Big[\int_\Omega A(x,t\varphi)|\nabla\varphi^+|^2\,dx + \int_\Omega A(x,t\varphi)|\nabla\varphi^-|^2\,dx\Big] - \int_\Omega F(\lambda,x,t\varphi)\,dx \\ &\le \frac{t^2}{t}\Big[\int_\Omega A(x,t\varphi)|\nabla\varphi^+|^2\,dx - \alpha\mu_1\int_\Omega(\varphi^+)^2\,dx - \alpha\mu_1\int_\Omega(\varphi^-)^2\,dx \\ &\quad + \int_\Omega A(x,t\varphi)|\nabla\varphi^-|^2\,dx\Big] + \beta\mu_1\int_\Omega(t\varphi)^2\,dx - \int_\Omega F(\lambda,x,t\varphi)\,dx < 0. \end{aligned}$$

Therefore, there is a path $L = tL_0$ (t sufficiently small) connecting $\overset{\circ}{P}$ to $-\overset{\circ}{P}$ such that $J_\lambda(\varphi) < 0$, for all $\varphi \in L$. Normalizing such $\varphi \in L$, it is possible to improve L in order to get curves without self intersection on ∂B_1 where B_1 is the unit ball in Y. $\square$

5. Proofs of the main results

5.1 Proof of Theorem 5. From Lemma 10, we know that there exist two pairs of strict subsolutions and supersolutions of (P_λ), $-\phi < -r\varphi_1 < 0$ and $0 < r\varphi_1 < \phi$ where φ_1 is the eigenfunction associated to the first eigenvalue of the Laplacian operator $-\triangle$ with 0-Dirichlet boundary conditions. From [10], (P_λ) has a positive solution u_2^+ and a negative solution u_4^-, such that $-\phi < u_4^- < -r\varphi_1$, $r\varphi_1 < u_2^+ < \phi$, u_2^+ and u_4^- are local minimizer of $J_\lambda(u)$ with $J_\lambda(u_2^+) < 0$ and $J_\lambda(u_4^-) < 0$. We may assume that u_2^+ is the minimal minimizer of $J_\lambda(u)$ and u_4^- is the maximal minimizer of $J_\lambda(u)$. From Theorem 9, J_λ has a mountain-pass point $u_5 \in M$ such that $u_5 \in [-\phi,\phi] \setminus ([-\phi,-r\varphi_1] \cup [r\varphi_1,\phi])$ and $u_4^- < u_5 < u_2^+$. Let us consider $\Gamma_r = \{u(t) : u(t) \in [-\phi,\phi], \forall\, t \in [0,1]\}$ such that $u(t) \in ([-\phi,\phi] \setminus ([-\phi,-r\varphi_1] \cup [r\varphi_1,\phi])$ if $x \in \left(\frac{1}{3},\frac{2}{3}\right)$, $u(t) = \eta\left(\frac{1}{3} - t, u\left(\frac{1}{3}\right)\right)$ if $0 \le t \le \frac{1}{3}$, $u\left(\frac{1}{3}\right) \in \partial[-\phi, r\varphi_1])$, $u(t) = \eta\left(t - \frac{2}{3}, u\left(\frac{2}{3}\right)\right)$ if $\frac{2}{3} \le t \le 1$ and $u\left(\frac{2}{3}\right) \in \partial[r\varphi_1,\phi]$. Hence, $\Gamma_r \neq \emptyset$ is a complete metric space in Y with the metric $\delta(u,v) = \max_{[0,1]}\|u(t) - v(t)\|_Y$. We may choose $b(t) \in C\left(\left[\frac{1}{3},\frac{2}{3}\right],\ [-\phi,\phi] - ([-\phi,-r\varphi_1] \cup [r\varphi_1,\phi])\right)$ such that $b\left(\frac{1}{3}\right) = -r\varphi_1$, $b\left(\frac{2}{3}\right) = r\varphi_1$ and $J_\lambda(b(t)) < 0$ for all $t \in \left[\frac{1}{3},\frac{2}{3}\right]$. Define

$$r_r(t) = \begin{cases} b(t) & \text{if } t \in \left[\frac{1}{3},\frac{2}{3}\right], \\ \eta\left(\frac{1}{3} - t, -r\varphi_1\right) & \text{if } t \in \left[0,\frac{1}{3}\right], \\ \eta\left(t - \frac{2}{3}, r\varphi_1\right) & \text{if } t \in \left[\frac{2}{3},1\right]. \end{cases}$$

Then, $e_r(t) \in \Gamma_r$ and $\sup_{t\in[0,1]} J_\lambda(e_r(t)) < 0$. By the definition of Γ_r and (7), we see that u_5 is a critical point given by the critical value $\inf_{u\in\Gamma}\sup_{t\in[0,1]} J_\lambda(u(t))$

< 0. Hence, $J_\lambda(u_5) < 0$ and $u_5 \neq 0$. Since we may assume that $u = 0$ is an isolated critical point in $B(0,r)$, letting r go to zero, we get that u_5 must be sign-changing. Let

$$D = \{u \in M : -\phi \leq u(x) \leq \phi\}.$$

From Theorem 8, D and $\overset{\circ}{D}$ are positively invariant under the descend flow of J_λ (negative pseudo-gradient flow). Let $U = \{h \in M : \exists\, t_h > 0$ such that $\eta(t_h, h) \in \overset{\circ}{D}\}$ where $\eta(t,u)$ is the unique solution of the boundary value problem

$$\frac{\partial \eta(t,u)}{\partial t} = v(\eta(t,u)),$$
$$\eta(t,0) = 0.$$

Then, U is an open set in Y and a positively invariant set under the negative pseudo-gradient flow of J_λ in M. Since $\eta(t,u)$ has continuous dependence on the initial value h, it is also easy to prove that ∂U is an invariant set under the negative pseudo-gradient flow. Moreover, J_λ is bounded from below on ∂U and satisfies the compactness condition (C) and the deformation property in M.

From Lemma 19, we know that there exists a path $L \subset M$ connecting $\overset{\circ}{P}$ and $-\overset{\circ}{P}$ such that $J_\lambda(u) < 0$ for all $u \in L$. Define $Z = \{tu : t > 0$ and $u \in L\}$ and note that Z is homeomorphic to the set $\{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ in $\mathbb{R}^2$. We already know that P and $-P$ are positively invariant sets under the negative pseudo-gradient flow. Thus, $U \cap Z$ is a bounded and relatively open set in Z and $\partial(U \cap Z) \neq \emptyset$. We may assume that $U \cap Z$ is connected, since otherwise, we consider a connected component $Z' \subset Z$ of $U \cap Z$, with $(0,0) \in Z$ instead of $U \cap Z$ and, by result of [18, Chapter 4], there exists at least one connected component $C \subset \partial Z$ such that $C \cap \overset{\circ}{P} \neq \emptyset$, $C \cap -\overset{\circ}{P} \neq \emptyset$ and $C \cap (M - (-P \cup P)) \neq \emptyset$. To apply the result of [18], we consider the homeomorphic image of U in $\mathbb{R}^{2+}$, add its reflection to the other half plane, obtaining so an open set in $\mathbb{R}^2$. Moreover, L can be chosen arcwise connected.

Since P and $-P$ are positively invariant sets under the negative pseudo-gradient flow J_λ, let

$$V_P = \{h \in M : \exists\, t_h \geq 0 \ \text{ such that } \ \eta(t_h, h) \in \overset{\circ}{P}\}, \ \text{ and}$$
$$V_{-P} = \{h \in M : \exists\, t_h \geq 0 \ \text{ such that } \ \eta(t_h, h) \in -\overset{\circ}{P}\}.$$

From the strongly order preserving property of the inverse operator K, the sets V_P, V_{-P}, ∂V_P and ∂V_{-P} are invariant open sets of the negative pseudo-gradient flow $\eta(t,u)$ in M. Let

$$C_+ = \inf_{u\in\partial U\cap P} J_\lambda(u),$$
$$C_- = \inf_{u\in\partial U\cap -P} J_\lambda(u),$$
$$C_0 = \inf_{u\in\partial U\cap V_P} J_\lambda(u).$$

Then, C_+, C_- and C_0 attain their minima say u_1^+, u_3^- and u_6 respectively. Since $\partial U\cap P\neq\emptyset$, $\partial U\cap -P\neq\emptyset$ and $\partial U\cap\partial V_0\neq\emptyset$ are invariant sets under the flow $\eta(t,u)$, we claim that u_1^+, u_3^- and u_6 are critical points of J_λ for any fixed direction in J. In fact, $J_\lambda(u_1^+)\neq 0$ implies that $J_\lambda(\eta(t,u_1^+)) < C_+$ for $t>0$, but $\eta(t,u_1^+)\in\partial U\cap P$ for all $t>0$, so we get a contradiction with the definition of C_+. Similar arguments are used to prove that u_3^- and u_6 are critical points of J_λ. We have $u_1^+ > u_2^+ > 0$, $u_3^- < u_4^- < 0$. From the strong maximum principle and the fact that $u_6\in\partial U\cap\partial V$, it follows that u_6 is sign-changing.

5.2 Proof of Theorem 7. From (f_4') and using similar argument as in [4], we find $N = N(\lambda) > 0$ satisfying

$$-\triangle(Ne) > f(\lambda,x,Ne).$$

By the continuity of the function $A(x,u)$ with respect to u for a.e. $x\in\Omega$ and by (A_1), we have

$$-\operatorname{div}(A(x,Ne)\nabla(Ne)) > -\beta\triangle(Ne) > f(\lambda,x,Ne).$$

Thus, Ne is a strict supersolution of (P_λ), and for $r>0$ small enough, $r\varphi_1 < Ne$ is a pair of subsolution and supersolution, $-Ne < -r\varphi_1$ is another one. By similar argument as in the proof of Theorem 5, the problem (P_λ) has a negative solution u_4^- and a positive solution u_2^+ such that $-Ne < u_4^- < -r\varphi_1$, $r\varphi_1 < u_2^+ < Ne$, u_2^+ is the minimal positive minimizer, u_4^- is the maximal negative minimizer of J_λ, and J_λ has a mountain-pass point $u_5\in M$ such that $u_5\in[-Ne,Ne]\setminus([-Ne,-r\varphi_1]\cup[r\varphi_1,Ne])$, $u_4^- < u_5 < u_2^+$, $u_5\neq 0$ and is sign-changing.

From (f_2'), there exists $u_0>0$ such that $f(\lambda,x,u)>0$ and $\frac{f(\lambda,x,u)}{F(\lambda,x,u)}\geq\frac{\theta}{u}$ for all $u\in\max\{u_0,R\}$ for some $R>0$. Hence, $F(\lambda,x,u)\geq Cu^\theta$ for some $C>0$. Therefore, there exists a constant $C_1>0$ such that

$$J_\lambda(u)\leq\frac{\beta}{2}\bigg(\int_\Omega\bigg(|\nabla u|^2-\frac{2C}{\beta}u^\theta\bigg)\,dx\bigg)+C_1.$$

Since $\theta>2$, if we choose $u=t\varphi_1$ for $t>0$ then $J_\lambda(t\varphi_1)\to-\infty$ as $t\to\infty$. Hence, there exists $T_0>N$ such that

$$J_\lambda(T_0\varphi_1)<0,\quad \frac{d}{dt}J_\lambda(te)\big|_{t=T_0}<0, \tag{23}$$

and

$$\frac{d}{dt} J_\lambda(te)\big|_{t=N} < 0, \tag{24}$$

since $e(x)$ is fixed in M and $v(t) = te(x)$ is linear. Hence, by implicit functions theorem, there exists T_1 with $N < T_1 < T_0$ such that $T_1 e \in \partial U$. Therefore, $\partial U \cap \{u_2^+ + P\} \neq \emptyset$, where $w + P = \{u = w + v, v \in P\}$. By a similar argument we get $\partial U \cap \{u_2^+ - P\} \neq \emptyset$. Let φ_1 and φ_2 be the first and the second eigenfunctions of the problem

$$\begin{aligned} -\Delta u - \lambda u, \qquad & x \in \Omega, \\ u = 0, \qquad & x \in \partial\Omega. \end{aligned}$$

Substituting $v \in \operatorname{Span}\{\varphi_1, \varphi_2\}$ for $e(x)$ in (23) and (24), and using a similar argument, we get $\partial U \cap (M - (-P \cup P)) \neq \emptyset$. Since P, $-P$ are positively invariant under η, if V_P and V_{-P} are defined as in the proof of Theorem 5, then by similar argument, ∂V_P, V_P, ∂V_{-P}, V_{-P}, $u_2^+ + P$ and $u_4^- - P$ are invariant sets. Let

$$\begin{aligned} C_+ &= \inf_{u \in \partial U \cap (u_2^+ + P)} J_\lambda(u), \\ C_- &= \inf_{u \in \partial U \cap (u_4^- - P)} J_\lambda(u), \\ C_0 &= \inf_{u \in \partial U \cap V_P} J_\lambda(u). \end{aligned}$$

Using a similar argument as in the proof of Theorem 5, we get that our six solutions $u_1^+ > u_2^+ > 0$, $u_3^- < u_4^- < 0$, u_5 and u_6 are sign-changing. The proof is complete.

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