Delannoy and tetrahedral numbers

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Abstract. We establish an identity between Delannoy numbers and tetrahedral numbers of arbitrary dimension.

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Classification: Primary 05A10; Secondary 05A15, 05A19

The numbers $D(n,m)$ of Henri Auguste Delannoy (1833–1915), [4], count the lattice paths in $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to (m,n) with set of permitted steps $\{(0,1),(1,0),(1,1)\}$, i.e. north, east and north-east steps. They can conveniently be described as minimal king walks from the bottom left corner to the upper right corner on a $m\times n$ chess board (see Figure 3 for $m=n=2$). It is known since the times of Delannoy that

$$D(n,m)=\sum_{\nu=0}^{m}\binom{m}{\nu}\binom{n+\nu}{m}=\sum_{\nu=0}^{\min\{m,n\}}2^{\nu}\binom{m}{\nu}\binom{n}{\nu}. \tag{1}$$

Tetrahedral numbers have, by definition, a simple geometric meaning, too. In two dimensions they are the number of lattice points in an equilateral triangle, correspondingly in three dimension they count the number of points in a regular tetrahedron, see Figure 1. Generalization to higher dimensions is apparent and involves the d-dimensional simplex (hypertetrahedron). Their determination is even easier than the determination of Delannoy numbers, because they are built up inductively. If $T_d(n)$ denotes the number of points with integer coordinates in the d-dimensional hypertetrahedron of edge length[1] $n-1$, then $T_1(n)=n$ and $T_{d+1}(n)=\sum_{v=1}^{n}T_d(v)$. It happens that $T_1(n)=\binom{n}{1}$ and the triangular number

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[1] The edge length is the number of intervals created by n equidistant points, where start and end point of the edge carries a point. It is naturally 1 less than the number of points.

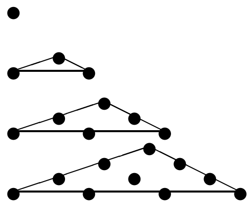

FIGURE 1: An example of the simplest 3-dimensional polytop, a discrete tetrahedron or 3-simplex, of edge length 3.

$T_2(n)$ equals $\frac{n(n+1)}{2} = \binom{n+1}{2}$. The discrete antiderivative of $\binom{x}{v}$ is $\binom{x}{v+1}$, i.e. $\Delta\binom{x}{v+1} = \binom{x}{v}$. Hence discrete integration can be used to show

$$T_d(n) = \binom{n+d-1}{d}. \tag{2}$$

In order to establish a link between Delannoy and tetrahedral numbers, we will need a result about crystal balls, which was first discovered by Vassilev & Atanassov [10].

Definition 1. (1) Let $\mathbf{x} = (x_1, x_2, ..., x_d) \in \mathbb{Z}^d$, $d \in \mathbb{N}$. The L^1-norm $|\mathbf{x}|_1$ of $\mathbf{x}$ is defined by $|\mathbf{x}|_1 := \sum |x_i|$.

(2) $\mathcal{S}_d(n) := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{Z}^d \text{ and } |\mathbf{x}|_1 = n\}$ is called $d-1$-dimensional crystal sphere of radius n. We set $S_d(n) := |\,\mathcal{S}_d(n)|$. The sequence $(S_d(n))_{n\in\mathbb{N}}$ is called coordination-sequence (or -numbers). The union $\bigcup_{\nu=0}^{n} \mathcal{S}_d(\nu) =: \mathcal{G}_d(n)$ is called d-dimensional crystal ball of radius n, see Figure 2. We put $|\,\mathcal{G}_d(n)| =: G_d(n) = \sum_{\nu=0}^{n} S_d(\nu)$.

(3) If $f : D \to \mathbb{Z}, D \subseteq \mathbb{Z}$, is a function then the (forward) difference operator Δ is defined by $\Delta f(n) = f(n+1) - f(n)$. Δ^m is defined by $\Delta^1 = \Delta$ and $\Delta^m = \Delta \circ \Delta^{m-1}$.

Theorem 2 ([10], [8])**.**

$$D(n,m) = \sum_{\mu=0}^{m} S_n(\mu) = G_n(m),$$

see Figure 3.

PROOF: We will follow the proof in Schröder [8], because it is considerably shorter. The GF of the Delannoy numbers $D(n,m)$ is known to be (cf. [9])

$$\sum_{n,m\geq 0} D(n,m)x^n y^m = \frac{1}{1-x-y-xy}.$$

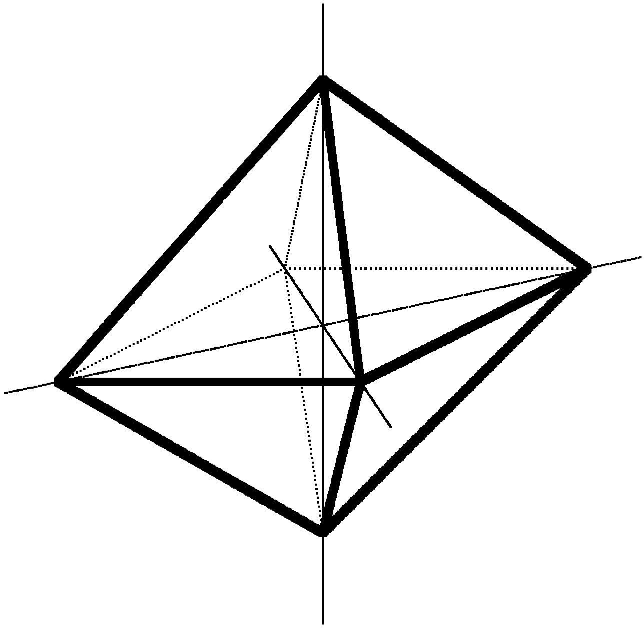

FIGURE 2: The shape of a crystal ball in 3 dimensions, also called regular octahedron (the discrete points are not drawn).

We have

$$\frac{1}{1-x-y-xy} = \frac{1}{1-y}\,\frac{1}{1-x\frac{1+y}{1-y}} = \frac{1}{1-y}\sum_{n\geq 0}\left(\frac{1+y}{1-y}\right)^n x^n$$

$$= \frac{1}{1-y}\sum_{n,m\geq 0} S_n(m)y^m x^n = \sum_{n,m\geq 0}\sum_{\mu=0}^{m} S_n(\mu)y^m x^n = \sum_{n,m\geq 0} G_n(m)y^m x^n.$$

Indeed, Conway & Sloane show in [3, p. 9, Equation (16)], that

$$S_d(n) = \sum_{k=0}^{d}\binom{d}{k}\binom{n+d-k-1}{d-1} \quad \left(= \sum_{k=0}^{d}\binom{d}{k}\binom{n+k-1}{d-1}\right)$$

is the coordination number of distance n in $\mathbb{Z}^d$ and their generating function is

$$\sum_{n\geq 0} S_d(n)y^n = \left(\frac{1+y}{1-y}\right)^d.$$

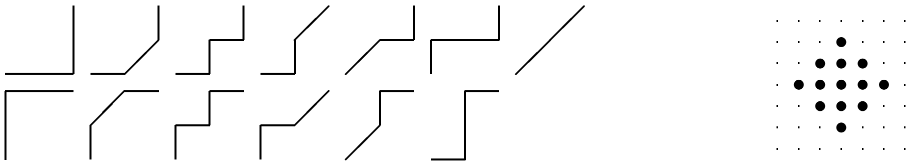

FIGURE 3: $D(2,2) = 13$ and $G_2(2) = 13$.

See also [6, p. 4, Aufg. 29]. $\square$

We are ready to state and prove the main theorem. The idea is to decompose a crystal ball into a number of hypertetrahedra. For instance, if we look at Figure 2, we can see that the top pyramid consists of 4 tetrahedra of type as depicted in Figure 1. Overall $\mathcal{G}_3(n)$ is composed out of 8 tetrahedra. Unfortunately these tetrahedra overlap. They have some faces and edges in common. Therefore, to get the correct number of points, we have to apply the principle of inclusion-exclusion. For instance, if we stay with Figure 1 and Figure 2, the first approximation to $G_3(3)$ is obtained by taking 8 times the number of lattice points in Figure 1. Then we have to subtract 12 times the number of points of a tetrahedral face, because 12 faces are common to 2 tetrahedra, add 6 times the number of points of a tetrahedral edge, because 6 edges in the coordinate axes' are contained in 4 tetrahedra and finally subtract 1 for the point in the center, which is common to all tetrahedra. We have shown $D(3,4-1) = G_3(4-1) = 8\binom{4+2}{3} - 12\binom{4+1}{2} + 6\binom{4}{1} - 1 = 8 \times 20 - 12 \times 10 + 6 \times 4 - 1 = 63$ and more generally

$$D(3, n-1) = G_3(n-1) = 8\binom{n+2}{3} - 12\binom{n+1}{2} + 6\binom{n}{1} - 1,$$

see Equation 2.

Theorem 3.

$$(3)\quad D(n,m) = \sum_{v=0}^{m} (-1)^{m-v} 2^v \binom{m}{v} T_v(n+1) = \sum_{v=0}^{m} (-1)^{m-v} 2^v \binom{m}{v} \binom{n+v}{v}$$

$$(4)\qquad D(n,m) = \Delta^m\, 2^x \binom{n+x}{x}\bigg|_{x=0} = \Delta^m\, 2^x T_x(n+1)\bigg|_{x=0}$$

$$(5)\qquad 2^m T_m(n+1) = 2^m \binom{n+m}{m} = \sum_{v=0}^{m} \binom{m}{v} D(n,v)$$

$$(6)\qquad 2^m \binom{n}{m} = \sum_{v=0}^{m} (-1)^{m-v} \binom{m}{v} D(n,v) = \Delta^m D(n,x)\bigg|_{x=0}$$

Proof: Equations 3, 4 and 5 are equivalent via binomial inversion and difference formula. Binomial inversion again shows the (known) equivalence of Equation 1 and Equation 6, which was added for completeness. In order to prove Equation 3, we have to determine the number of simplices with a given, common sub-simplex. The easiest method might be to use a linear scheme in which we record the quadrant of the point and the varying coordinates, as in Schröder [7]. Given a d-dimensional crystal ball $\mathcal{G}_d$, let $e = e_1e_2e_3\ldots e_d$ be a finite sequence, where $e_i \in \{+, 0, -\}$. To every point $p = (x_1, x_2, \ldots, x_d) \in \mathcal{G}_d$ we assign a sequence e by

$$e_i := \begin{cases} + & \text{if } x_i > 0 \\ - & \text{if } x_i < 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

Vice versa, every e defines a sub-simplex of $\mathcal{G}_d$, modulo size. An entry 0 in e has a special meaning, because it indicates a set of points which are common to more than 1 sub-simplex. For instance, in 3 dimensions, $e = + - +$ stands for the 3-simplex which lies in the octant with positive x- and z- coordinates and negative y- coordinate. $e = +0+$ describes the sub-simplex (triangle) with positive x- and z- coordinates and vanishing y- coordinate. It is the common face of $+++$ and $+-+$, see Figure 2. In d dimensions, the first approximation to $G_d(n)$ is $2^d T_d(n+1)$, because there are 2^d different $+$, $-$ sequences of length d, i.e. $\mathcal{G}_d$ is composed out of 2^d d-simplices. We have to subtract points in the common faces to get the second approximation. There are $\binom{d}{1}$ possibilities to insert 0 in a sequence of length d and 2^{d-1} possibilities to fill the remaining places with $+$ and $-$. Each case accounts for $T_{d-1}(n+1)$ points. In the next step we have to add $2^{d-2}\binom{d}{2}T_{d-2}(n+1)$ points, etc. Eventually we arrive at

$$D(n,d) = \sum_{v=0}^{d} (-1)^v 2^{d-v} \binom{d}{v} T_{d-v}(n+1) = \sum_{v=0}^{d} (-1)^v 2^{d-v} \binom{d}{v} \binom{n+d-v}{d-v}.$$

Substitution $d - v \to v$ produces Equation 3. $\square$

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