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Tanaka spaces and products of sequential spaces

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Abstract. We consider properties of Tanaka spaces (introduced in Mynard F., *More on strongly sequential spaces*, Comment. Math. Univ. Carolin. **43** (2002), 525–530), strongly sequential spaces, and weakly sequential spaces. Applications include product theorems for these types of spaces.

Keywords: Tanaka space, strongly sequential space, weakly sequential space, sequential space, k-space, inner-closed A-space

Classification: 54D50, 54D55, 54B10, 54B15

Introduction

We assume that all spaces are regular and T_1, and that all maps are continuous surjections.

In 1976, the author [23] investigated a characterization for the product of a sequential space with a first countable space to be sequential, introducing the following key condition:

(C) For every decreasing sequence (A_n) in X with $x \in \operatorname{cl} A_n$ for any $n \in \mathbb{N}$, there exist $x_n \in A_n$ such that $\{x_n : n \in \mathbb{N}\}$ converges to some point $p \in X$.

When $p = x$, such a space X is a *strongly Fréchet space* [21] (or, *countably bi-sequential space* [10]). Strongly Fréchet spaces have played an important role in studying products of Fréchet spaces.

The following question was essentially raised in [23]: For a sequential space X with property (C), is $X \times Y$ sequential for any first countable space Y ?

F. Mynard proved in [13] that a space X is strongly sequential if and only if $X \times Y$ is sequential for any first countable space Y, introducing "strongly sequential spaces", and then he showed in [14] that a strongly sequential space is exactly a sequential space with condition (C), which gives an affirmative answer to the above question.

A space is a *Tanaka space* [14] if it satisfies condition (C). A space X is *strongly sequential* [13] if, whenever (A_n) is a decreasing sequence in X with $x \in \operatorname{cl} A_n$ for any $n \in \mathbb{N}$, then the point x belongs to the (idempotent) *sequential closure* of the set A of limit points of convergent sequences $\{x_n : n \in \mathbb{N}\}$ with $x_n \in A_n$ (equivalently, X is a sequential space such that if $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence in X with $x \in \operatorname{cl} A_n$ for any $n \in \mathbb{N}$, then the point x belongs to the (usual) closure of the set A).

A strongly sequential space, or a locally sequentially compact space is a Tanaka space, but a Tanaka space need not be (strongly) sequential.

In this paper, we consider properties of Tanaka spaces, strongly sequential spaces, and weakly sequential spaces. As applications, we consider products of these types of spaces and sequential spaces. Also, we will pose some questions on these spaces and their product spaces.

For a cover $\mathcal{P}$ of a space X, $\mathcal{P}$ is a *determining cover* [29] (or, $\mathcal{P}$ *determines* X [5]), if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. Here, we can replace "open" by "closed". Obviously, every open cover is a determining cover. Recall that a space X is a *sequential space* (resp. a *k-space*, a *quasi-k-space*) if X has a determining cover by all compact metric sets (resp. compact sets, countably compact sets) in X. We note that we can replace "all" by "some", and also "compact metric sets" by "convergent sequences (containing its limit point) ". A space is *weakly sequential* [24] if it has a determining cover by sequentially compact sets. A strongly sequential space is sequential, and a sequential space is weakly sequential.

A map $f : X \to Y$ is *bi-quotient* [9] (resp. *countably bi-quotient* [21]) if, whenever $y \in Y$ and $\mathcal{U}$ is a cover (resp. a countable cover) of $f^{-1}(y)$ by open sets, then finitely many $f(U)$ with $U \in \mathcal{U}$ cover a neighborhood of y in Y. Then, open maps are bi-quotient, bi-quotient maps are countably bi-quotient, and countably bi-quotient maps are quotient maps. A (finite or infinite) product of bi-quotient maps is a bi-quotient map ([9]).

Many (local) topological properties (P) relevant to our purpose can be characterized as follows: A space Y has property (P) if and only if there exists a space X with property (Q) and a quotient map $f : X \to Y$ of the type p, where (Q) is a (nice) property stronger than (P) and p determines how good the quotient map is. We gather below such results from [10]. Recall that an M-space is exactly a space admitting a quasi-perfect map (i.e., closed with countably compact fibers) onto a metric space.

Characterizations: (A) A space X is respectively bi-sequential, bi-k, bi-quasi-k if and only if it is a bi-quotient image of a metric space, a paracompact M-space, an M-space. Here, the latter two domains can be chosen to be a subset of $X \times M$ for some metric space M (the same holds in (B) and (C) below).

(B) A space X is respectively countably bi-sequential, countably bi-k, countably bi-quasi-k if and only if it is a countably bi-quotient image of a metric space, a paracompact M-space, an M-space.

(C) A space X is respectively sequential, k, quasi-k if and only if it is a quotient image of a metric space, a paracompact M-space, an M-space.

Results

Proposition 1. (1) *Let X be a weakly sequential or Tanaka space. Then a subset S of X is sequentially compact* (*resp. weakly sequential*) *if and only if S is countably compact* (*resp. quasi-k*).

(2) *The following are equivalent:*

- (a) *X is weakly sequential;*
- (b) *X is a quasi-k-space whose countably compact sets are sequentially compact;*
- (c) *X is a quotient image of a Tanaka M-space.*

(3) *The following are equivalent:*

- (a) *X is sequential;*
- (b) *X is a weakly sequential space whose countably compact sets are closed in X;*
- (c) *X is a quotient image of a metric space.*

Proof: (1). The "only if" part is obvious. For the "if" part, any infinite sequence in a countably compact set in X can be assumed to be not closed in X. Then, since X is a weakly sequential or Tanaka space, we see that a subset S of X is sequentially compact if S is countably compact. Thus, the parenthetic part also holds.

(2). (a)$\Leftrightarrow$(b) holds by (1). For (b)$\Rightarrow$(c), let T be the topological sum of all countably compact (hence, sequentially compact) sets in X. Then T is a Tanaka M-space, and X is the quotient image of T. For (c)$\Rightarrow$(a), an M-space is quasi-k by Characterization (C), thus a Tanaka M-space is weakly sequential by (1). Then X is weakly sequential, for the weak sequentiality is obviously preserved by quotient maps.

(3). (a)$\Leftrightarrow$(c) holds by Characterization (C). For (a)$\Rightarrow$(b), a countably compact set K in X is closed in X, because $K \cap L$ is closed in X for every convergent sequence L in X. For (b)$\Rightarrow$(a), let $F \subset X$, and assume that $F \cap L$ is closed in X for every convergent sequence L in X. To show that F is closed in X, let C be a countably compact set in X. Then C is sequentially compact by (1), thus $F \cap C$ is sequentially compact by the assumption, hence, $F \cap C$ is closed in X by (b). This implies that F is closed in X since X is a quasi-k-space. □

Recall that a space X is a *GO-space* if X is a subspace of a LOTS, that is, a linearly ordered space. In a GO-space X, every countably compact set K is sequentially compact, because any sequence in K has a monotone, hence convergent, subsequence in K. Thus, the following holds by Proposition 1(2).

Corollary 2. *A GO-space is weakly sequential if and only if it is a quasi-k-space.*

Recall that a space X *has countable tightness*, denoted $t(X) \leq \omega$, if whenever $x \in \operatorname{cl} A$, then $x \in \operatorname{cl} C$ for some countable $C \subset A$ (equivalently, X has a determining cover by countable sets (cf. [10])). As is well-known, a sequential space or a hereditarily separable space has countable tightness.

A space X is an *A-space* [11] if, whenever (A_n) is a decreasing sequence in X with $x \in \mathrm{cl}(A_n - \{x\})$, then there exist $B_n \subset A_n$ such that $\bigcup\{\mathrm{cl}\, B_n : n \in \mathbb{N}\}$ is not closed in X. Also, X is an *inner-closed A-space* (resp. an *inner-one A-space*) when the B_n are closed sets (resp. singletons). For the decreasing sequence (A_n), if we assume $\bigcap\{A_n : n \in \mathbb{N}\} = \emptyset$, then such a space X is respectively *A'-space*, *inner-closed A'-space*, *inner-one A'-space*. For a space X with non-measurable cardinality or $t(X) \leq \omega$, X is an A-space if and only if X is an A'-space. We can add prefixes "inner-closed", and "inner-one" ([11]).

In the following proposition, (1) and (2) are routinely shown, using Proposition 1(1), and (3) is due to [14].

Proposition 3. (1) *A Tanaka space is inner-one A'. Thus, a Tanaka space of non-measurable cardinality or of countable tightness is inner-one A.*

(2) (i) *A space is weakly sequential and inner-closed A' if and only if it is weakly sequential and inner-one A' if and only if it is Tanaka and quasi-k.*

(ii) *A space X is weakly sequential and inner-closed A if and only if it is weakly sequential and inner-one A. Such a space is a Tanaka and quasi-k-space. The converse is true under the assumption that the cardinality of X is non-measurable or that $t(X) \leq \omega$.*

(3) *The following are equivalent:*

(a) *X is a sequential inner-closed A-space;*
(b) *X is a sequential inner-one A-space;*
(c) *X is a sequential Tanaka space;*
(d) *X is a strongly sequential space.*

Remark 4. (1) A compact space (hence inner-one A-space) need not be a weakly sequential or Tanaka space in view of Proposition 1(1). Also, a countable inner-one A-space need not be a Tanaka or quasi-k-space. Indeed, the compact sequential space Ψ^* in [3, Example 7.1] contains a subset $S = \mathbb{N} \cup \{\infty\}$. Since Ψ^* is sequential inner-closed A, any subset of Ψ^* is inner-one A by [11, Proposition 5.4], hence so is S. But, S is not a Tanaka or quasi-k-space, for any compact set in S is finite.

(2) Under (MC) (i.e., the existence of a measurable cardinal), the space X^* (with the only non-isolated point x^*) in [10, Example 10.16] is a (hereditarily) Tanaka space, but it is not an inner-one A-space with $t(X^*) > \omega$, not a quasi-k-space. Indeed, if $x^* \in \mathrm{cl}\, A_n$ ($A_n \supset A_{n+1}$), $\bigcap\{A_n : n \in \mathbb{N}\} \neq \emptyset$, and no countable subset in X^* has an accumulation point in X^*, as is shown in [10]. Also, a Tanaka space or even a sequentially compact space need not be a k-space by [2, 3.10.I]. Without (MC), the author does not know whether a Tanaka space is a quasi-k-space. In other words, he has the following question in view of Proposition 1: *Is a weakly sequential space exactly a quotient image of a Tanaka space* ?

The following follows from Propositions 1(1) & 3(2), and Corollary 2.

Corollary 5. *Let X be an inner-closed A and quasi-k-space, in particular, a countably bi-quasi-k-space* ([10]).
(1) *The following are equivalent:*

(a) *X is a Tanaka space;*
(b) *X is weakly sequential;*
(c) *Any countably compact set in X is sequentially compact.*

(2) *If X is a sequential space or a GO-space, then X is a Tanaka space.*

In the following lemma, (i) in (1) follows from [11, Theorem 6.3 and Proposition 2.4], and (ii) follows from [11, Theorem 6.7 and Proposition 3.1] (cf. [10, Theorem 9.9]). (2) follows from [10, Theorem 9.5 and Lemma 9.6].

Lemma 6. (1) *Let $f : X \to Y$ be a closed map. Then the following hold.*

(i) *f is countably bi-quotient if Y is an A'-space.*
(ii) *Each boundary $\partial f^{-1}(y)$ is countably compact if X is normal or countably paracompact, and Y is inner-closed A.*

(2) *A quotient map $f : X \to Y$ is bi-quotient if f is Lindelöf and X is paracompact* (*resp. f is an s-map and X is meta-Lindelöf*), *and Y is an inner-closed A-space with $t(Y) \leq \omega$. Here, a map is Lindelöf* (*resp. an s-map*) *if each fibers is Lindelöf* (*resp. separable*).

Recall that a cover $\mathcal{P}$ of a space X is a *k-network* if, for any compact set K of X and any open set $V \supset K$, $K \subset \cup \mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}$. When K is any convergent sequence (containing its limit point), some $P \in \mathcal{P}$ with $P \subset U$ contains a subsequence of K (not necessarily containing the limit point), then such a cover $\mathcal{P}$ is a *wcs*-network* [7] which is a useful generalization of k-networks. For a survey on k-networks, see [28].

A space X is a *q-space* [8] if each $x \in X$ has a q-sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods of x (i.e., if $x_n \in V_n$, $\{x_n : n \in \mathbb{N}\}$ has an accumulation point). A q-space is characterized as an open image of an M-space; see [10].

The *Arens' space S_2* is a quotient space obtained from the topological sum of $\{L_n : n \in \mathbb{N}\}$, where L_n are the convergent sequence $\{1/n : n \in \mathbb{N}\} \cup \{0\}$, by identifying each $1/n \in L_1$ with $0 \in L_n$ $(n \geq 2)$. The quotient space S_2/L_1 is the *sequential fan S_ω*.

Proposition 7. (1) *Let X be a Fréchet space, a sequential hereditarily normal space, or a space with G_δ-points* (*i.e., a space whose points are G_δ-sets*). *Then X is a Tanaka space if and only if it is strongly Fréchet.*

(2) *Let X be a space with a point-countable wcs*-network. If X is a k-space or $t(X) \leq \omega$, then X is a Tanaka space if and only if it is first countable* (*actually, X has a point-countable base*).

(3) *Let X be a closed image of a countably bi-quasi-k-space* (*resp. a closed image of an M-space*). *Then X is a Tanaka space if and only if it is a weakly sequential and countably bi-quasi-k-space* (*resp. a weakly sequential and q-space*).

(4) *Let X be a quotient Lindelöf image of a paracompact bi-quasi-k-space* (*resp. a quotient s-image of a meta-Lindelöf bi-quasi-k-space*). *If $t(X) \leq \omega$, then X is a Tanaka space if and only if it is a weakly sequential and bi-k-space* (*resp. a weakly sequential and bi-quasi-k-space*).

Proof: The "if" parts of (1)$\sim$(4) hold by Corollary 5, so let X be a Tanaka space for (1)$\sim$(4).

(1). For X being a Fréchet space or a space with G_δ-points, X is strongly Fréchet by [23, Lemma 2.1]. For X being a sequential hereditarily normal space, since X contains no closed copy of S_2, X is Fréchet by [6, Corollary 2.3], hence strongly Fréchet.

(2). Let $\mathcal{P}$ be a point-countable wcs^*-network for X. Then $\mathcal{P}$ is a k-network by [26, Proposition 1.2(1)] and Proposition 1(1). Thus X has a point-countable k-network. Since X is a k-space or $t(X) \leq \omega$, X has a point-countable base in view of the proofs of [5, Corollaries 3.5 and 3.6], replacing "countably bi-k-space" by "Tanaka space of countable tightness" there.

(3). Since X is Tanaka and quasi-k, X is weakly sequential by Proposition 1(1). Since X is an A'-space by Proposition 3(1), any closed map onto X is countably bi-quotient by Lemma 6(1). Thus, X is countably bi-quasi-k in view of [10]. For the parenthetic part, let $f : S \to X$ be a closed map with S an M-space. Since the M-space S is countably bi-quasi-k, X is countably bi-quasi-k, hence inner-closed A. While, S is countably paracompact, for every M-space is countably paracompact. Thus, each boundary $\partial f^{-1}(x)$ is countably compact by Lemma 6(1). Since f is a closed map, for some closed set F in S, $g = f|F$ is a quasi-perfect map onto X. Thus, since F is an M-space, X is a q-space by [12, Theorem 4.2].

(4). X is weakly sequential, for X is Tanaka and quasi-k. Since X is inner-closed A by Proposition 3(1), X is a bi-quotient image of a bi-k-space (resp. a bi-quasi-k-space) by Lemma 6(2), here a paracompact bi-quasi-k-space is bi-k ([10, p. 94]). Thus X is bi-k (resp. bi-quasi-k) in view of [10]. □

Theorem 8. (1) *Let $t(X) \leq \omega$. Then X is a hereditarily Tanaka space if and only if it is strongly Fréchet.*

(2) *Let X be a Tanaka space. A subset S of X is a Tanaka space if S is a closed or open set in X, or S is a quasi-k-space such that S is inner-closed A or $t(X) \leq \omega$.*

(3) *A countably bi-quotient image of a Tanaka space is a Tanaka space.*

(4) *Let X be a Tanaka space, and Y be a Tanaka bi-quasi-k-space. Then $X \times Y$ is a Tanaka space.*

Proof: (1). The "if" part is obvious. For the "only if" part, any countable set in X is a Tanaka space, hence strongly Fréchet by Proposition 7(1). Thus, since

$t(X) \leq \omega$, X is strongly Fréchet by [10, Proposition 8.7].

(2). For S being open in X, S is obviously a Tanaka space (for, X is regular). So, let S be quasi-k. Then S is weakly sequential by Proposition 1(1). Thus, when S is inner-closed A, S is a Tanaka space by Proposition 3(2). When $t(X) \leq \omega$, since X is inner-one A by Proposition 3(1), S is inner-closed A by [11, Proposition 5.3]. Thus S is also a Tanaka space.

(3). This holds, because for a countably bi-quotient map $f : X \to Y$, if (A_n) is a decreasing sequence in Y with $y \in \mathrm{cl}(A_n - \{y\})$, then $(f^{-1}(A_n))$ is a decreasing sequence in X with $x \in \mathrm{cl}((f^{-1}(A_n)) - \{x\})$ for some $x \in f^{-1}(y)$ ([21]).

(4). First, let Y be a q-space. To see that $X \times Y$ is a Tanaka space, let (A_n) be a decreasing sequence in $X \times Y$ with $(x, y) \in \mathrm{cl}\, A_n$. Since Y is a q-space, there exists a q-sequence $\{V_n : n \in \mathbb{N}\}$ of neighborhoods of y. Let $B_n = A_n \cap (X \times V_n)$ and let $p : X \times Y \to X$ denote the projection. Then $(p(B_n))$ is a decreasing sequence of subsets of X such that $x \in \mathrm{cl}\, p(B_n)$. But, Y is a Tanaka and q-space, so that Y is weakly sequential by Proposition 1(1). Thus, if $y_n \in V_n$, the sequence $\{y_n : n \in \mathbb{N}\}$ has a convergent subsequence. Hence, since X is a Tanaka space, there exist $p_n \in B_n$ such that $\{p_n : n \in \mathbb{N}\}$ converges to a point in $X \times Y$. Then $X \times Y$ is a Tanaka space. Next, let Y be a bi-quasi-k-space. Then Y is a bi-quotient image of a q-space T, where T is a subspace of $Y \times M$ for some metric space M by Characterization (A). Thus $Y \times M$ is a Tanaka space by the above. But, T is a q-space, hence an inner-one A and quasi-k-space. Thus T is a Tanaka space by (2). Since X and T are Tanaka spaces, and T is a q-space, $X \times T$ is a Tanaka space by the above. Since $X \times Y$ is a bi-quotient image of $X \times T$, $X \times Y$ is a Tanaka space by (3). □

Remark 9. (1) Let X be a Tanaka space. A subset S of X which is inner-one A need not be a Tanaka space by Remark 4(1). While, every sequential and inner-one A-space is a Tanaka space by Proposition 3(3). But, the author has the following question: *Is any sequential subset of a Tanaka space a Tanaka space* ?

(2) A quotient finite-to-one image X of a countable metric space need not be a Tanaka space (by taking X as the space S_2).

(3) A product of countable strongly Fréchet spaces need not be a Tanaka space. Indeed, there exists a countable strongly Fréchet space whose square is not Fréchet (cf. [16]), hence the square is not a Tanaka space by Proposition 7(1). Moreover, under (CH), there exist countable strongly Fréchet spaces X and Y such that $X \times Y$ is Fréchet, but $X \times Y$ is not strongly Fréchet ([20]), hence it is not a Tanaka space.

The following corollary is a consequence of Propositions 3(3) and 7, and Theorem 8. For (3), note that a subset of a sequential space is sequential if it is quasi-k (by Proposition 1(1) and (3)). (4) can be also obtained by means of [13, Theorem 3.3].

Corollary 10. (1) (i) *Let X be a Fréchet space, a hereditarily normal space, or a space with G_δ-points. Then X is strongly sequential if and only if it is strongly Fréchet.*

(ii) *Let X be a space with a point-countable wcs^*-network. Then X is strongly sequential if and only if it is first countable.*

(iii) *Let X be a closed image of a countably bi-quasi-k-space* (*resp. a closed image of an M-space*)*. If X is sequential, then X is strongly sequential if and only if it is countably bi-quasi-k* (*resp. q*)*.*

(iv) *Let X be a quotient Lindelöf image of a paracompact bi-quasi-k-space* (*resp. a quotient s-image of a meta-Lindelöf bi-quasi-k-space*)*. If X is sequential, then X is strongly sequential if and only if it is bi-k* (*resp. bi-quasi-k*)*.*

(2) *A hereditarily strongly sequential space is exactly strongly Fréchet.*

(3) *A subset of a strongly sequential space is strongly sequential if and only if it is quasi-k.*

(4) *A countably bi-quotient image of a strongly sequential space is strongly sequential.*

(5) *A countable product of spaces with G_δ-points is a Tanaka space if and only if it is strongly Fréchet* (*equivalently, strongly sequential*)*.*

Remark 11. A square X^2 can be sequential but not strongly sequential (even not a Tanaka space). For instance, if $X = S_\omega$ or S_2, then X^2 is known to be sequential by [9, (7.5)] but not a Tanaka space. Also, $X \times Y$ can be sequential but not a Tanaka space, even if X and Y are both strongly sequential (by Remark 9(3)). However, under these circumstances, the author has the following question: *Let X and Y be both* (*strongly*) *sequential. If $X \times Y$ is a Tanaka space, is $X \times Y$ sequential ?*

The square of a compact Fréchet space (hence, strongly Fréchet space) need not be Fréchet ([19]). But for a strongly Fréchet space X and a countably compact space Y, if $X \times Y$ is Fréchet, it is a strongly Fréchet ([16]). Also, the square of a countable strongly Fréchet space need not be quasi-k under $(2^{\aleph_0} < 2^{\aleph_1})$ ([18]). While, a product of a strongly Fréchet space with a sequential bi-quasi-k-space is sequential ([14]). More generally, the following theorem holds ((2)(i) should be compared to the above result in [14]).

Theorem 12. (1) (i) *If X is strongly Fréchet and Y is bi-sequential, then $X \times Y$ is* (*strongly*) *Fréchet. Conversely, if $X \times Y$ is Fréchet, then X is strongly Fréchet or Y is discrete.*

(ii) *Let X be Fréchet, and Y be bi-quasi-k. Then $X \times Y$ is Fréchet if and only if $X \times Y$ is strongly Fréchet, or Y is discrete.*

(2) (i) *Let X be strongly sequential, and let Y be sequential bi-quasi-k. Then*

$X \times Y$ is strongly sequential.

(ii) *Let X be sequential, and Y be a sequential bi-k-space (resp. a bi-k-space). Then $X \times Y$ is a sequential space (resp. a k-space) if and only if X is a Tanaka space, or Y is locally countably compact.*

Proof: (1). (i) is due to [10], here the latter part holds by the proof of [10, Proposition 4.D.5]. For (ii), let $X \times Y$ be Fréchet with Y non-discrete. Then X is strongly Fréchet by (a). Thus $X \times Y$ is a Tanaka space by Theorem 8(4) and Corollary 5. Then $X \times Y$ is strongly Fréchet by Proposition 7(1).

(2). For (i), $X \times Y$ is sequential by [27, Corollary 9], and it is a Tanaka space by means of Theorem 8(4). Then $X \times Y$ is strongly sequential by Proposition 3(3). Part (ii) holds by means of [27, Corollary 9 and Lemma 16]. □

Remark 13. In (1) in Theorem 12, we cannot replace "bi-sequential (or bi-quasi-k)" with "strongly Fréchet" by Remark 9(3). For (2), the author has the following question: *Let X be sequential, and Y be sequential bi-quasi-k. Does the conclusion in* (2)(ii) *remain true* ?

Theorem 14. (1) *Let X be weakly sequential. If X^ω is a quasi-k-space, then it is weakly sequential.*

(2) *Let X be weakly sequential and $t(X) \leq \omega$. If X^ω is a k-space (resp. a quasi-k-space), then X^ω (resp. X) is a Tanaka space.*

(3) *Let X be sequential. If X^ω is a quasi-k-space, then it is strongly sequential.*

Proof: (1). Since X^ω is quasi-k, it has a determining cover $\{K; K$ is countably compact in $X^\omega\}$. But, for each countably compact set K in X^ω, $K \subset \Pi K_i$ for some countably compact sets K_i in X. Thus X^ω has a determining cover $\mathcal{C} = \{\Pi K_i : K_i$ is countably compact in $X\}$. But, the countably compact sets K_i in X are sequentially compact by Proposition 1(1), then so is ΠK_i by [2, 3.10.35]. Hence, X^ω is weakly sequential.

(2). Since X^ω is quasi-k and $t(X) \leq \omega$, X is inner-one A by refering to [25, Theorem 4.13]. Thus, X is a Tanaka space by Proposition 3(2). Let $Y = X^\omega$. Then $Y^\omega (\cong X^\omega)$ is a k-space and Y is weakly sequential by (1). Then, it suffices to show that $t(Y) \leq \omega$, hence we show that any compact set K of Y has countable tightness by [25, Corollary 1.13(2)], for Y is a k-space. But, $K \subset \Pi K_i$ and the compact set ΠK_i has countable tightness in view of [2, 3.12.8], then K has countable tightness.

(3). Since X is sequential, by Proposition 1(3), any countably compact set in X is closed, hence sequential. Thus, by [15, Theorem 4.5], the elements ΠK_i of the determining cover $\mathcal{C}$ of X^ω in (1) are sequential since each K_i is countably compact sequential. Then X^ω is sequential, and thus X^ω is a Tanaka space by (2). Hence X^ω is strongly sequential by Proposition 3(3). □

Remark 15. In Theorem 14(2), "$t(X) \leq \omega$" is essential. Indeed, under (CH), there exists a (weakly sequential) space X which is an image of a locally compact Lindelöf space under a closed map f, and X^ω is a k-space, but X is not locally compact ([1]). Since X is not locally compact, the closed map f is not bi-quotient, hence not countably bi-quotient. Then X is not an A'-space by Lemma 6(1), hence not a Tanaka space.

The square of a countably compact space need not be a quasi-k-space ([10, Example 10.7]), nor an inner-one A-space by the proof there, using [11, Example 11.17]. But, the following holds.

Corollary 16. *Let X be a bi-quasi-k-space. Then the following hold.*

(1) *If X is a Tanaka space* (*equivalently, a weakly sequential space*), *then X^ω is a Tanaka* (*weakly sequential*) *bi-quasi-k-space.*

(2) *If X is sequential, then X^ω is a strongly sequential bi-quasi-k-space.*

Proof: (1). By Characterization (A), X is a bi-quotient image of an M-space S, where $S \subset X \times M$ for some metric space M. Since X is a Tanaka space, any countably compact set in X is sequentially compact by Proposition 1(1), hence so is any countably compact set in $X \times M$, hence in S. Thus, S^ω is an M-space by [22, Corollary 1]. While, X^ω is a bi-quotient image of S^ω. Therefore X^ω is bi-quasi-k, thus quasi-k. Then, by Theorem 14(1) X^ω is weakly sequential (equivalently, a Tanaka space by Corollary 5).

(2). This holds by Theorem 14(3), for X^ω is quasi-k by (1). □

Corollary 17. *Let X be sequential. Concerning equivalent relations among the following* (a)$\sim$(g), (1), (2), *and* (3) *below hold:*

(a) *X^ω is quasi-k;*
(b) *X^ω is sequential;*
(c) *X^ω is strongly sequential;*
(d) *X^ω is Fréchet;*
(e) *X^ω is strongly Fréchet;*
(f) *X^ω is countably bi-quasi-k;*
(g) *X is bi-quasi-k.*

(1) (a), (b), *and* (c) *are equivalent, and so are* (d) *and* (e).

(2) (i) *If X is a space with G_δ-points, then* (a)$\sim$(f) *are equivalent.*

(ii) *If X is a space with a point-countable wcs^*-network, then* (a)$\sim$(g) *are equivalent.*

(iii) *Suppose that X is a closed image of an M-space, a quotient Lindelöf image of a paracompact bi-quasi-k-space, or a quotient s-image of a meta-Lindelöf bi-quasi-k-space. Then* (a), (b), (c), (f) *and* (g) *are equivalent.*

In (ii) *and* (iii), *for* (g), *X is respectively first countable, q, bi-k if X is a space with a point-countable wcs^*-network, a closed image of an M-space, a quotient Lindelöf image of a paracompact bi-quasi-k-space.*

(3) *If* X *is a closed image of a normal bi-*k*-space, then* (a), (b), (c), *and* (f) *are equivalent.*

Proof: (1). (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) holds by Theorem 14(3). (d)$\Rightarrow$(e) holds by Corollary 10(1), for X^ω is strongly sequential and Fréchet.

(2). This holds by Corollaries 10 and 16(2).

(3). Let $f : S \to X$ be a closed map with S normal bi-k. Let X^ω be sequential (hence, X is strongly sequential). Since S is normal and X is inner-closed A, each $\partial f^{-1}(x)$ is countably compact by Lemma 6(1). Thus, we can assume that the map f is quasi-perfect (by the proof of Proposition 7(3)). Since S is a k-space and X^ω is sequential, by [22, Theorem 2], f^ω is quasi-perfect, hence countably bi-quotient. Then X^ω is a countably bi-quotient image of S^ω. Since S^ω is bi-k ([10]), X^ω is countably bi-k. $\square$

Remark 18. (1) Related to Corollary 17, it is also given in [13] that (b) implies X is strongly sequential. For a closed image X of a countably bi-quasi-k-space, (b) implies X is countably bi-quasi-k by Corollary 10(1), but the author has the following question: *Let* X *be a closed image of a countably bi-quasi-*k*-space. If* X^ω *is sequential, is* X^ω (*or* X) *bi-quasi-*k ?

(2) In Corollary 17, (c) need not imply (d). Indeed, there exists a compact Fréchet space X such that X^2 is not Fréchet ([19]), but X^ω is strongly sequential by Corollary 16(2). Also, (e) need not imply (g). Indeed, under (CH) there exists a countable space X such that X^ω is strongly Fréchet, but X is not bi-sequential ([17]), hence not bi-quasi-k by [10, Theorem 7.3].

(3) A product X^ω need not be a Tanaka or quasi-k-space even if any finite product X^n is strongly Fréchet. Indeed, under (MA) there exists a countable space X such that any X^n is strongly Fréchet, but X^ω is not Fréchet ([4]), hence not a Tanaka or quasi-k-space by Corollaries 10(5) and 17.

Finally, we give the following questions on products. (1), (2), (3), and (4) relate respectively to Theorem 12(2), Theorem 14 with Remark 15, Corollary 16 with Remark 18(1), and Remark 18(3).

Question 19. (1) Let X be a Tanaka k-space, and Y be a metric or bi-k-space. Is $X \times Y$ a k-space ?

(2) Let X^ω be a k-space with X a Tanaka space. Is X^ω a Tanaka or A'-space ?

(3) Let X^ω be sequential. Is X^ω (or X) countably bi-quasi-k ?

(4) Let X^ω be sequential with any X^n $(n \in \mathbb{N})$ Fréchet. Is X^ω Fréchet ?

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