A categorical account of the localic closed subgroup theorem

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Abstract. Given an axiomatic account of the category of locales the closed subgroup theorem is proved. The theorem is seen as a consequence of a categorical account of the Hofmann-Mislove theorem. The categorical account has an order dual providing a new result for locale theory: every compact subgroup is necessarily fitted.

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1. Introduction

The closed subgroup theorem in locale theory ([IKPR]) is remarkable as it shows a clear distinction between locale theory and ordinary point set topology. It is not the case that every subgroup of a topological group is closed; it is the case that every subgroup of a localic group is a closed sublocale. In its proof the localic subgroup theorem uses the excluded middle. Since a motivation for locale theory is to develop a theory that is true in any topos of sheaves (i.e. one that is slice stable) it is natural that a version of the closed subgroup theorem was developed that did not depend on the excluded middle. The theorem became: for any localic subgroup $H \leq G$ then provided H is open as a locale, the subgroup inclusion is weakly closed. In the presence of the excluded middle all locales are open and weakly closed is the same condition as closed and so the original result is recovered. In a recent paper ([T05]) an axiomatic account of the category of locales is developed. Since this axiomatic account is strong enough to discuss the relevant concepts (open maps, weakly closed) an investigation as to whether the closed subgroup theorem held seemed worthwhile. The purpose of this paper is to show that indeed the closed subgroup does hold. Further, what was found is (a) the closed subgroup theorem is a consequence of a categorical account of the Hofmann-Mislove theorem and (b) the theorem has an order dual which is that every compact localic subgroup is fitted.

Let us outline the contents of the paper. In the next section we recall various axioms that are to be put on a category $\mathcal{C}$ so as to make it behave like the category of locales. Next we discuss how to describe weak-closure of a subobject in such a category and develop some lemmas about how this closure operation works (for example, relative denseness is pullback stable). The next section proves the (weakly) closed subgroup theorem. The final section describes the order duality which exists with the axiomatic approach offered, this leads to an interpretation of the order dual of the closed subgroup theorem in the case $\mathcal{C} = \mathbf{Loc}$, the category of locales.

The reader is assumed to be familiar with lattice theory and category theory (see [J82] and [MacL71] respectively). Since the proof is axiomatic the reader does not necessarily need to fully understand locale theory though they must be aware that the category of locales is a category set up to behave like the familiar category of topological spaces.

2. Categorical axioms for locales

Here are the categorical axioms that we will place on a category $\mathcal{C}$ so that it behaves like a category of locales.

I. $\mathcal{C}$ is order enriched, has finite limits, finite coproducts and finite products distribute over coproducts,

II. there exists an order internal distributive lattice, the *Sierpiński object*, $\mathbb{S}$, such that for any morphism $\alpha : \mathbb{S}^X \to \mathbb{S}$

$$\sqcap_{\mathbb{S}}(\alpha \times \mathrm{Id}) \sqsubseteq \alpha \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^{!^X}),$$

III. for any two objects X and Y of $\mathcal{C}$ the map $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \xrightarrow{\mathbb{S}^{\pi_1} \times \mathbb{S}^{\pi_2}} \mathbb{S}^{X \times Y} \times \mathbb{S}^{X \times Y} \xrightarrow{\sqcap} \mathbb{S}^{X \times Y}$ is a universal join bilinear map,

IV. there is a KZ-monad (P_L, η_L, μ_L) on $\mathcal{C}$, the *lower power monad*, such that there exists a natural order isomorphism,

$$\mathcal{C}(Y, P_L(X)) \cong \{\alpha : \mathbb{S}^X \to \mathbb{S}^Y \mid \alpha \text{ is a join semilattice homomorphism}\},$$

V. for any equalizer $E \overset{e}{\hookrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{C}$ and any join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Z$ such that

$$\alpha \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^f) = \alpha \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^g)$$

there exists a unique join semilattice homomorphism β such that $\beta \mathbb{S}^e = \alpha$; and,

VI. the map $f \longmapsto \mathbb{S}^f$ reflects isomorphisms.

We do not make the assumption that $\mathbb{S}^X$ exists as an object of $\mathcal{C}$. The object $\mathbb{S}^X$ is the presheaf

$$\mathcal{C}(\_ \times X, \mathbb{S}) : \mathcal{C}^{op} \to \mathbf{Set}$$

and so, for example, the morphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Z$ is a natural transformation. Since join semilattice homomorphisms $\mathbb{S}^Y \to \mathbb{S}^X$ are in order isomorphism with suplattice homomorphisms $\Omega Y \to \Omega X$ in the case $\mathcal{C} = \mathbf{Loc}$ (see [TV02]) all of these axioms are elementary observations from locale theory (Chapter II of [J82], or [V89]).

To proceed we are going to need to use topological definitions relative to the category $\mathcal{C}$.

Definition 1. (a) A morphism $f : X \to Y$ of $\mathcal{C}$ is *open* if and only if $\mathbb{S}^f : \mathbb{S}^Y \to \mathbb{S}^X$ has a left adjoint $\exists_f : \mathbb{S}^X \to \mathbb{S}^Y$ such that the *Frobenius condition*

$$\sqcap_{\mathbb{S}^Y}(\exists_f \times \mathrm{Id}) = \exists_f \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^f)$$

holds,

(b) an object X of $\mathcal{C}$ is *open* if and only if $!^X : X \to 1$ is open,

(c) $i : X_0 \hookrightarrow X$ is a *subobject* provided it is a regular monomorphism; and,

(d) a subobject $i : X_0 \hookrightarrow X$ is weakly closed provided if it is a lax equalizer of a diagram $f, g : X \rightrightarrows Y$ universally setting $g \sqsubseteq f$ where f factors via the terminal object 1.

See [JT84] for background on open maps in locales. Trivially the composition of two open maps is open. If $f : X \to Y$ is an open morphism and is a split epimorphism (split by $i : Y \to X$ say) then $\exists_f \mathbb{S}^f = \mathrm{Id}$. This is because $\mathbb{S}^f \exists_f \mathbb{S}^f = \mathbb{S}^f$ for any adjunction and $\mathbb{S}^f$ is a split monomorphism (split by $\mathbb{S}^i$). We will call upon this in the proof of the main theorem. Further note that by Axiom II it is sufficient to construct a left adjoint to $\mathbb{S}^{!^X} : \mathbb{S} \to \mathbb{S}^X$ to prove that an object X is open.

It can be shown axiomatically, as in the localic case, that open maps are pullback stable and the Beck-Chevalley condition holds for the pullback square. There is a uniqueness aspect to this assertion which is that given any pullback square

$$\begin{tikzcd} X \times_Y Z \arrow[r, "p_2"] \arrow[d, "f^*p"'] & Z \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in $\mathcal{C}$ with p an open morphism then there exists a unique join semilattice homomorphism $\beta : \mathbb{S}^{X \times_Y Z} \to \mathbb{S}^X$ which satisfies the Frobenius condition for f^*p and for which the Beck-Chevalley condition

$$\beta \mathbb{S}^{p_2} = \mathbb{S}^f \exists_p$$

holds. This uniqueness assertion follows by an application of Axiom V, treating the pullback as an equalizer in the usual manner.

In order to avoid a detailed exposition on the axiomatic approaches available when studying $\mathbf{Loc}$ we will rely, without proof, on the following result which is proved axiomatically in [T05].

Theorem 2. *For any object X, $\mathcal{C}(1, P_L X)$ is naturally order isomorphic to $\{X_0 \hookrightarrow X \mid X_0$ open and $X_0 \hookrightarrow X$ a weakly closed subobject$\}$.*

This result is an order dualization (see Section 5) of a categorical proof of the Hofmann-Mislove theorem, see [T05] for details. The order isomorphism of this theorem is found, in one direction, by sending $i : X_0 \hookrightarrow X$ to the map

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S}$$

and applying Axiom IV. In the other direction any morphism $p : 1 \to P_L X$ gives rise to a lax equalizer setting $p!^X \sqsupseteq (\eta_L)_X$ which is, by definition, weakly closed.

The (weakly) closed subgroup theorem relative to $\mathcal{C}$ will be established by showing that any subgroup, which is open as a subobject, is isomorphic to its weak closure. To make this work we are going to need to develop the notion of weak closure and so will need to work out what it means, axiomatically, for a subobject to be dense. This is the subject of the next section.

3. Dense subobjects

In this section we propose a new definition of dense sublocale. In the presence of the excluded middle the new definition is exactly the usual definition. It is shown, axiomatically, that any subobject $X_0 \hookrightarrow X$ in $\mathcal{C}$ with X_0 open, factors uniquely as a dense subobject followed by a weakly closed subobject. The remainder of the section proves basic results about this weak closure operation.

Recall that the data for a locale map $f : X \to Y$ is given by a frame homomorphism $\Omega f : \Omega Y \to \Omega X$ [J82]. A locale map is *dense* if and only if $\Omega f(a) = 0$ implies $a = 0$ for every $a \in \Omega X$. For a category $\mathcal{C}$ satisfying the axioms the following definition is proposed:

Definition 3. A morphism $f : X \to Y$ of $\mathcal{C}$ is *dense* if and only if X and Y are open and

$$\begin{tikzcd} \mathbb{S}^X \arrow[dr, "\exists_X"'] & \mathbb{S}^Y \arrow[l, "\mathbb{S}^f"'] \arrow[d, "\exists_Y"] \\ & \mathbb{S} \end{tikzcd}$$

commutes.

Whilst we have given the definition in terms of a general map, we shall not be looking at situations where f is not a subobject. It is immediate that the composition of two dense morphisms is dense.

Lemma 4. *When $\mathcal{C} = \mathbf{Loc}$ and the excluded middle is true, the two definitions of dense are equivalent.*

PROOF: If the excluded middle is true then all locales are open since for any locale X the map $\exists_X : \Omega X \to \Omega$, defined by $\exists_X(a) = 0$ if and only if $a = 0$, provides a left adjoint for $\Omega!^X$. Verifying this lemma then becomes trivial. $\square$

Enough definitions have now been given to enable us to state and prove the main technical result.

Proposition 5. *If $\mathcal{C}$ is a category satisfying the axioms then any subobject $i : X_0 \hookrightarrow X$, with X_0 open, factors uniquely as*

$$X_0 \overset{j}{\hookrightarrow} \overline{X_0} \overset{\bar{i}}{\hookrightarrow} X$$

where j is dense and $\bar{i}$ is weakly closed.

PROOF: Let $\overline{X_0} \overset{\bar{i}}{\hookrightarrow} X$ be the weakly closed subobject of X corresponding to the point of $P_L(X)$ given by the join semilattice homomorphism,

$$\alpha : \mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S}.$$

This weakly closed subobject exists by Theorem 2 and further, by that theorem, $\overline{X_0}$ is an open object. $\overline{X_0} \overset{\bar{i}}{\hookrightarrow} X$ is constructed as the lax equalizer setting $p_\alpha !^X \sqsupseteq (\eta_L)_X$ where

$$p_\alpha : 1 \longrightarrow P_L(X)$$

is the point corresponding to α (Axiom IV). To prove that i factors via $\bar{i}$ we must check that

[A] $$p_\alpha !^X i \sqsupseteq (\eta_L)_X i.$$

Note that $!^X i = !^{X_0}$ and that $(\eta_L)_X$ must correspond to the identity $\mathrm{Id} : \mathbb{S}^X \longrightarrow \mathbb{S}^X$ via Axiom IV. So to check [A], by Axiom IV this amounts to checking that $\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0}$ is less than

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S} \xrightarrow{\mathbb{S}^{!^{X_0}}} \mathbb{S}^{X_0}$$

which is immediate as $\exists_{X_0}$ is left adjoint to $\mathbb{S}^{!^{X_0}}$. Therefore there exists unique $j : X_0 \hookrightarrow \overline{X_0}$ such that $\bar{i}j = i$. Now the proof of Theorem 2 (see the comments above after the statement of the theorem) shows that

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^{\bar{i}}} \mathbb{S}^{\overline{X_0}} \xrightarrow{\exists_{\overline{X_0}}} \mathbb{S}$$

equals

$$\alpha : \mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S}$$

and so, since $\mathbb{S}^{\overline{i}}$ is epimorphic (Axiom V) and $\mathbb{S}^j\mathbb{S}^{\overline{i}} = \mathbb{S}^i$ we obtain that

$$\begin{tikzcd}
\mathbb{S}^{X_0} \arrow[dr, "\exists_{X_0}"'] & \mathbb{S}^{\overline{X_0}} \arrow[l, "\mathbb{S}^j"'] \arrow[d, "\exists_{\overline{X_0}}"] \\
& \mathbb{S}
\end{tikzcd}$$

commutes. Since we have observed already that $\overline{X_0}$ is open this is sufficient to prove that j is dense.

Let us now tackle the uniqueness assertion. Say that i also factors is

$$X_0 \overset{j'}{\hookrightarrow} \overline{X_0'} \overset{\overline{i'}}{\hookrightarrow} X.$$

What we will show is that if j' is dense then $\overline{i'}$ factors via $\overline{i}$; this more general assertion will be of use later. Next we will show that if $\overline{i'}$ is weakly closed (with open domain) then $\overline{i}$ factors via $\overline{i'}$; this is sufficient to establish the uniqueness assertion of the theorem.

Say j' is dense. To prove that $\overline{i'}$ factors via $\overline{i}$ we must check that

$$p_\alpha !^X \overline{i'} \sqsupseteq (\eta_L)_X \overline{i'}$$

which amounts to checking that $\mathbb{S}^X \xrightarrow{\mathbb{S}^{\overline{i'}}} \mathbb{S}^{\overline{X_0'}}$ is less than

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S} \xrightarrow{\mathbb{S}^{!^{\overline{X_0'}}}} \mathbb{S}^{\overline{X_0'}}.$$

But $\mathbb{S}^i = \mathbb{S}^{j'}\mathbb{S}^{\overline{i'}}$ and as j' is dense we have that $\exists_{X_0}\mathbb{S}^{j'} = \exists_{\overline{X_0'}}$ and so this is immediate as $\exists_{\overline{X_0'}}$ is left adjoint to $\mathbb{S}^{!^{\overline{X_0'}}}$.

On the other hand, say $\overline{i'}$ is weakly closed with open domain. Then via Theorem 2 it is the lax equalizer setting

$$p_{\alpha'} !^X \sqsupseteq (\eta_L)_X$$

where α' is the map $\mathbb{S}^X \xrightarrow{\mathbb{S}^{\overline{i'}}} \mathbb{S}^{\overline{X_0'}} \xrightarrow{\exists_{\overline{X_0'}}} \mathbb{S}$. So to prove that $\overline{i}$ factors via $\overline{i'}$ we must check that

$$p_{\alpha'} !^X \overline{i} \sqsupseteq (\eta_L)_X \overline{i}$$

which amounts to checking that $\mathbb{S}^X \xrightarrow{\mathbb{S}^{\bar{i}}} \mathbb{S}^{\overline{X_0}}$ is less than

[B] $$\mathbb{S}^X \xrightarrow{\mathbb{S}^{\overline{i'}}} \mathbb{S}^{\overline{X'_0}} \xrightarrow{\exists_{\overline{X'_0}}} \mathbb{S} \xrightarrow{\mathbb{S}^{!_{\overline{X_0}}}} \mathbb{S}^{\overline{X_0}}.$$

Since j' is dense $\mathbb{S}^{\overline{X'_0}} \xrightarrow{\exists_{\overline{X'_0}}} \mathbb{S}$ factors as $\mathbb{S}^{\overline{X'_0}} \xrightarrow{\mathbb{S}^{j'}} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S}$ and so [B] is equal to

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^{\bar{i}}} \mathbb{S}^{\overline{X_0}} \xrightarrow{\mathbb{S}^{j}} \mathbb{S}^{X_0} \xrightarrow{\exists_{X_0}} \mathbb{S} \xrightarrow{\mathbb{S}^{!_{\overline{X_0}}}} \mathbb{S}^{\overline{X_0}}$$

since $\overline{i'}j' = i = \bar{i}j$. But then by denseness of j we have that [B] equals

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^{\bar{i}}} \mathbb{S}^{\overline{X_0}} \xrightarrow{\exists_{\overline{X_0}}} \mathbb{S} \xrightarrow{\mathbb{S}^{!_{\overline{X_0}}}} \mathbb{S}^{\overline{X_0}}$$

and so we are done, again by the fact that $\exists_{\overline{X_0}}$ is left adjoint. □

It is natural to make the following definition for relative denseness.

Definition 6. Given morphisms $j : X_0 \longrightarrow X$ and $f : X \to Y$ in $\mathcal{C}$, we say that j is *dense over* f if and only if both f and fj are open maps and the diagram

$$\begin{tikzcd} \mathbb{S}^{X_0} \arrow[dr, "\exists_{fj}"'] & \mathbb{S}^X \arrow[l, "\mathbb{S}^j"'] \arrow[d, "\exists_f"] \\ & \mathbb{S}^Y \end{tikzcd}$$

commutes.

With a slice stable account of $\mathcal{C}$ (see [T05]) this definition is equivalent to the requirement that j is dense in the category $\mathcal{C}/Y$. Clearly, if $j : X_0 \longrightarrow X$ is dense over some $f : X \to Y$ and $g : Y \to Z$ is some other open morphism then $j : X_0 \longrightarrow X$ is dense over $gf : X \to Z$. Notice therefore that if $j : X_0 \longrightarrow X$ is dense over any $f : X \to Y$ and Y is open, then j is dense.

Let us now check a lemma which is trivial but key to the main theorem below:

Lemma 7. *Given morphisms $X_0 \xrightarrow{j} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ in $\mathcal{C}$ together with automorphisms $\phi : X \to X$ and $\phi_0 : X_0 \to X_0$ such that $\phi j = j\phi_0$ and $f\phi = g$ then j is dense over f if and only if it is dense over g.*

Proof: If f is open then so is g since ϕ is an isomorphism; $\exists_g = \exists_f \exists_\phi$ where $\exists_\phi = \mathbb{S}^{\phi^{-1}}$. If fj is open then so is gj since $gj = fj\phi_0$ and ϕ_0 is an isomorphism.

Therefore also $\exists_{gj} = \exists_{fj}\mathbb{S}^{\phi_0^{-1}}$ and so

$$\begin{aligned}
\exists_{gj}\mathbb{S}^j &= \exists_{fj}\mathbb{S}^{\phi_0^{-1}}\mathbb{S}^j \\
&= \exists_{fj}\mathbb{S}^j\mathbb{S}^{\phi^{-1}} \\
&= \exists_f\mathbb{S}^{\phi^{-1}} \\
&= \exists_g,
\end{aligned}$$

the second last line based on an assumption that j is dense over f. This shows that j is dense over g given the assumption that it is dense over f.

The 'if' way round follows symmetrically. $\square$

To end the section here are two technical lemmas on the relative notion of denseness just introduced: (i) it is pullback stable and (ii) the intersection of two dense subobjects is dense.

Lemma 8. *We are given $j : A_0 \longrightarrow A$ and $f : A \to Y$ and a morphism $p : Z \to Y$ in $\mathcal{C}$ and consider the pullback squares*

$$\begin{tikzcd}
B_0 \arrow[r, "k"] \arrow[d, "g"'] & B \arrow[r, "g"] \arrow[d, "r"] & Z \arrow[d, "p"] \\
A_0 \arrow[r, "j"] & A \arrow[r, "f"] & Y
\end{tikzcd}$$

Then if j is dense over f, k is dense over g.

Proof: By the pullback stability of open morphisms we have, under the assumption that j is dense over f, both g and gk are open. It remains to prove that

$$\begin{tikzcd}
\mathbb{S}^{B_0} \arrow[dr, "\exists_{gk}"'] & \mathbb{S}^B \arrow[l, "\mathbb{S}^k"'] \arrow[d, "\exists_g"] \\
& \mathbb{S}^Y
\end{tikzcd}$$

commutes. However, by the uniqueness of morphisms satisfying Beck-Chevalley on the right hand pullback square, it is sufficient to verify that $\exists_{gk}\mathbb{S}^k$ satisfies the Frobenius condition on g and the Beck-Chevalley condition. Both assertions are easy. For the Frobenius condition note that $\mathbb{S}^k$ is always a meet semilattice homomorphism as it is a distributive lattice homomorphism. For the Beck-Chevalley condition:

$$\begin{aligned}
(\exists_{gk}\mathbb{S}^k)\mathbb{S}^r &= (\exists_{gk}\mathbb{S}^q)\mathbb{S}^j \\
&= (\mathbb{S}^p\exists_{fj})\mathbb{S}^j \\
&= \mathbb{S}^p\exists_f
\end{aligned}$$

where the second last line is using Beck-Chevalley on the outer square and the last line is by denseness of j over f. □

Lemma 9. *Say $f : X \to Y$ is a morphism in $\mathcal{C}$. Then if $X_0 \hookrightarrow X$ and $X_1 \hookrightarrow X$ are two subobjects of X, both dense over f, then their intersection is a dense subobject over f.*

Proof: By definition of subobject we have two equalizers

$$X_0 \overset{i_{X_0}}{\hookrightarrow} X \underset{a_2}{\overset{a_1}{\rightrightarrows}} A \tag{x}$$

and

$$X_1 \overset{i_{X_1}}{\hookrightarrow} X \underset{b_2}{\overset{b_1}{\rightrightarrows}} B \tag{y}$$

and therefore the intersection of X_0 and X_1 is found by constructing the equalizer

$$X_0 \cap X_1 \overset{i_{X_0 \cap X_1}}{\hookrightarrow} X \underset{(a_2,b_2)}{\overset{(a_1,b_1)}{\rightrightarrows}} A \times B.$$

Notice that we can find a join semilattice homomorphism $\alpha : \mathbb{S}^{X_0 \cap X_1} \to \mathbb{S}^Y$ satisfying

$$\begin{tikzcd} \mathbb{S}^{X_0 \cap X_1} \arrow[dr, "\alpha"'] & \mathbb{S}^X \arrow[l, "\mathbb{S}^{i_{X_0 \cap X_1}}"'] \arrow[d, "\exists_f"] \\ & \mathbb{S}^Y \end{tikzcd}$$

by applying Axiom V since it can be verified that $\exists_f : \mathbb{S}^X \to \mathbb{S}^Y$ composes equally with

$$\mathbb{S}^X \times \mathbb{S}^{A \times B} \underset{\sqcap(\mathrm{Id} \times \mathbb{S}^{(a_2,b_2)})}{\overset{\sqcap(\mathrm{Id} \times \mathbb{S}^{(a_1,b_1)})}{\rightrightarrows}} \mathbb{S}^X$$

by using Axiom V on the equalizers (x) and (y), and applying Axiom III. To complete it therefore remains to check that $fi_{X_0 \cap X_1}$ is open with $\exists_{fi_{X_0 \cap X_1}} = \alpha$. We clearly have that $\alpha \mathbb{S}^{fi_{X_0 \cap X_1}} \sqsubseteq \mathrm{Id}$ since $\exists_f \mathbb{S}^f \sqsubseteq \mathrm{Id}$. Also $\mathbb{S}^{fi_{X_0 \cap X_1}} \alpha \mathbb{S}^{i_{X_0 \cap X_1}} \sqsupseteq \mathbb{S}^{i_{X_0 \cap X_1}}$ since $\mathbb{S}^f \exists_f \sqsupseteq \mathrm{Id}$ and so $\mathbb{S}^{fi_{X_0 \cap X_1}} \alpha \sqsupseteq \mathrm{Id}$ by application of the uniqueness part of Axiom V. Therefore α is left adjoint to $\mathbb{S}^{fi_{X_0 \cap X_1}}$ and it remains only to check the Frobenius condition. In fact since $\mathbb{S}^{i_{X_0 \cap X_1} \times \mathrm{Id}}$ is a surjection (Axiom V again), and using the fact that $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \to \mathbb{S}^{X \times Y}$ is universal bilinear, it is sufficient to check

$$\alpha \sqcap (\mathbb{S}^{i_{X_0 \cap X_1}} \times \mathbb{S}^{i_{X_0 \cap X_1}} \mathbb{S}^f) = \sqcap(\alpha \mathbb{S}^{i_{X_0 \cap X_1}} \times \mathrm{Id})$$

which is immediate. □

4. Closed subgroup theorem

The weak closure operation in $\mathcal{C}$ extends to the category of internal monoids in $\mathcal{C}$. Once this is established in the next lemma it will be trivial to extend the result to groups. But it will then become clear, for groups, that the weak closure operation is trivial: all subgroups, that are open as objects, are already weakly closed.

Lemma 10. *If $(M, *_M, e)$ is an internal monoid in $\mathcal{C}$ and $i_{M_0} : M_0 \hookrightarrow M$ is a submonoid with M_0 an open object, then $\overline{i_{M_0}} : \overline{M_0} \hookrightarrow M$ is a submonoid of M.*

Proof: By pulling back $M_0 \overset{j}{\hookrightarrow} \overline{M_0} \xrightarrow{!^{\overline{M_0}}} 1$ along itself we see that

$$M_0 \times M_0 \xhookrightarrow{j \times \mathrm{Id}} \overline{M_0} \times M_0$$

is dense over $\pi_2 : \overline{M_0} \times M_0 \to M_0$ by Lemma 8. It is therefore dense. By pulling back $M_0 \overset{j}{\hookrightarrow} \overline{M_0} \xrightarrow{!^{\overline{M_0}}} 1$ along $\overline{M_0} \xrightarrow{!^{\overline{M_0}}} 1$ we see that

$$\overline{M_0} \times M_0 \xhookrightarrow{\mathrm{Id} \times j} \overline{M_0} \times \overline{M_0}$$

is dense over $\pi_1 : \overline{M_0} \times \overline{M_0} \to \overline{M_0}$. It is therefore dense. Hence $M_0 \times M_0 \xhookrightarrow{j \times j} \overline{M_0} \times \overline{M_0}$ is dense as it is the composition of two dense morphisms. This is then sufficient, by use of Proposition 5, to show that there exists $*_{\overline{M_0}}$ making the diagram

$$\begin{tikzcd}
M_0 \times M_0 \arrow[r, hook, "j \times j"] \arrow[d, "*_{M_0}"'] & \overline{M_0} \times \overline{M_0} \arrow[r, hook, "\overline{i_{M_0}} \times \overline{i_{M_0}}"] \arrow[d, "*_{\overline{M_0}}"'] & M \times M \arrow[d, "*_M"] \\
A_0 \arrow[r, hook, "j"] & \overline{M_0} \arrow[r, hook, "\overline{i_{M_0}}"] & M
\end{tikzcd}$$

commute. □

We can now prove the main result.

Theorem 11. *If G is an internal group in $\mathcal{C}$ and $i_H : H \hookrightarrow G$ is a subgroup with H an open object in $\mathcal{C}$ then i_H is weakly closed.*

Proof: Firstly the previous lemma extends to groups; since $j : H \hookrightarrow \overline{H}$ is dense the map

$$\overline{H} \xhookrightarrow{\overline{i_H}} G \xrightarrow{\mathrm{inv}_G} G$$

factors through $\overline{H} \overset{\overline{i_H}}{\hookrightarrow} G$ where inv_G is the group inverse. This is by application of Proposition 5.

Next consider the pullbacks

$$\begin{tikzcd}
H \times \overline{H} \arrow[r, hook, "j \times \mathrm{Id}"] \arrow[d, "\pi_1"'] & \overline{H} \times \overline{H} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \overline{H} \arrow[d] \\
H \arrow[r, hook, "j"] & \overline{H} \arrow[r] & 1
\end{tikzcd}$$

By Lemma 8, $H \times \overline{H} \overset{j\times \mathrm{Id}}{\hookrightarrow} \overline{H} \times \overline{H}$ is dense over $\pi_2 : \overline{H}\times\overline{H} \to \overline{H}$. Symmetrically $H \times \overline{H} \overset{\mathrm{Id}\times j}{\hookrightarrow} \overline{H} \times \overline{H}$ is dense over $\pi_1 : \overline{H} \times \overline{H} \to \overline{H}$. Now, finally using the fact that we have a group,

$$(\pi_1, *_{\overline{H}}) : \overline{H} \times \overline{H} \to \overline{H} \times \overline{H}$$

is an automorphism with $\pi_2(\pi_1, *_{\overline{H}}) = *_{\overline{H}}$ and

$$(*_{\overline{H}}, \pi_2) : \overline{H} \times \overline{H} \to \overline{H} \times \overline{H}$$

is an automorphism with $\pi_1(*_{\overline{H}}, \pi_2) = *_{\overline{H}}$. (Indeed a monoid is a group if and only if these maps are automorphisms.) The inverses are given by $(\pi_1, *_{\overline{H}}(\mathrm{inv}_{\overline{H}}\pi_1, \pi_2))$ and $(*_{\overline{H}}(\pi_1, \mathrm{inv}_{\overline{H}}\pi_2), \pi_2)$ respectively. Further $(j \times \mathrm{Id})(\pi_1, *_{\overline{H}}) = (\pi_1, *_{\overline{H}})(j \times \mathrm{Id})$ and so by Lemma 7 $H \times \overline{H} \overset{j\times \mathrm{Id}}{\hookrightarrow} \overline{H} \times \overline{H}$ is dense over $*_{\overline{H}}$. Also $(\mathrm{Id}\times j)(*_{\overline{H}}, \pi_2) = (*_{\overline{H}}, \pi_2)(\mathrm{Id}\times j)$ and so by the same lemma, $\overline{H} \times H \overset{\mathrm{Id}\times j}{\hookrightarrow} \overline{H} \times \overline{H}$ is dense over $*_{\overline{H}}$. It follows by the stability of relative denseness proved above that $H \times H \overset{j\times j}{\hookrightarrow} \overline{H} \times \overline{H}$ is dense over $*_{\overline{H}}$ as it is the intersection of $\overline{H} \times H \overset{\mathrm{Id}\times j}{\hookrightarrow} \overline{H} \times \overline{H}$ and $H \times \overline{H} \overset{j\times \mathrm{Id}}{\hookrightarrow} \overline{H} \times \overline{H}$.

Intuitively we are now done since $*_{\overline{H}}$ is a split epimorphism and so the denseness assertion just obtained shows that $*_{\overline{H}}(j \times j)$ is epimorphic which is sufficient as this morphism factors through $j : H \hookrightarrow \overline{H}$, a monomorphism, so forcing it to be an isomorphism.

Let us furnish the details. As $*_{\overline{H}}$ is a split epimorphism and open we have that $\exists_{*_{\overline{H}}}\mathbb{S}^{*_{\overline{H}}} = \mathrm{Id}$. Recalling that of course we have a commuting square,

$$\begin{tikzcd}
H \times H \arrow[r, hook, "j\times j"] \arrow[d, "*_H"'] & \overline{H} \times \overline{H} \arrow[d, "*_{\overline{H}}"] \\
H \arrow[r, hook, "j"] & \overline{H}
\end{tikzcd}$$

then,

$$\begin{aligned}(\exists_{*_{\overline{H}}(j\times j)}\mathbb{S}^{*_H})\mathbb{S}^{j} &= \exists_{*_{\overline{H}}(j\times j)}\mathbb{S}^{j\times j}\mathbb{S}^{*_{\overline{H}}}\\ &= \exists_{*_{\overline{H}}}\mathbb{S}^{*_{\overline{H}}}\\ &= \mathrm{Id}\,.\end{aligned}$$

But then we also have, $\mathbb{S}^{j}(\exists_{*_{\overline{H}}(j\times j)}\mathbb{S}^{*_H})\mathbb{S}^{j} = \mathbb{S}^{j}$ and so $\mathbb{S}^{j}(\exists_{*_{\overline{H}}(j\times j)}\mathbb{S}^{*_H}) = \mathrm{Id}$ applying Axiom V, available since j is a regular monomorphism. $\mathbb{S}^{j}$ is therefore an isomorphism and we are done by Axiom VI. □

5. The order dual result

Not only does the category **Loc** satisfy the axioms, but so too does $\mathbf{Loc}^{co}$, the category of locales with reversed order enrichment. Axiom I is clearly order dual since it does not say anything about the order enrichment other than that it exists. Axiom II becomes, “for any meet semilattice homomorphism, $\alpha : \mathbb{S}^X \to \mathbb{S}$,

$$\alpha \sqcup_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^{!^X}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times \mathrm{Id}).\text{”}$$

This can be verified as morphisms $\mathbb{S}^X \to \mathbb{S}$ are in bijection with Scott continuous maps (i.e. dcpo homomorphisms) $\Omega X \to \Omega$ ([TV02]) and $\Omega!^X : \Omega \to \Omega X$ is given by

$$i \longmapsto \bigvee\nolimits^{\uparrow}\{0_{\Omega X}\} \cup \{1_{\Omega X} \mid i = 1_{\Omega}\}.$$

Axiom III becomes the assertion that the map $\odot : \mathbb{S}^X \times \mathbb{S}^Y \xrightarrow{\mathbb{S}^{\pi_1}\times\mathbb{S}^{\pi_2}} \mathbb{S}^{X\times Y} \times \mathbb{S}^{X\times Y} \xrightarrow{\sqcup} \mathbb{S}^{X\times Y}$ is a universal meet bilinear map. This is exactly the preframe tensor description of locale product made explicit in [JV91].

That there is a coKZ-monad (P_U, η_U, μ_U) enjoying

$$\mathbf{Loc}(Y, P_U(X)) \cong \{\alpha : \mathbb{S}^X \to \mathbb{S}^Y \mid \alpha \text{ is a meet semilattice homomorphism}\},$$

is the well known upper power locale construction (e.g. Chapter 11 of [V89]).

Axiom V, applied to $\mathbf{Loc}^{co}$, is a consequence of the preframe coverage theorem of [JV91] and Axiom VI is clear as it is order dual.

So we can immediately deduce by the main theorem that if $i^H : H \hookrightarrow G$ is a subgroup in **Loc** (i.e. exactly a subgroup in $\mathbf{Loc}^{co}$) and $!^H : H \to 1$ is open in $\mathbf{Loc}^{co}$ then i^H is weakly closed relative to $\mathbf{Loc}^{co}$. Now $!^H$ is open in $\mathbf{Loc}^{co}$ if and only if $\mathbb{S}^{!^H}$ has a right adjoint, and this is well known to be equivalent to compactness as such a right adjoint corresponds to a dcpo homomorphism. Of course i^H is weakly closed relative to $\mathbf{Loc}^{co}$ if an only if it is a lax equalizer of a diagram $f, g : X \rightrightarrows Y$ universally setting $f \sqsubseteq g$ where f factors via the terminal object 1. But this condition is exactly *fitted*, see [I72], where a sublocale is fitted if and only if it is an intersection of open sublocales. So for no extra work we get what appears to be a new result for locale theory:

Theorem 12. *Any compact subgroup of a localic group is fitted.*

6. Final comments

With a categorical account of locales we are able to prove the closed subgroup theorem. That the closed subgroup theorem can be shown by using essentially categorical arguments is clear from the proof offered as Theorem C5.3.1 in [J02] (indeed there we see a result about groupoids which is also recoverable using our framework as the category of locales is slice stable). So that a categorical proof of the main theorem exists is not overly surprising. What is new is the use of the order dual of the Hofmann-Mislove to provide the appropriate factorization system for subobjects with open domains. It is really the factorization system of Proposition 5 (and by implication our chosen definition of dense) that is new. With this we are showing a new application of the Hofmann-Mislove theorem, seeing it as a key step in the proof of the closed subgroup theorem.

The second new aspect is the order duality which leads, as we have just shown, to a new result about compact localic subgroups: they are always fitted.

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