I-weight of compact and locally compact LOTS

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Abstract. Ramírez-Páramo proved that under GCH for the class of compact Hausdorff spaces i-weight reflects all cardinals [*A reflection theorem for i-weight*, Topology Proc. **28** (2004), no. 1, 277–281]. We show that in ZFC i-weight reflects all cardinals for the class of compact LOTS. We define local i-weight, then calculate i-weight of locally compact LOTS and paracompact spaces in terms of the extent of the space and the i-weight of open sets or the local i-weight. For locally compact LOTS we find a necessary and sufficient condition for i-weight to reflect cardinal κ.

Keywords: i-weight, reflection, T_1-separating weight, LOTS, compact

Classification: 54A25, 54F05

1. Introduction

Tkachenko began the study of reflection in [6], and a systematic study was made by Hodel and Vaughan in [4]. Hajnal and Juhász proved that weight reflects every infinite cardinal [2]. Ramírez-Páramo proved that under GCH for the class of compact Hausdorff spaces, i-weight reflects all infinite cardinals [5]. In the second section of this paper we prove that for compact linearly ordered spaces i-weight reflects all infinite cardinals. We show that the point-separating weight must reflect, which implies that i-weight must reflect. In Section 3, we find necessary and sufficient conditions for i-weight to reflect in the class of locally compact linearly ordered spaces. The lemmas used to determine under what conditions i-weight will reflect for these spaces provide a means of calculating the i-weight of an ordinal space. In the last section, we define local i-weight, and show that for paracompact spaces, i-weight is determined by the extent of the space and the local i-weight. This paper concludes with a necessary condition for i-weight to reflect in the class of paracompact linearly ordered spaces and an example of an hereditarily paracompact linearly ordered space for which i-weight does not reflect.

All the spaces considered in this paper are assumed to be at least Tychonoff. We begin with some definitions which may be found in [5], [4].

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Definition. A cardinal function ϕ is said to *reflect* cardinal κ, if when $\phi(X) \geq \kappa$ there is a subset Y of X so that $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

Definition. We say that X is *condensed onto* Z if there is a continuous bijection from $f : X \to Z$. Commonly, Z is regarded as a copy of X, and the topology on Z is considered to be contained in the topology on X. We say that X is *condensed into* Y if there is a subspace Z of Y that X may be condensed onto.

Definition. For a Tychonoff space X the *i-weight of* X is the minimum weight of a Tychonoff space onto which X may be condensed.

So for example, the i-weight of Tychonoff space X is ω if and only if X has a weaker separable metric topology.

Definition. We say that $\mathcal{V} \subseteq \mathcal{P}(X)$ is T_1*-separating* if for each pair $(x, y) \in X^2$, $x \neq y$, there is an $U \in \mathcal{V}$ so that $x \in U$ and $y \notin U$. By T_1*-separating weight*, denoted $sw(X)$, we mean $\min\{|\mathcal{V}| : \mathcal{V}$ is a separating open cover of $X\}$.

Definition. A *tree* is a partially ordered set $(T, \preceq)$ with the property that for each $t \in T$, the set $\{s \in T : s \prec t\}$ is well ordered by $\preceq$. The order type of the set $\{s \in T : s \prec t\}$, denoted $\mathrm{lv}(t)$, is called the level of t. For any ordinal α, $T_\alpha = \{t \in T : \mathrm{lv}(t) = \alpha\}$; the *height* of T is the first ordinal so that $T_\alpha = \emptyset$.

Two members of T_α are equivalent if they have the same set of strict predecessors. This gives an equivalence relation on T_α. We call the equivalent classes, *nodes*. For each node, N of T, we choose a linear order $<_N$ of the points of N.

A *path* in T is a subset $P \subseteq T$ so that $\{s \in T : s \prec t\} \subseteq P$ whenever $t \in P$, and the height of P, denoted $\mathrm{ht}(P)$, is the least ordinal so that $P \cap T_\alpha = \emptyset$. A *branch* of T is a maximal path. For a branch v of T, let $v(\alpha)$ be the unique point of $v \cap T_\alpha$.

By $\mathrm{Br}(T)$ we denote the set of all branches in T. If v and w are distinct branches of T, let $\beta(v, w)$ be the least ordinal so that $v(\beta(v, w)) \neq w(\beta(v, w))$. Then $v(\beta(v, w))$ and $w(\beta(v, w))$ are members of the same node N. Define $v \sqsubseteq w$ if and only if $v(\beta(v, w)) <_N w(\beta(v, w))$. This gives a linear order for $\mathrm{Br}(T)$, which we use to define an order topology on the set.

2. Compact linearly ordered spaces

In this section we show that for compact linearly ordered spaces, T_1-separating weight and i-weight reflect all cardinals. For the reader's comfort we now briefly outline how we intend to show this.

For a cardinal κ, a κ^+-Aronszajn tree is a tree of height κ^+ such that every chain and every level is of cardinality $< \kappa^+$. For a κ^+-Aronszajn tree T we show that the space $\mathrm{Br}(T)$ has no separating open cover of cardinality less than κ^+. For each compact linearly ordered space we construct the tree $\mathrm{Tr}(X)$. If every subset of X that has cardinality κ^+ may be separated by κ or fewer open sets, then $\mathrm{Tr}(X)$ will contain κ^+-Aronszajn tree A for which the points of $\mathrm{Br}(A)$ may be separated by κ or fewer open sets, which contradicts the earlier result.

If a node N has only finitely many points, then each of the points $\min((N, <_N))$ and $\max((N, <_N))$ are defined. Let A be a tree in which each node contains finitely many points. For any $t \in A$, let $l(t)$ be the branch of A that satisfies

(1) $t \in l(t)$,
(2) if $\mathrm{lv}(t) < \alpha$ and if $l(t) \cap T_\alpha \neq \emptyset$, then $l(t)(\alpha) = \min((N, <_N))$ where N is the node to which $l(t)(\alpha)$ belongs.

Informally, we think of $l(t)$ as the left-most branch passing through t. We define $r(t)$, the right-most branch passing through t, analogously using $\max((N, <_N))$.

Lemma 1. *Let A be a κ^+-Aronszajn tree such that*

(1) *A has one point at level* 0*,*
(2) *every node has either* 0 *or* 2 *points.*

Then less than or equal to κ many open sets in $\mathrm{Br}(A)$ *cannot separate the pairs* $\{\{l(t), r(t)\} : t \in A\}$.

Proof: Let A be as above. Consider in $\mathrm{Br}(A)$ the branches $l(t)$ and $r(t)$ for each $t \in A$. Suppose that $\mathcal{V}$ is a collection of $\leq \kappa$-many open sets that point separate $r(t)$ and $l(t)$ for $t \in A$. For each $t \in A$, let $\langle V_t, W_t\rangle$ be a pair from $\mathcal{V}$ that separates $l(t)$ and $r(t)$ with $l(t) \in V_t$, $r(t) \in W_t$, $r(t) \notin V_t$ and $l(t) \notin W_t$. Each $V \in \mathcal{V}$ can be decomposed into disjoint convex subsets, so for each $t \in A'$ let (v_t, v'_t) be the neighborhood of $l(t)$ from the decomposition of V_t.

For the set (v_t, v'_t) to contain $l(t)$ and not $r(t)$, the branch v_t must contain a point a_t that is immediately to the "left" of a point below t. That is for some point $b_t \preceq t$, we have that $a_t <_N b_t$, where N is the node containing a_t and p. The branch v'_t must contain the point t.

Let S be a stationary subset of κ^+, and for each $t \in A$ so that $\mathrm{lv}(t) \in S$, let $p(t) = a_t$. Now we define a map f from S into κ^+. First, for each $s \in S$, pick $t^s \in A$ in the s^{th} level of A, then let $f(s) = \alpha$ if and only if the level of $(p(t^s))$ is α. By the Pressing Down Lemma, there is a level of the tree, α, so that $f^{-1}\{\alpha\}$ is a stationary subset of κ^+. Since each level has cardinality less than or equal to κ this means that there is a point a in level α so that $|p^{-1}\{a\}| = \kappa^+$. Consider $\{V_t : t \in p^{-1}\{a\}\} \subseteq \mathcal{V}$. Since $|\mathcal{V}| \leq \kappa$, there is a V so that $V = V_t$ for κ^+-many $t \in p^{-1}\{a\}$. Since a is to the left of a point below each $t \in p^{-1}\{a\}$, there is one convex set (v, v') from the disjoint decomposition of V so that $v = v_t$ and $v' = v'_t$ for each $t \in p^{-1}\{a\}$. Let β be the height of branch v', and note that $|\beta| \leq \kappa$. Then the number of points t so that $l(t) \in (v, v')$ is less than $|\beta| \leq \kappa$, yet $p^{-1}(\{a\}) \subseteq (v, v')$, contradiction. □

Lemma 2. *For a compact Hausdorff space X, $iw(X) = w(X) = nw(X) = sw(X)$.*

Proof: By [3], we know that $sw(X) \leq nw(X)$ and that if X is compact $psw(X) = nw(X) = w(X)$. The proof that if X is compact Hausdorff then $sw(X) \geq nw(X)$ closely follows the proof that $psw(X) \geq nw(X)$ for compact Hausdorff X, in [3]. □

A cardinal function ϕ is called *monotone* if and only if $\phi(Y) \leq \phi(X)$ whenever $Y \subseteq X$.The following lemma and its proof are found in [4].

Lemma 3. *If ϕ is a monotone cardinal function that reflects successor cardinals, then ϕ reflects all infinite cardinals.*

Next, we need to observe that T_1-separating weight is monotone. We combine this with a similar lemma for i-weight.

Lemma 4. *I-weight is monotone, and for compact spaces T_1-separating weight is monotone.*

Theorem 5. *For a compact linearly ordered space, T_1-separating weight reflects κ^+. Hence, for compact linearly ordered spaces, T_1-separating weight reflects all infinite cardinals.*

Proof: Suppose X is a compact linearly ordered space and $sw(X) \geq \kappa^+$, but every subset Y of X such that $|Y| \leq \kappa^+$ has T_1-separating weight at most κ.

We construct a tree $\mathrm{Tr}(X)$ from the space X.

Let $X = I_\emptyset$, then divide the space X into two closed intervals with at most one point in common, $I_{\langle 0\rangle}$ and $I_{\langle 1\rangle}$, so that every point of $I_{\langle 0\rangle}$ is less than or equal to every point of $I_{\langle 1\rangle}$. For a reason that will only be important near the end of this construction, we choose to split the interval at a non-isolated point, if the interval contains a non-isolated point. Next, form $I_{\langle 0,0\rangle}$, $I_{\langle 0,1\rangle}$, $I_{\langle 1,0\rangle}$ and $I_{\langle 1,1\rangle}$ by dividing each of $I_{\langle 0\rangle}$ and $I_{\langle 1\rangle}$ into two parts, again at a non-isolated point, if possible. This process is done at each successor level of the tree. At the level ω, consider $\sigma : w \to 2$. If $\bigcap_{n<\omega} I_{\sigma\restriction n}$ is a non-degenerate interval, then I_σ is a point at the ω level. Likewise at each limit level, points appear only if the intersection of preceding intervals is a non-degenerate interval. If at some successor level an interval should have only one point in it, then that point (in the tree) does not branch. The ordering on the tree is that $I_\sigma \leq I_\tau$ if and only if $\sigma \subseteq \tau$.

Claim 1. *For each $\alpha < \kappa^+$, the cardinality of level α is less than κ^+.*

Suppose that at some level less than κ^+, there are at least κ^+ many non-degenerate intervals. Let α be the least such level.

Since $|\alpha| \leq \kappa$, the collection of points that precede the α level has size not more than κ. Let $\{I_\gamma : \gamma < \kappa^+\} = \{[l_\gamma, r_\gamma] : \gamma < \kappa^+\}$ index the κ^+ many non-degenerate intervals at the level α. Assume that there is a collection $\mathcal{V}$ of at most κ-many open sets in X that T_1 separate points r_γ and l_γ for $\gamma < \kappa^+$. Then for each $\gamma < \kappa^+$ there is a $W_\gamma \in \mathcal{V}$ so that, $r_\gamma \in W_\gamma$ and $l_\gamma \notin W_\gamma$. There is a set W and some $J \in [\kappa^+]^{\kappa^+}$ so that $W = W_\gamma$ for $\gamma \in J$.

Next, decompose W into disjoint convex subsets, and for each $\gamma \in J$ let $(w_\gamma, w'_\gamma) \subset W$ be the convex neighborhood of r_γ from this disjoint decomposition. The assignment $\gamma \mapsto (w_\gamma, w'_\gamma)$ is one-to-one since if $(w_\gamma, w'_\gamma) = (w_\beta, w'_\beta)$, then either l_γ or l_β is in $(w_\gamma, w'_\gamma) \subset W$. Now each r_γ is the limit of a sequence of right end points of the intervals that precede I_γ in the ordering of the tree. Therefore, for each $\gamma \in J$, there is a right end point r'_γ of an interval preceding I_γ that is contained in (w_γ, w'_γ). However, there are only κ-many points below the α level of this tree, and therefore only κ-many potential r'_γ for κ^+-many disjoint (w_γ, w'_γ), contradiction. Therefore, we conclude that every level of $\mathrm{Tr}(X)$ below the κ^+ level has cardinality less than κ^+.

Claim 2. *Every branch has length* $< \kappa^+$.

Suppose this tree contains a branch of length κ^+. Then either the sequence of left endpoints or the sequence of right endpoints of intervals forming the branch in the tree has cardinality κ^+. Without loss of generality, the set of right end points $\{r_\alpha : \alpha < \kappa^+\}$ has cardinality κ^+. Then $\{r_\alpha\}_{\alpha<\kappa^+}$ is a non-increasing sequence, and must converge to $r \geq \sup\{l_\alpha : \alpha < \kappa^+\}$. For each $\alpha < \kappa^+$, let $U_\alpha \in \mathcal{V}$ be the open set that contains r and not r_α. Then some U is U_α for κ^+-many α. However, for some $\beta < \kappa^+$ we will have $r_\gamma \in U$ whenever $\gamma > \beta$, contradiction.

We have now shown that every level of $\mathrm{Tr}(X)$ has cardinality not more than κ and that every chain has length less than or equal to κ. So either $\mathrm{Tr}(X)$ is a κ^+-Aronszajn tree or the height of $\mathrm{Tr}(X)$ is less than κ^+. It follows from Lemma 1 that $\mathrm{Tr}(X)$ cannot be a κ^+-Aronszajn tree. If $\mathrm{Tr}(X)$ were a κ^+-Aronszajn tree, then consider $\mathrm{Br}(\mathrm{Tr}(X))$ and the points corresponding to $r(t)$ and $l(t)$ for the points $t \in \mathrm{Tr}(X)$. By our assumptions, in X we are able to separate these $\leq \kappa^+$ points in X with κ-many open sets, contradiction.

The left and right endpoints of the intervals contained as points in the tree form a subset of X, call this collection Y. Since the height of the tree is less than κ^+ have that $|Y| \leq \kappa$. We claim that this subset together with the isolated points is dense in X. Consider a nonempty open convex subset (a, b) of X. Either (a, b) contains a left or right endpoint of some interval contained in the tree, or (a, b) is contained in an interval from each level of the tree. Then let β be the height of the tree, and for each $\alpha < \beta$ let J_α be the interval from level α that contains (a, b). Then $(a, b) \subseteq \bigcap_{\alpha<\beta} J_\alpha$, while $|\bigcap_{\alpha<\beta} J_\alpha| = 1$. So nonempty $(a, b) = \{x\}$. So Y together with the isolated points is a dense set in X.

We now claim that the set of isolated points has cardinality at most κ. Let $\{J_m : m \in M\}$ be the collection of minimal intervals contained in the tree which we were not able to split at a non-isolated point, meaning each point of J_m is isolated. We claim each J_m can contain at most countably many isolated points. Let $[l_m, r_m] = J_m$, then pick $a_m \in J_m$; we have $[l_m, a_m]$ is a closed set, and is therefore compact. Pick any open neighborhood W of l_m, and then $\{W\} \cup \{\{x\} : x \in [l_m, a_m] \setminus W\}$ covers $[l_m, a_m]$. Therefore, all but finitely many of the points of $[l_m, a_m]$ must be in W. Since W is arbitrary, this implies that$[l_m, a_m]$ contains countably many isolated points. By symmetry, so does $[a_m, r_m]$. Now it remains to note that every isolated point of X is contained in J_m for some $m \in M$, and since $|M| \leq \kappa$, we have at most κ many isolated points.

We use Y to construct a base of cardinality at most κ for X. Let $\mathcal{B} = \{(a,b) : a,b \in Y, a < b\} \cup \{\{x\} : x \text{ is isolated}\}$. Let $Y' = Y \cup \{x : x \text{ is isolated}\}$.

Let x be a point in X and (u,u') be a convex neighborhood of x. Unless x is isolated, we have that at least one of (u,x) and (x,u') is nonempty. Without loss of generality, assume that (u,x) is nonempty, and choose $a \in Y' \cap (u,x)$. If (x,u') is nonempty, then choose b similarly, and $x \in (a,b) \subseteq (u,u')$. So assume instead that (x,u') is empty.

Consider an increasing sequence of points in the tree (which is a decreasing sequence of intervals in X) that contain the point x, say $\{K_\alpha : \alpha < \beta\}$. Then $\bigcap_{\alpha<\beta} K_\alpha = \{x\}$. Let γ be the least ordinal so that for r_γ, the right end point of K_γ, $r_\gamma \leq u'$. If $r_\gamma = u'$, then $u' \in Y$, and the set we need is (a,u'). If $r_\gamma = x$, then there is a second increasing sequence of intervals in the tree that contain x, but for this sequence x is a left end point. Then consider all the members of this second sequence that also contain u'. The intersection of them would be $[x,u']$, and would be a point of the tree. Therefore, $u' \in Y$, and again the set we want is (a,u'), for then $x \in (a,u') \subseteq (u,u')$. Therefore, X has weight at most κ, and hence T_1-separating weight at most κ, contradiction. □

Corollary 6. *For compact linearly ordered spaces, i-weight reflects all cardinals.*

Proof: Since $iw(X) = sw(X)$ for each compact Hausdorff space, if $iw(X) \geq \kappa$ then $sw(X) \geq \kappa$. Therefore, there is $Y \subseteq X$ so that $|Y| \leq \kappa$ yet $sw(Y) \geq \kappa$. Any base for a Tychonoff topology, would also be a separating open cover, therefore, $iw(Y) \geq sw(Y) \geq \kappa$. □

3. Locally compact linearly ordered spaces

In this section we prove reflection theorems for locally compact linearly ordered spaces and i-weight. We begin with several lemmas that build toward the main result. We determine that the i-weight of an ordinal space is the cardinality of the ordinal. Also, we determine the i-weights of subspaces of ordinal spaces. We find necessary and sufficient conditions for i-weight to reflect cardinal κ in the class of locally compact linearly ordered space. This section ends with two examples.

We use the following definitions throughout the rest of this section.

Definition. For a locally compact linearly ordered space X and $a,b \in X$ we write $a \sim b$ if and only if either $[a,b]$ or $[b,a]$ is compact. Then $\sim$ is an equivalence relation. Let $\tilde{a} = \{b \in X : a \sim b\}$ denote the equivalence class of a. Then for a locally compact linearly ordered space X, we define $E(X)$ to be the number of distinct equivalence classes under the $\sim$ relation, i.e., $E(X) = |\{\tilde{a} : a \in X\}|$.

Let D_κ denote the discrete space of cardinality κ. Also, by $\log(\kappa)$ we denote $\min\{\lambda \leq \kappa : 2^\lambda \geq \kappa\}$.

Lemma 7. *For a locally compact linearly ordered space X, each $\tilde{a}$ is an open subset of X.*

Proof: To see that $\tilde{a}$ is open, let $p \in \tilde{a}$. We assume that $a < p$, and the proof for the case with $p < a$ is analogous. Then since X is linearly ordered and locally compact there is an open interval (c, d) of X so that $p \in (c, d)$ and $\overline{(c, d)} = [c, d]$ is compact. Then either $a < c$ or $c \leq a \leq d$. If $a < c$, then $[a, d] = [a, p] \cup [p, d]$. Since $[p, d]$ is a closed subset of the compact space $[c, d]$, $[p, d]$ is compact and $[a, d]$ is compact as a union of finitely many compact sets. Also, for each $b \in (c, d)$ the set $[a, b]$ is compact. Therefore, $(c, d) \subseteq \tilde{a}$, which implies that $\tilde{a}$ is open.

If $c \leq a \leq d$ then for each $b \in (c, d)$, either $[a, b]$ or $[b, a]$ is nonempty and compact as a closed subset of $[c, d]$, so in this case $(c, d) \subseteq \tilde{a}$. □

Lemma 8. *For a locally compact linearly ordered space X, if $E(X)$ is infinite then $E(X) = e(X)$.*

Proof: Notice first that $e(X) \geq E(X)$. We form a closed discrete set C of cardinality $E(X)$ by taking one point from each equivalence class. We have shown that for each $a \in X$ the set $\tilde{a}$ is open, so C is discrete. Also, $\tilde{a} \setminus \{a\}$ is open, so C is closed.

Next, suppose that $e(X) > E(X)$. Since $E(X)$ is infinite, $e(X) \geq \omega_1$. Then there is at least one equivalence class, call it $\tilde{a}$, that contains at least ω_1-many members of a closed discrete set C. Choose a point $p' \in C \cap \tilde{a}$. At least one of $P = \{c \in C \cap \tilde{a} : c < p'\}$ and $S = \{c \in C \cap \tilde{a} : c > p'\}$ is uncountable. We assume the set P is uncountable, as the proof for the case that S is uncountable is analogous. We claim that we can find $p < p' \in \tilde{a}$ so that $|C \cap [p, p']| \geq \omega$. For $c \in P$, let $A_c = \{d \in P : c < d\}$. If A_c were finite for each $c \in P$, then P would be an increasing union of sets which are all finite, and so $|P| \leq \omega$, contradiction. Hence there is a p so that $|A_p| \geq \omega$, then $[p, p'] \cap C$ is infinite, $[p, p']$ is compact and cannot contain an infinite closed discrete set, contradiction. □

Lemma 9. *For any topological spaces X and Y, $iw(X \times Y) = \max\{iw(X), iw(Y)\}$.*

Proof: Suppose that X and Y are topological spaces, and consider $X \times Y$. If $\mathcal{B}_X$ and $\mathcal{B}_Y$ are bases for X and Y, then $\mathcal{B}_X \times \mathcal{B}_Y$ is a base for $X \times Y$.

So $w(X \times Y) \leq |\mathcal{B}_X \times \mathcal{B}_Y| = |\mathcal{B}_X||\mathcal{B}_Y|$. Therefore, $iw(X \times Y) \leq iw(X)iw(Y) = \max\{iw(X), iw(Y)\}$.

Next, suppose that $\mathcal{B}$ is a base for a Tychonoff topology on $X \times Y$ which is coarser than the product topology. Fix $y_0 \in Y$ and consider $\mathcal{U}_X = \{U \cap (X \times \{y_0\}) \neq \emptyset : U \in \mathcal{B}\}$. Then $\pi_1(\mathcal{U}_X)$ is a base for a Tychonoff topology on X.

The above argument is symmetric with respect to x and y, so the i-weights of X and Y are not more than $|\mathcal{B}|$. Therefore, i-weights of X and Y are not more than i-weight of $X \times Y$. □

Theorem 10. *Let κ be a regular cardinal, and let A be a stationary subset of κ. Then A with the subspace topology inherited from the order topology on κ has i-weight κ.*

Proof: Let κ be a regular cardinal and assume that A is a stationary subset of κ. Suppose by way of a contradiction that $\mathcal{B}$ is a base for a Tychonoff topology on A contained in the relative order topology so that $|\mathcal{B}| < \kappa$. For each $U \in \mathcal{B}$, there is an open subset U' of κ so that $U' \cap A = U$. Let $\mathcal{B}' = \{U' : U \in \mathcal{B}\}$. Since A is stationary, A contains stationarily many limit ordinals. Let S denote the limit ordinals contained in A.

For each $s \in S$, let p_s be any element of A so that $p_s > s$. Also, for each $s \in S$ let $(U_s, V_s) \in \mathcal{B}^2$ be such that $s \in U_s$, $p_s \in V_s$ and $U_s \cap V_s = \emptyset$. Since U_s' must be open in the order topology, we know that each U_s' contains a convex segment containing s. Let $g(s)$ be an ordinal less than s so that $(g(s), s] \subseteq U_s'$. Then since S is stationary, there is a γ so that $g^{-1}(\gamma)$ is stationary. Because $|\mathcal{B}| < \kappa$ and $|g^{-1}(\gamma)| = \kappa$, there is (U^*, V^*) so that $(U^*, V^*) = (U_s, V_s)$ for κ-many different $s \in g^{-1}(\gamma)$. Then $s \in (\gamma, s] \subseteq U^*$ and $p_s \notin U^*$ for each $s \in g^{-1}(\gamma)$. For any fixed $s \in g^{-1}(\gamma)$, let $p^* = p_s$. We claim that p^* is an upper bound on the set $g^{-1}(\gamma)$, else if there is a s' so that $p^* \leq s'$ then $p^* \in (\gamma, s'], \subseteq U^*$. □

Corollary 11. *For any cardinal κ, the i-weight of the ordinal space κ is κ.*

Proof: If κ is not regular then κ must be a limit ordinal, since each successor ordinal is regular. Each limit cardinal is the limit of the preceding regular cardinals. Therefore, let $L = \{\alpha < \kappa : \alpha \text{ is a regular cardinal}\}$, and notice that κ is equal to $\bigcup_{\alpha \in L} \alpha$. Then $|\kappa| \geq iw(\kappa) \geq \sup\{iw(\alpha) : \alpha \in L\} = \kappa$. □

Corollary 12. *The i-weight of any ordinal space κ is $|\kappa|$.*

Proof: Assume that κ is an ordinal but not a cardinal. Then, as ordinals, $|\kappa| < \kappa$. By monotonicity of i-weight we know that $iw(|\kappa|) \leq iw(\kappa)$. Also, $iw(\kappa) \leq w(\kappa) = |\kappa| = iw(|\kappa|)$. □

Lemma 13. *A Tychonoff space X with $iw(X) \leq \lambda$ can be condensed into I^λ.*

Proof: Any Tychonoff space of weight m can be embedded in I^m. So if a space X has i-weight $m \leq \lambda$, then X has a Tychonoff topology τ so that (X, τ) is homeomorphic to a subset of I^m. Call the corresponding embedding f.

Then we embed each I^m into I^γ by the defining $h : I^m \to I^\gamma$ as follows. Let $x \in I^m$ be denoted as $(x_i)_{i<m}$; then $h(x) = (x_i)_{i<m} {}^\frown (0)_{m \leq j < \gamma}$. Let $G : X \to I^\gamma$ be defined by $h \circ f$. Clearly, G is one-to-one and continuous. □

Theorem 14. *Let κ be an ordinal, and C a club subset of κ. Suppose that $\kappa \setminus C = \bigcup_{i<\gamma}(a_i, b_i)$ so that $\gamma \leq |\kappa|$ and $(a_i, b_i) \cap (a_j, b_j) \neq \emptyset$ if and only if $i = j$. Then, $iw(\kappa \setminus C) = \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$.*

Recall that by D_γ we denote the discrete space of cardinality γ.

Proof: Let κ and C be as above, i.e. $\kappa \setminus C = \bigcup_{i<\gamma}(a_i, b_i)$. Then by monotonicity of i-weight, we have $iw(\kappa \setminus C) \geq iw((a_i, b_i))$ for each $i < \gamma$. Also, since the (a_i, b_i) are pairwise disjoint, and each is nonempty and open, we may choose $x_i \in (a_i, b_i)$ so that $X = \{x_i : i < \gamma\}$ is a discrete space homeomorphic to D_γ. Then, invoking monotonicity once again, $iw(\kappa \setminus C) \geq iw(X)$.

We prove that $iw(\kappa \setminus C) \leq \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$ by considering two cases.

Suppose that there is some $\lambda < |\kappa|$ so that $\lambda = \sup\{iw((a_i, b_i)) : i < \gamma\}$. We condense each (a_i, b_i) into I^λ. Then, $\kappa \setminus C$ can be condensed to a subset of $D_\gamma \times I^\lambda$. So by Lemma 9, $iw(D_\gamma \times I^\lambda) = \max\{iw(D_\gamma), \lambda\}$. Therefore, $iw(\kappa \setminus C) \leq \{iw(D_\gamma),\ \sup\{iw((a_i, b_i) : i < \gamma\}\}$. Suppose on the other hand that $\sup\{iw((a_i, b_i)) : i < \gamma\} = |\kappa|$. Since $|\kappa| = iw(\kappa) \geq iw(\kappa \setminus C) \geq \sup\{iw((a_i, b_i)) : i < \gamma\}$, then $iw(\kappa \setminus C) = |\kappa| = \max\{iw(D_\gamma),\ \sup\{iw((a_i, b_i)) : i < \gamma\}\}$. □

Corollary 15. *The i-weight of D_κ is $\log(\kappa)$.*

Proof: From Theorem 4.2 in [3], we know that for any Hausdorff topology on X, $|X| \leq w(X)^{\psi(X)} \leq w(X)^{w(X)} = 2^{w(X)}$. This gives us a means of bounding the i-weight of a space, in particular, $\kappa \leq 2^{iw(D_\kappa)}$. So $iw(D_\kappa) \in \{\lambda : 2^\lambda \geq \kappa\}$. Notice that for each λ so that $2^\lambda \geq \kappa$, we may consider κ to be a subset of 2^λ. Then under the product topology on 2^λ, the weight of 2^λ is λ. Also in the product topology, the set of all limit ordinals less than κ are a discrete space of cardinality κ and is therefore homeomorphic to D_κ. Which means that $iw(D_\kappa) \leq \lambda$. So $iw(D_\kappa) = \min\{\lambda : 2^\lambda \geq \kappa\} = \log(\kappa)$. □

Lemma 16. *If X is a locally compact linearly ordered space so that for each pair $a, b \in X$ with $a < b$ the set $[a, b]$ is compact, then $iw(X) = w(X)$. Moreover, for such a space X, i-weight reflects all cardinals.*

Proof: Let X be as above. Then pick $a \in X$. Either $(-\infty, a]$ or $[a, \infty)$ has the same weight as X. Without loss of generality, let $w([a, \infty)) = w(X)$.

We intend to show that $w([a, \infty)) = iw([a, \infty)) = \max\{\mathrm{cf}([a, \infty)), \sup\{w([a, b]) : b > a\}\}$.

First, suppose that $\mathcal{B}$ is a base for a Tychonoff topology on $[a, \infty)$ which is coarser than the order topology. Since weight equals i-weight for compact Hausdorff spaces, we know that $iw([a, b]) = w([a, b])$ and by monotonicity of weight, we know that $w([a, \infty)) \geq \sup\{w([a, b]) : b > a\}$. Also, suppose that $\mathrm{cf}([a, \infty)) = \kappa$. We construct a set C that is homeomorphic to κ. Let $c_0 = a$. Suppose that for $j \leq i$ each c_j has been defined, and pick $c_{i+1} > c_i$. If α is a limit ordinal so that for each $j < \alpha$, c_j has been defined, define $c_\alpha = \sup\{c_j : j < \alpha\}$. Since the cofinality of $[a, \infty)$ is κ, c_i is defined for each $i < \kappa$. Let $C = \{c_i : i < \kappa\}$. If i is a successor ordinal, $(c_{i-1}, c_{i+1}) \cap C = \{c_i\}$ and is open. If α is a limit ordinal then $(c_i, c_\alpha] \cap C = \{c_j : i < j \leq \alpha\}$ is open for $i < \alpha$. We map C homeomorphically to

κ by $h(c_i) = i$. Then the i-weight of C is κ, the i-weight of κ. This implies that $iw([a,\infty)) \geq \kappa = \mathrm{cf}([a,\infty))$.

Next, we observe that $w([a,\infty)) \leq \max\{\mathrm{cf}([a,\infty)),\ \sup\{w([a,b]) : b > a\}\}$. Let K be a cofinal subset of $[a,\infty)$ of cardinality $\mathrm{cf}([a,\infty))$; so $K = \{k_i : i < \kappa\}$ and $k_i < k_j$ iff $i < j$. The set $\{[a,\kappa_\alpha) : \alpha < \kappa\}$ is an open cover of $[a,\infty)$. Also, $w([a,\kappa_\alpha)) \leq w([a,\kappa_\alpha])$. Let $\mathcal{B}_\alpha$ be a base for $[a,\kappa_\alpha)$ under the subspace topology for the order topology on $[a,\infty)$. Then $\mathcal{B} = \bigcup_{\alpha<\kappa}\mathcal{B}_\alpha$ is a base for $[a,\infty)$. The cardinality of $\mathcal{B}$ is less than $\max\{\kappa,\ \sup\{w([a,k_\alpha)) : \alpha < \kappa\}\} \leq \max\{\kappa,\ \sup\{w([a,k_\alpha]) : \alpha < \kappa\}\} \leq \max\{\kappa,\ \sup\{w([a,b]) : b > a\}\}$.

So $w([a,\infty)) = iw([a,\infty))$.

Now we will show that i-weight reflects. Suppose that $\gamma \leq iw([a,\infty))$. Then consider several quick cases.

(1) If $\gamma \leq \mathrm{cf}([a,\infty))$, then let C be the cofinal subset above in this proof. Then $Y = \{c_i : i < \gamma\}$ is a subset of $[a,\infty)$ that is homeomorphic to γ, hence Y has i-weight γ.

(2) If there is a $b > a$ so that $iw([a,b]) \geq \gamma$, then take Y to be a subset of $[a,b]$ that reflects γ.

(3) Now assume that $\gamma > \mathrm{cf}([a,\infty))$ and that $iw([a,b]) < \gamma$ for each $b > a$. Then let $Y_i \subseteq [a,k_i]$ so that $|Y_i| \leq iw([a,k_i]) = iw(Y_i)$. Take $Y = \bigcup_{i<\kappa} Y_i$. We claim $iw([a,\infty)) \geq iw(Y) \geq \sup\{w([a,k_i]) : i < \kappa\} = \sup\{w([a,b]) : b > a\} = \gamma$. It is clear that $\sup\{w([a,k_i]) : i < \kappa\} = \sup\{w([a,b]) : b > a\}$ since K is cofinal. To verify that $\sup\{w([a,b]) : b > a\} = \gamma$ recall that $w([a,b]) = iw([a,b])$ and that $iw([a,b]) < \gamma \leq iw([a,\infty))$, so $\sup\{w([a,b]) : b > a\} = \gamma$. So $iw(Y) \geq \gamma$, because $iw(Y) = \max\{\kappa, \sup\{iw([a,\kappa_i]) : i < \kappa\}\} = \max\{\kappa,\lambda\}$ and $\kappa \leq \lambda$.

Therefore, for $[a,\infty)$, i-weight reflects all cardinals. Recall, that $w([a,\infty)) = w(X) \geq iw(X) \geq iw([a,\infty)) = w([a,\infty))$. So the i-weight of X is the i-weight of $[a,\infty)$; therefore, i-weight reflects for the space X. □

Theorem 17. *Let X be a locally compact linearly ordered space. Then $iw(X) = \max\{iw(D_{E(X)}),\ \sup\{iw(\tilde{a}) : a \in X\}\} = \max\{\log(e(X)),\ \sup\{iw(\tilde{a}) : a \in X\}\}$.*

Proof: By monotonicity, we know that $iw(X) \geq \max\{iw(D_{E(X)}),\ \sup\{iw(\tilde{a}) : a \in X\}\}$. Now suppose that $\lambda = \sup\{iw(\tilde{a}) : a \in X\}$. Then there is a condensation of X into $D_{E(X)} \times I^\lambda$, which has i-weight $\max\{iw(D_{E(X)}),\ \lambda\}$. So $iw(X) = \max\{iw(D_{E(X)}),\ \sup\{iw(\tilde{a}) : a \in X\}\}$.

Also $iw(D_{E(X)}) = \log|E(X)| = \log(e(X))$, therefore, $iw(X) = \max\{\log(e(X)),\ \sup\{iw(\tilde{a}) : a \in X\}\}$. □

Theorem 18. *Let X be a locally compact linearly ordered space. If $iw(X) = iw(D_{E(X)}) = \log(e(X))$, then i-weight reflects the cardinal κ if and only if either $2^\lambda < \kappa$ for all $\lambda < \kappa$ or $\kappa \leq \sup\{iw(\tilde{a}) : a \in X\}$. Hence, if $iw(X) = \sup\{iw(\tilde{a}) : a \in X\}$, then i-weight reflects all cardinals.*

Proof: Suppose first that X is as above, and $iw(X) = iw(D_{E(X)})$. Assume that i-weight reflects the cardinal κ and $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. There is a Y contained in X so that $|Y| \leq \kappa$ and $iw(Y) \geq \kappa$. Since $iw(Y \cap \tilde{a}) \leq iw(\tilde{a})$ we have that $iw(Y \cap \tilde{a}) < \kappa$ for each $a \in X$. Also, since $|Y| \leq \kappa$, $Y \cap \tilde{a}$ is nonempty for only κ-many different equivalence classes. So let $\{\tilde{a_i} : i < \kappa\} = \{\tilde{a} : \tilde{a} \cap Y \neq \emptyset\}$. Then for we may condense Y into $\kappa \times I^{\sup\{iw(\tilde{a_i}):i<\kappa\}}$, which has i-weight $iw(k)$ since $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. Therefore, $\kappa = iw(Y) \leq iw(\kappa) \leq \kappa$. So, for each $\lambda < \kappa$, we have $2^\lambda < \kappa$; else, if $2^\lambda \geq \kappa$ for some $\lambda < \kappa$, the i-weight of κ would be λ.

Next, assume that i-weight reflects κ and $2^\lambda \geq \kappa$ for some $\lambda < \kappa$. Aiming for a contradiction, further assume that $\kappa > \sup\{iw(\tilde{a}) : a \in X\}$. Since i-weight reflects κ, there is a set Y so that $|Y| \leq \kappa$ and $iw(Y) \geq \kappa$. Then Y can be condensed into $\kappa \times I^{\sup\{iw(\tilde{a}):a\in X\}}$ which has i-weight $\max\{\lambda,\ \sup\{iw(\tilde{a}) : a \in X\}\} \leq \kappa$, contradiction.

Now we prove the reverse direction.

Assume that $2^\lambda < \kappa$ for all $\lambda < \kappa$ and $\kappa \leq iw(X) = iw(E(X)) \leq E(X)$. Pick κ-many different a_i so that $\{\tilde{a_i} : i < \kappa\}$ is a collection of pairwise disjoint open sets. Then for each $i < \kappa$, pick $y_i \in \tilde{a_i}$ and define $Y = \{y_i : i < \kappa\}$. So Y is a discrete space of cardinality κ, so the i-weight of Y is $\log(\kappa) = \kappa$.

If $\kappa \leq \sup\{iw(\tilde{a}) : a \in X\}$, then consider two cases. First, if $\kappa < \sup\{iw(\tilde{a}) : a \in X\}$, then let $x \in X$ be so that $iw(\tilde{x}) \geq \kappa$. Then for $\tilde{x}$, i-weight reflects κ.

Suppose that $\kappa > iw(\tilde{a})$ for each $a \in X$. Then for some $\gamma \leq \kappa$ let $\{a_i : i < \gamma\}$ be a subset of X so that $\{iw(\tilde{a_i}) : i < \gamma\}$ is cofinal in κ. Then for each $i < \gamma$ pick $Y_i \subseteq \tilde{a_i}$ so that $|Y| \leq iw(\tilde{a_i}) = iw(Y_i)$. Let $Y = \bigcup_{i<\gamma} Y_i$. Then $|Y| \leq \kappa$ and $iw(Y) = \sup\{iw(\tilde{a_i}) : i < \gamma\} = \kappa$. □

By examining the proof above we can see that for all spaces if $\kappa < \log(e(X))$ and $2^\lambda < k$ for all $\lambda < \kappa$, then i-weight reflects the cardinal κ.

Next, we present some examples of linearly ordered spaces for which i-weight does not reflect some cardinal, which is possible because $E(X) > iw(X)$. So the conditions on $E(X)$ in the preceding theorem may not be omitted. On the other hand, we can also see that $D_{2^\kappa} \times (\kappa+1)$ is an example of a locally compact linearly ordered space for which $E(X) = 2^\kappa > iw(X) = \kappa$ and yet i-weight will reflect all cardinals less than κ.

Lemma 19. *Any infinite discrete space is homeomorphic to a linearly ordered space.*

Theorem 20 (GCH)**.** *There is a locally compact linearly ordered space X with $iw(X) = \omega_1$, yet for X i-weight does not reflect ω_1.*

Proof: Consider $X = D_{2^{\omega_1}}$. Then by [3], we know that $2^{\omega_1} \leq 2^{iw(X)}$. Since the order topology is coarser than the discrete topology, $iw(X) = \omega_1$.

Take any subset Y of X so that $|Y| \leq \omega_1$. Since Y is discrete, Y may be condensed onto a subset of the real line. Thus $iw(Y) = \omega$. $\square$

We may eliminate the need for GCH if we are willing to allow the i-weight to exceed ω_1. Instead, let $X = D_{2^{2^{\omega_1}}}$, then $iw(X) \geq \omega_1$, yet each subset of cardinality ω_1 will have i-weight ω.

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