A quest for nice kernels of neighbourhood assignments

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Abstract. Given a topological property (or a class) $\mathcal{P}$, the class $\mathcal{P}^*$ dual to $\mathcal{P}$ (with respect to neighbourhood assignments) consists of spaces X such that for any neighbourhood assignment $\{O_x : x \in X\}$ there is $Y \subset X$ with $Y \in \mathcal{P}$ and $\bigcup\{O_x : x \in Y\} = X$. The spaces from $\mathcal{P}^*$ are called *dually* $\mathcal{P}$. We continue the study of this duality which constitutes a development of an idea of E. van Douwen used to define D-spaces. We prove a number of results on duals of some general classes of spaces establishing, in particular, that any generalized ordered space of countable extent is dually discrete.

Keywords: neighbourhood assignment, duality, weak duality, Lindelöf space, weakly Lindelöf space

Classification: Primary 54H11, 54C10, 22A05, 54D06; Secondary 54D25, 54C25

1. Introduction

We take a closer look at dually discrete spaces which constitute the first natural generalization of the class of van Douwen's D-spaces. It is well-known that no countably compact non-compact space is a D-space while it is not so easy to find spaces which are not dually discrete; we give examples of such spaces. The main result of this paper is Theorem 3.1 which states that any generalized ordered space of countable extent is dually discrete. We also solve Problems 4.2 and 4.5 from [vMTW] as well as Problems 4.2, 4.5, 4.6, 4.7 and 4.8 from [ATW] in ZFC and give a consistent solution of Problem 4.13 from [ATW].

All spaces under consideration are assumed to be Tychonoff. Given a space X, the family $\tau(X)$ is its topology. If $x \in X$, then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. If $\mathcal{O} = \{O_x : x \in X\}$ is a neighbourhood assignment in a space X then, for any subspace $Y \subset X$ let $\mathcal{O}(Y) = \bigcup\{O_x : x \in Y\}$; if $\mathcal{O}(Y) = X$ then Y is called *a kernel of* $\mathcal{O}$. The space $\mathbb{D}$ is the doubleton $\{0, 1\}$ with the discrete topology and $\mathbb{R}$ is the real line with the usual order topology. The rest of the notation can be found in [En], [vMTW] and [ATW].

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2. Dual classes and their categorical behaviour

The duals of quite a few important classes proved to be interesting so we formulate some easy results on their categorical properties and relationship with well-known classes. It seems to be of special importance to study dually discrete spaces because they are a straightforward generalization of van Douwen's *D*-spaces while representing a much wider class.

2.1 Proposition. (1) *Any space is dually scattered.*

(2) *There exists a scattered space Y which is neither dually left-separated nor dually metalindelöf. In particular, Y is not dually paracompact and hence not every space is dually discrete* (*or even dually metrizable*).

Proof: The proof of (1) is standard and can be omitted. It is a ZFC result (see [To, Theorem 0.5]) that there exists a space X such that $\kappa = hd(X) < hl(X)$. Take a subspace $Y = \{x_\alpha : \alpha < \kappa^+\} \subset X$ which is right-separated by the indexation, i.e., the set $Y_\alpha = \{x_\beta : \beta < \alpha\}$ is open in Y for any $\alpha < \kappa^+$. It is easy to show that Y is the promised example. □

2.2 Proposition. (1) *If a class $\mathcal{P}$ is invariant under continuous images then its dual class $\mathcal{P}^*$ is also invariant under continuous images.*

(2) *If a class $\mathcal{P}$ is inverse invariant under perfect maps then so is its dual class $\mathcal{P}^*$.*

(3) *If κ is a cardinal and a class $\mathcal{P}$ is invariant under unions of at most κ-many subspaces then so is its dual class $\mathcal{P}^*$.*

(4) *If $\mathcal{P}$ is a class which is closed-invariant then $\mathcal{P}^*$ is also closed-invariant.*

2.3 Corollary. (1) *The classes of dually Lindelöf spaces, dually σ-compact spaces, dually Lindelöf Σ-spaces, dually K-analytic spaces and dually* (*hereditarily*) *separable spaces are preserved by continuous mappings.*

(2) *The classes of dually Lindelöf spaces, dually σ-compact spaces, dually Lindelöf Σ-spaces, dually K-analytic spaces are inverse invariant under perfect maps.*

Although the class of discrete spaces is not inverse invariant under perfect maps, it turns out that discrete duality still behaves nicely in this situation.

2.4 Proposition. *If Y is a dually discrete space and X maps perfectly onto Y then X is also dually discrete.*

2.5 Theorem. *Under Jensen's Axiom $\diamondsuit$, the space $\mathbb{R}^\kappa$ is not dually discrete for any uncountable cardinal κ.*

Proof: It was established in [Os] that, under $\diamondsuit$, there exists an S-space Y such that $|Y| = \omega_1$. The space $\mathbb{R}$ being second countable, $\mathbb{R} \times Y$ is also an S-space. Since $\diamondsuit$ implies CH, we can choose a bijection $\varphi : \mathbb{R} \to Y$. Its graph

$X = \{(x,\varphi(x)) : x \in \mathbb{R}\}$ is also an S-space which condenses on $\mathbb{R}$ so X is realcompact. It is easy to see that X is not dually discrete.

Let $C(X)$ be the set of all real-valued continuous functions on X; it follows from separability of X that $|C(X)| = \mathfrak{c} = \omega_1$. The diagonal product $\mu : X \to \mathbb{R}^{C(X)}$ of the family $C(X)$ embeds X in the space $\mathbb{R}^{C(X)} = \mathbb{R}^{\omega_1}$ as a closed subspace which is not dually discrete so $\mathbb{R}^{\omega_1}$ is not dually discrete either (see Proposition 2.2). Since $\mathbb{R}^{\omega_1}$ embeds in $\mathbb{R}^\kappa$ as a closed subspace, we can apply Proposition 2.2 again to conclude that $\mathbb{R}^\kappa$ is not dually discrete. □

Recall that the *Alexandroff double* $AD(X)$ of a space X is the set $X \times \mathbb{D}$ with a topology τ such that all points of the set $X \times \{1\}$ are isolated in $(AD(X),\tau)$ and, for any point $z = (x,0)$ the family $\{(U \times \mathbb{D})\backslash\{(x,1)\} : U \in \tau(x,X)\}$ is a local base of $AD(X)$ at z. If $\pi : X \times \mathbb{D} \to X$ is the projection then it is straightforward that π is a perfect map.

2.6 Proposition. *Given an infinite cardinal κ and a space X, if $l(X) > \kappa$ then $wl(AD(X)) > \kappa$, i.e., the weak Lindelöf number of the Alexandroff double of X is strictly greater than κ.*

Proof: Take an open cover $\mathcal{U}$ of the space X which has no subcover of cardinality $\leqslant \kappa$. The family $\mathcal{V} = \{U \times \mathbb{D} : U \in \mathcal{U}\}$ is an open cover of $AD(X)$. If $\mathcal{V}'$ is a subfamily of $\mathcal{V}$ of cardinality at most κ then there exists $\mathcal{U}' \subset \mathcal{U}$ such that $|\mathcal{U}'| \leqslant \kappa$ and $\mathcal{V}' = \{U \times \mathbb{D} : U \in \mathcal{U}'\}$. Since $l(X) > \kappa$, there is a point $x \in X\backslash(\bigcup\mathcal{U}')$. It is clear that the non-empty open set $\{(x,1)\}$ witnesses that the union of the family $\mathcal{V}'$ is not dense in $AD(X)$ and hence $wl(AD(X)) > \kappa$. □

2.7 Proposition. *Assume that κ is an infinite cardinal. Then*

(a) *if X a dually hereditarily κ-separable space then $AD(X)$ is also dually hereditarily κ-separable.*
(b) *If X is dually κ-separable and* $\mathrm{ext}(X) \leqslant \kappa$ *then $AD(X)$ is dually κ-separable.*

Proof: If X is dually hereditarily κ-separable then it follows from Proposition 2.2 that every closed subspace of X is dually hereditarily κ-separable; an immediate consequence is that $\mathrm{ext}(X) \leqslant \kappa$. Proceeding with the proof for both (a) and (b) take any neighbourhood assignment $\mathcal{O} = \{O_z : z \in AD(X)\}$ of the space $AD(X)$. There exists a (hereditarily) κ-separable $Y \subset X$ such that $U = \mathcal{O}(Y) \supset X \times \{0\}$. The set $D = AD(X)\backslash U$ is closed and discrete in $AD(X)$; if $|D| > \kappa$ then $|\pi(D)| > \kappa$ while $\pi(D)$ is closed and discrete in X because π is a perfect map. This gives a contradiction because we have $\mathrm{ext}(X) \leqslant \kappa$ in both (a) and (b). Thus $|D| \leqslant \kappa$ so $Z = Y \cup D$ is a (hereditarily) κ-separable kernel of $\mathcal{O}$. □

The following corollary gives a negative answer to Problems 4.2 and 4.5 from the paper [vMTW].

2.8 Corollary. *There exist dually separable spaces which are not weakly Lindelöf.*

Proof: Take a set $Q \subset \mathbb{R}^{\omega_1}$ homeomorphic to ω_1 with its interval topology. It is easy to see that Q is nowhere dense in $\mathbb{R}^{\omega_1}$ so we can find a countable dense subspace $A \subset \mathbb{R}^{\omega_1} \backslash \overline{Q}$. The space $X = A \cup Q$ is separable while $\mathrm{ext}(X) = \omega$ and $l(X) > \omega$. Proposition 2.6 shows that the Alexandroff double $AD(X)$ of the space X is not weakly Lindelöf; applying Proposition 2.7 we conclude that the space $AD(X)$ is dually separable. □

The following corollary gives a consistent answer to Problem 4.6 of [ATW].

2.9 Corollary. *In any model of ZFC containing an S-space there exist dually hereditarily separable spaces which are not weakly Lindelöf.*

Proof: If X is an S-space then apply Propositions 2.6 and 2.7 to see that $AD(X)$ is a dually hereditarily separable space which is not weakly Lindelöf. □

The corollary that follows solves Problems 4.5, 4.7 and 4.8 from [ATW].

2.10 Corollary. *There exist dually σ-compact spaces which are not weakly Lindelöf.*

Proof: It was shown in [ATW] that there exists a non-Lindelöf dually σ-compact space X. Since $AD(X)$ is a perfect preimage of X, the space $AD(X)$ is dually σ-compact by Corollary 2.3. Applying Proposition 2.6 we conclude that $AD(X)$ is not weakly Lindelöf. □

2.11 Proposition. *Given an infinite cardinal κ, if a space X is dually κ-separable, then $l(X) \leqslant 2^\kappa$.*

Proof: For any neighbourhood assignment $\mathcal{O}$ of the space X there exists a kernel K for $\mathcal{O}$ with $d(K) \leqslant \kappa$ and hence $hl(K) \leqslant w(K) \leqslant 2^\kappa$. Applying Theorem 2.8 of [vMTW] we convince ourselves that $l(X) \leqslant 2^\kappa$. □

2.12 Theorem. *Suppose that X is a space of countable tightness such that any subset $A \subset X$ with $|A| = \omega_1$ has a complete accumulation point. Then X is dually separable.*

Proof: Take any neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X. Pick a point $x_0 \in X$ arbitrarily; proceeding by induction assume that $\beta < \omega_1$ and we have chosen points $\{x_\alpha : \alpha < \beta\}$ and let $F_\beta = \overline{\{x_\alpha : \alpha < \beta\}}$. If $\mathcal{O}(F_\beta) = X$ then F_β is a separable kernel of $\mathcal{O}$. If not then choose a point $x_\beta \in X \backslash \mathcal{O}(F_\beta)$. If $\mathcal{O}(F_\beta) = X$ for some $\beta < \omega_1$ then $\mathcal{O}$ has a separable kernel as required. If not, then our inductive procedure gives us a set $P = \{x_\alpha : \alpha < \omega_1\}$ such that $x_\beta \notin \mathcal{O}(F_\beta)$ for any $\beta < \omega_1$.

By our assumption about X, the set P has a complete accumulation point x. Since $t(X) = \omega$, there is $\beta < \omega_1$ such that $x \in F_\beta$; as a consequence, $\mathcal{O}(F_\beta)$ is a neighbourhood of x which does not meet the set $\{x_\alpha : \alpha \geqslant \beta\}$, i.e., x is not a complete accumulation point of P which is a contradiction. Therefore some separable subspace F_β is a kernel of $\mathcal{O}$ and hence X is dually separable. □

2.13 Corollary. *Any linearly Lindelöf space of countable tightness is dually separable.*

2.14 Theorem. *There exist dually σ-compact spaces that are not dually separable. In particular, a linearly Lindelöf space need not be dually separable.*

Proof: Let $\xi_0 = \omega$ and $\xi_{n+1} = 2^{\xi_n}$ for all $n \in \omega$; we will need the cardinal $\xi = \sup\{\xi_n : n \in \omega\}$. Consider the subspace $X = \{x \in \mathbb{D}^\xi : |x^{-1}(1)| < \xi\}$ of the space $\mathbb{D}^\xi$. It was proved in [ATW] that X is dually σ-compact. If $l(X) < \xi$ then there exists $n \in \omega$ such that $l(X) \leqslant \xi_n$; the point $a \in \mathbb{D}^\xi$ defined by $a(\alpha) = 1$ for every $\alpha < \xi$ does not belong to X so there exists a G_{ξ_n}-subset H of the space $\mathbb{D}^\xi$ such that $a \in H \subset \mathbb{D}^\xi \backslash X$. It is easy to see that there exists a set $Q \subset \xi$ for which $|Q| \leqslant \xi_n$ and the set $G = \{x \in \mathbb{D}^\xi : x(Q) = \{1\}\} \subset H$. If $x(\alpha) = 1$ for all $\alpha \in Q$ and $x(\alpha) = 0$ whenever $\alpha \in \xi \backslash Q$ then $x \in G \cap X \subset H \cap X$ which is a contradiction. Therefore $l(X) = \xi > \mathfrak{c} = \xi_1$ and hence X cannot be dually separable by Proposition 2.11. Every dually σ-compact space is linearly Lindelöf by [vMTW, Proposition 2.7] so a linearly Lindelöf space is not necessarily dually separable. □

The following fact shows that the positive answer to Problem 4.13 from [ATW] is consistent with ZFC.

2.15 Proposition. *In every model of ZFC in which there are no S-spaces, any space X which is in the dual class of spaces of countable spread, is Lindelöf. In particular, if X is first countable and every neighbourhood assignment of X has a kernel of countable spread then $|X| \leqslant \mathfrak{c}$.*

Proof: It suffices to note that, in the absence of S-spaces, every space of countable spread is hereditarily Lindelöf (see [Ro, Proposition 3.3]) and apply Theorem 2.8 of [vMTW].

□

The result that follows gives a consistent negative answer to Problem 4.2 from the paper [ATW].

2.16 Theorem. *Denote the cardinal ω_ω by μ. In any model of ZFC in which $2^\mu = \mathfrak{c}$ there exists a dually Lindelöf space which is not dually Lindelöf Σ.*

Proof: In the space $\mathbb{D}^\mu$ consider the subspace $L = \{x : |x^{-1}(1)| < \mu\}$ and let $I = [0,1]$. For any $A \subset \mu$ define a point $u_A \in \mathbb{D}^A$ by $u_A(\alpha) = 0$ for all $\alpha \in A$. It was proved in [AB, Example 15] that the equality $2^\mu = \mathfrak{c}$ implies existence of a linearly Lindelöf subspace $X \subset L \times I$ such that the projection $p : X \to I$ is injective while the projection $q : X \to L$ is a surjective map.

To see that X is dually Lindelöf consider an arbitrary neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X. Since $w(X) \leqslant \mu$, we can choose a subspace $Y \subset X$ such that $|Y| \leqslant \mu$ and $X = \bigcup\{O_y : y \in Y\}$. It takes a straightforward induction to construct an injection $\varphi : Y \to \mu$ such that $|q(y)^{-1}(1)| \geqslant \omega_n$ implies $\varphi(y) \geqslant \omega_n$. Let $Y_n = \{y \in Y : \varphi(y) \leqslant \omega_n\}$ for every $n \in \omega$; then $Y = \bigcup_{n\in\omega} Y_n$. Fix any $n \in \omega$; it follows from the choice of φ that $|q(y)^{-1}(1)| \leqslant \omega_n$ for any $y \in Y_n$. Since also $|Y_n| \leqslant \omega_n$, we conclude that the cardinality of the set $A_n = \bigcup\{q(y)^{-1}(1) : y \in Y_n\}$ does not exceed ω_n and hence the compact set $K_n = \mathbb{D}^{A_n} \times \{u_{\mu\setminus A_n}\}$ is contained in L. The set $M_n = (K_n \times I) \cap X$ is closed in X so it is linearly Lindelöf; since also $w(M_n) \leqslant \omega_n$, every space M_n is Lindelöf. Furthermore, $Y_n \subset M_n$ for all $n \in \omega$ so $Y \subset M = \bigcup_{n\in\omega} M_n$ while M is Lindelöf and $\mathcal{O}(M) \supset \mathcal{O}(Y) = X$, i.e., M is a Lindelöf kernel of $\mathcal{O}$.

Finally observe that if every neighbourhood assignment $\mathcal{O}$ in the space X has a Lindelöf Σ-kernel K then $p|K$ condenses K onto a subset of I; an immediate consequence is that $nw(K) \leqslant \omega$ and, in particular, K is hereditarily Lindelöf so it follows from [vMTW, Theorem 2.8] that X is Lindelöf and hence L is Lindelöf being a continuous image of X. Since L is pseudocompact (see [AB, Example 15]) and not compact, we obtain a contradiction which shows that the space X is not dually Lindelöf Σ. □

2.17 Corollary. *If, for the cardinal $\mu = \omega_\omega$, we have $2^\mu = \mathfrak{c}$ then a dually dually σ-compact space need not be dually Lindelöf Σ. This gives a partial answer to Problem 4.15 from [ATW].*

Proof: Since any Lindelöf space is dually countable, any dually Lindelöf space is dually dually countable and hence dually dually σ-compact. We saw that, under our assumptions, there exists a dually Lindelöf space X which is not dually Lindelöf Σ. Therefore X is the promised example. □

3. Dual discreteness in generalized ordered spaces

Recall that X is a generalized ordered space (or GO space) if it is embeddable in a linearly ordered topological space. It was proved in [vMTW, Example 2.3] that the ordinal ω_1 with its interval topology is dually discrete. It turns out that this result holds for a much wider class of generalized ordered spaces.

3.1 Theorem. *Any GO space of countable extent is dually discrete.*

Proof: Suppose that X is a GO space with $\text{ext}(X) = \omega$. It is standard that there exists a compact linearly ordered space $(K, <)$ such that X is densely embedded in K and, for any $z \in K\setminus X$, the point z can be only reached from X from one side, i.e., either z has a successor in K or z is a successor of a point from K.

Consider the space $Y = X \cup \{x \in K\setminus X : \chi(x, K) > \omega\}$; then Y is realcompact (because the complement of the union of G_δ-subsets of a realcompact space is realcompact) so Y is paracompact by [Lu, Theorem 4.4]. It is an easy consequence of countability of extent of X and collectionwise normality of Y that $\mathrm{ext}(Y) = \omega$ and hence Y is actually Lindelöf but this will not be needed.

Fix an arbitrary neighbourhood assignment $\mathcal{O} = \{O_x : x \in X\}$ in the space X. We can consider, without loss of generality, that every O_x is the intersection with X of a convex open subset B_x of the space Y. We want to extend the family $\{B_x : x \in X\}$ to a neighbourhood assignment on the space Y so fix a point $y \in Y \backslash X$. By our choice of K, we have a symmetric situation about the side from which y can be reached from X so we can consider that y does not have a predecessor and hence it does not belong to the closure of the set $(y, \to)_K$. We claim that

($*$) there exists a point $a_y < y$, $a_y \in X$ and a discrete subspace $C_y \subset X \cap (\leftarrow, y)_K$ such that C_y is cofinal in $(\leftarrow, y)_K$ and $(a_y, z]_X \subset O_z$ for any $z \in C_y$.

Let us show first that there exists a point $a < y$ and a set $A \subset (\leftarrow, y)_K \cap X$ such that $a \in X$, the set A is cofinal in $(\leftarrow, y)_K$ and $(a, z]_X \subset O_z$ for all $z \in A$. Indeed, if this is false then, for any $x < y$ there exists a point $b \in (x, y)_K \cap X$ such that $O_z \subset (x, \to)_X$ for any $z \in (b, y)_K \cap X$.

It follows from $\chi(y, K) > \omega$ that no countable subset of $X \cap (\leftarrow, y)_K$ is cofinal in $(\leftarrow, y)_K$ and $(x, y)_K \cap X$ is non-empty for any $x < y$ so we can construct inductively an ω_1-sequence $B = \{b_\alpha : \alpha < \omega_1\} \subset (\leftarrow, y)_K \cap X$ such that $\alpha < \beta < \omega_1$ implies $b_\alpha < b_\beta$ and $O_z \subset (b_\alpha, \to)_X$ for any $\alpha < \omega_1$ and $z \in X \cap (b_\beta, y)_K$. It follows from $\mathrm{ext}(X) \leqslant \omega$ that we can find a cluster point $x \in X$ for the set B. Since B is increasing, the point x does not belong to the closure of the set $(x, \to)_X \cap B$ so x is a cluster point of the set $(\leftarrow, x)_X \cap B$. Therefore we can find $\alpha, \beta \in \omega_1$ such that $b_\alpha < b_\beta < x$ and $\{b_\alpha, b_\beta\} \subset O_x$. It follows from the choice of B that we must have $O_x \subset (b_\alpha, \to)$ which is a contradiction. The space K being discretely generated (see [DTTW, Theorem 3.10]) we can choose a point $a_y < y$, $a_y \in X$ and a discrete subspace $C_y \subset (\leftarrow, y]_K \cap X$ which contains y in its closure and $(a_y, x]_X \subset O_x$ for any $x \in D$, i.e., ($*$) is proved.

Analogously, if the point $y \in K \backslash X$ is a successor then we can choose a discrete subspace $C_y \subset (y, \to)_X$ and $a_y > y$, $a_y \in X$ such that $y \in \overline{C_y}$ and $[x, a_y)_X \subset O_x$ for any $x \in C_y$.

If $y \in K \backslash X$ has a successor then let $B_y = (a_y, y]_Y$; if y is a successor then $B_y = [y, a_y)_Y$. The neighbourhood assignment $\{B_y : y \in Y\}$ has a closed discrete kernel D because Y is a D-space by [vDL, Theorem 1.2]. Since Y is collectionwise normal, we can choose a discrete family $\mathcal{W} = \{W_d : d \in D\}$ of convex open subsets of Y such that $d \in W_d$ for any $d \in D$. It is an easy exercise that the subspace $E = (D \cap X) \cup (\bigcup \{C_y \cap W_y : y \in D \backslash X\})$ is a discrete kernel of $\mathcal{O}$ so X is dually discrete. $\square$

3.2 Corollary. *Any countably compact GO space is dually discrete.*

3.3 Corollary. *Every locally compact GO space is dually discrete.*

Proof: Let X be a locally compact GO space; we can assume that X is a dense subspace of a compact linearly ordered space $(K, <)$. Let us prove first that

(1) $\operatorname{ext}(U) = \omega$ for every convex open set U in the space $(K, <)$.

Assume toward a contradiction that $D \subset U$ is an uncountable closed and discrete subset of U. There is no loss of generality to assume that there exists a point $a \in U$ such that the set $D' = D \cap (a, \to)_U$ is uncountable. For any $b \in (a, \to)_U$ the set $[a, b]_U = [a, b]_K$ is compact so $D' \cap [a, b]_U$ is finite. Now, it is easy to construct by induction a set $E = \{d_n : n \in \omega\} \subset D'$ such that $d_n < d_{n+1}$ for all $n \in \omega$.

If E is not cofinal in the set U then $E \subset [a, b]_U$ for some $b \in (a, \to)_U$ which is a contradiction. Thus the set E is cofinal in $(U, <)$ so we have the equality $[a, \to)_U = [a, d_0]_U \cup (\bigcup\{[d_i, d_{i+1}]_U : i \in \omega\})$. This, together with $D' \subset [a, \to)_U$ shows that either $D' \cap [a, d_0]_U$ is uncountable or $|[d_i, d_{i+1}]_U \cap D'| > \omega$ for some $i \in \omega$. Since we obtain a contradiction in all possible cases, the property (1) is proved.

It follows from local compactness of X that X is open in K so we have the equality $X = \bigcup\{X_t : t \in T\}$ for some disjoint family $\{X_t : t \in T\}$ of open convex subsets of K. The property (1) shows that $\operatorname{ext}(X_t) \leqslant \omega$ and hence X_t is dually discrete for every $t \in T$. Finally, observe that X is homeomorphic to $\bigoplus\{X_t : t \in T\}$ so X is also dually discrete. $\square$

3.4 Corollary. *If α is an ordinal with its interval topology then α is dually discrete.*

4. Open problems

As usual, the more problems one solves, the more unsolved problems arise. The topic of this paper is by no means an exception; to illustrate this, we present below a list of problems which might require new methods for their solution.

4.1 Problem. *Is any linearly ordered space dually discrete?*

4.2 Problem. *Is any GO space dually discrete?*

4.3 Problem. *Must every monotonically normal space be dually discrete?*

4.4 Problem. *Suppose that X is a GO space of locally countable extent, i.e., every $x \in X$ has a neighbourhood of countable extent. Must X be dually discrete?*

4.5 Problem. *Must every dually metrizable space be dually discrete?*

4.6 Problem. *Suppose that X is first countable and every neighbourhood assignment in a space X has a kernel of countable spread. Is it true in ZFC that $|X| \leqslant \mathfrak{c}$?*

4.7 Problem. *Is there a model of ZFC in which $\mathbb{R}^{\omega_1}$ is dually discrete?*

4.8 Problem. *Suppose that a space X is dually discrete. Must every perfect image of X be dually discrete?*

4.9 Problem. *Suppose that $X = \bigcup_{n\in\omega} X_n$ and every X_n is a discrete subspace of X. Must X be dually discrete?*

4.10 Problem. *Must any Lindelöf monotonically normal space be a D-space?*

4.11 Problem. *Must any paracompact monotonically normal space be a D-space?*

4.12 Problem. *Suppose that a space X is dually finally discrete, i.e., any neighbourhood assignment in X has a kernel which is a finite union of discrete subspaces of X. Must X be dually discrete?*

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